8.3 - Chebyshev Polynomials

Chebyshev polynomials

Definition

Chebyshev polynomial of degree $n \geq= 0$ is defined as

$T_n(x) = \cos(n \arccos x)$, $\quad x \in [-1, 1]$, or, in a more instructive form,

$T_n(x) = \cos n\theta$, $\quad x = \cos\theta$, $\quad \theta \in [0, \pi]$.

Recursive relation of Chebyshev polynomials

$$T_0(x) = 1 \; , \quad T_1(x) = x \; ,$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \; , \quad n \geq 1 \; .$$

Thus

$$T_2(x) = 2x^2 - 1 \; , \; T_3(x) = 4x^3 - 3x \; , \; T_4(x) = 8x^4 - 8x^2 + 1 \quad \cdots$$

$T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1} for $n \geq 1$.

Orthogonality

Chebyshev polynomials are orthogonal w.r.t. weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. Namely,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if } \ m \neq n \\ \frac{\pi}{2} & \text{if } \ n = m \ \text{ for each } \ n \geq 1 \end{cases} \tag{1}$$

Theorem (Roots of Chebyshev polynomials)

The roots of $T_n(x)$ of degree $n \geq 1$ has n simple zeros in $[-1, 1]$ at $\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, $\;$ for each $\;k = 1, 2 \cdots n$.
Moreover, $T_n(x)$ assumes its absolute extrema at $\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right)$, $\;$ with $\;T_n(\bar{x}'_k) = (-1)^k$, $\;$ for each $k = 0, 1, \cdots n$.

For $k = 0, \cdots n$, $T_n(\bar{x}'_k) = \cos\left(n\cos^{-1}\left(\cos(k\pi/n)\right)\right) = \cos(k\pi) = (-1)^k$.

Definition

A monic polynomial is a polynomial with leading coefficient 1.

The monic Chebyshev polynomial $\tilde{T}_n(x)$ is defined by dividing $T_n(x)$ by $2^{n-1}, n \geq 1$. Hence,

$$\tilde{T}_0(x) = 1, \quad \tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x) \ , \quad \text{for each } n \geq 1$$

They satisfy the following recurrence relations
$\tilde{T}_2(x) = x\tilde{T}_1(x) - \frac{1}{2}\tilde{T}_0(x)$
$\tilde{T}_{n+1}(x) = x\tilde{T}_n(x) - \frac{1}{4}\tilde{T}_{n-1}(x) \quad \text{for each } n \geq 2$
The location of the zeros and extrema of $\tilde{T}_n(x)$ coincides with those of $T_n(x)$, however the extrema values are
$\tilde{T}_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}}$, at $\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right)$, $\quad k = 0, \cdots n.$

Definition

Let $\tilde{\prod}_n$ denote the set of all monic polynomials of degree n.

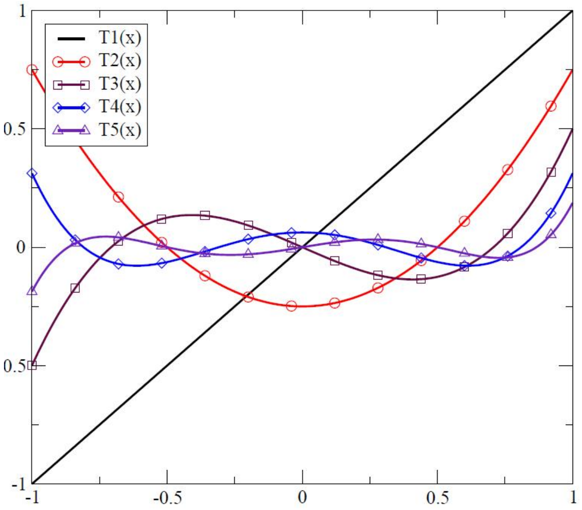
Theorem

(Min-Max Theorem) The monic Chebyshev polynomials $\tilde{T}_n(x)$, when $n \geq 1$, have the property that

$$\frac{1}{2^{n-1}} = max_{x\in[-1,1]}|\tilde{T}_n(x)| \leq max_{x\in[-1,1]}|P_n(x)| \ , \quad \text{for all} \quad P_n(x) \in \tilde{\prod}_n \ .$$

Moreover, equality occurs only if $P_n \equiv \tilde{T}_n$.

Diagrams of monic Chebyshev polynomials



Optimal node placement in Lagrange interpolation

If $x_0, x_1, \cdots x_n$ are distinct points in the interval $[-1, 1]$ and $f \in C^{n+1}[-1, 1]$, and $P(x)$ the nth degree interpolating Lagrange polynomial, then $\forall x \in [-1, 1]$, $\exists \xi(x) \in (-1, 1)$ so that

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{k=0}^{n} (x - x_k)$$

We place the nodes in a way to minimize the maximum $\prod_{k=0}^{n}(x - x_k)$. Since $\prod_{k=0}^{n}(x - x_k)$ is a monic polynomial of degree $(n+1)$, the min-max is obtained when the nodes are chosen so that

$$\prod_{k=0}^{n}(x - x_k) = \tilde{T}_{n+1}(x) \ , \qquad \text{i.e.} \quad x_k = \cos\left(\frac{2k+1}{2(n+1)}\pi\right)$$

for $k = 0, \cdots, n$. Min-Max theorem implies that
$\frac{1}{2^n} = max_{x\in[-1,1]} |(x - \bar{x}_1) \cdots (x - \bar{x}_{n+1})| \leq max_{x\in[-1,1]} \prod_{k=0}^{n} |(x - x_k)|$

Theorem

Suppose that $P(x)$ is the interpolating polynomial of degree at most n with nodes at the zeros of $T_{n+1}(x)$. Then
$\max_{x\in[-1,1]}|f(x)-P(x)| \leq \frac{1}{2^n(n+1)!}\max_{x\in[-1,1]}|f^{(n+1)}(x)|$,
for each $f \in C^{n+1}[-1,1]$.

Extending to any interval: The transformation $\tilde{x} = \frac{1}{2}[(b-a)x + (a+b)]$ transforms the nodes x_k in $[-1,1]$ into the corresponding nodes $\tilde{x}_k$ in $[a,b]$.