4.8 Multiple Integrals and Monte Carlo Integration

- Composite Trapezoidal rule to approximate double Integral: $\iint_R f(x,y)dA$ where $R = \{(x,y) | a \le x \le b, c \le y \le d\}$ for some constants a, b, c and d.

- Let $k = (d-c)/2, h = (b-a)/2$.

- $\iint_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy\right) dx$

$$\approx \int_a^b \left(\frac{d-c}{4}\left[f(x,c) + 2f\left(x, \frac{c+d}{2}\right) + f(x,d)\right]\right) dx$$

$$\approx \int_a^b \left(\frac{d-c}{4} \left[f(x,c) + 2f\left(x, \frac{c+d}{2}\right) + f(x,d) \right] \right) dx$$

$$= \frac{(b-a)}{4} \frac{(d-c)}{4} \left[f(a,c) + 2f\left(a, \frac{c+d}{2}\right) + f(a,d) \right]$$

$$+ \frac{(b-a)}{4} \frac{(d-c)}{4} 2 \left[f\left(\frac{a+b}{2}, c\right) + 2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, d\right) \right]$$

$$+ \frac{(b-a)}{4} \frac{(d-c)}{4} \left[f(b,c) + 2f\left(b, \frac{c+d}{2}\right) + f(b,d) \right]$$

The approximation is of order $O((b-a)(d-c)[(b-a)^2 + (d-c)^2])$

Monte Carlo Simulation

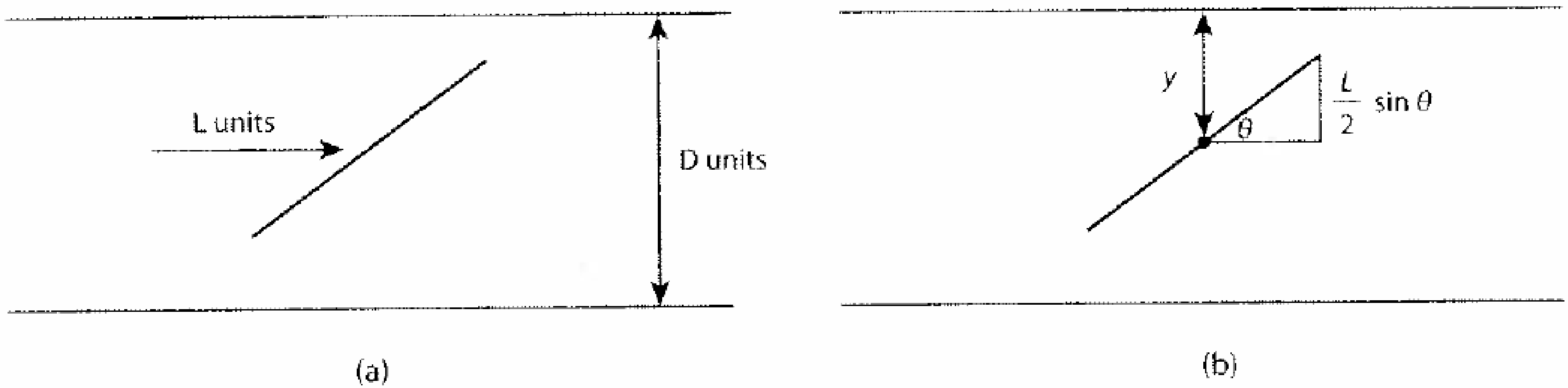
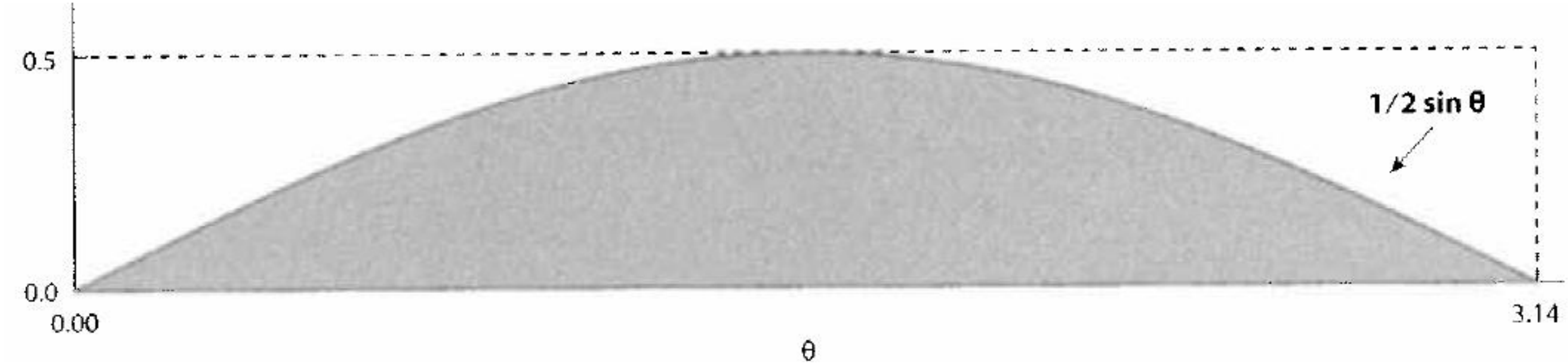


Figure 3.6. Buffon's needle experiment: (a) depicts the experiment where a needle of length L is randomly dropped between two lines a distance D apart. In (b), y denotes the distance between the needle's midpoint and the closest line; θ is the angle of the needle to the horizontal.

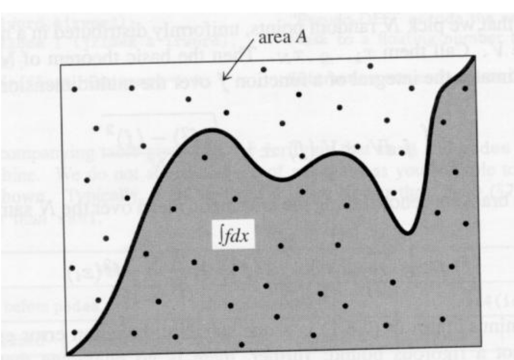
The needle crosses a line if $y \leq L/2\sin(\theta)$

Q: What's the probability p that the needle will intersect on of these lines?

- Let y be the distance between the needle's midpoint and the closest line, and θ be the angle of the needle to the horizontal.
- Assume that y takes uniformly distributed values between 0 and D/2; and θ takes uniformly distributed values between 0 and π.



- Let $L = D = 1$.
- The probability is the ratio of the area of the shaded region to the area of rectangle.
- $p = \frac{\int_0^{\pi} \frac{1}{2} sin\theta d\theta}{\pi/2} = 2/\pi$

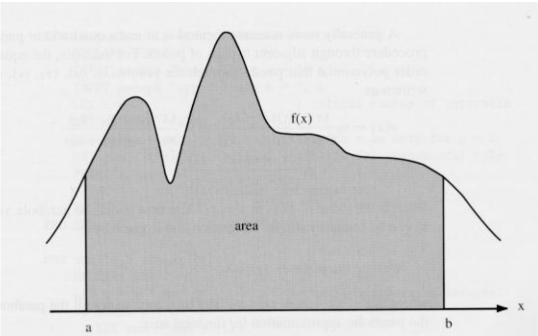


Hit and miss method:
The volume of the external region is V_e and the fraction of hits is f_h. Then the volume of the region to be integrated is $V = V_e f_h$.

Algorithm of Monte Carlo

- Define a domain of possible inputs.
- Generate inputs randomly from a probability distribution over the domain.
- Perform a deterministic computation on the inputs.
- aggregate the results from all deterministic computation.

Monte Carlo Integration (sampling)



$$A = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x$$

Where $\Delta x = \frac{b-a}{N}, x_i = a + (i - 0.5)\Delta x$.

$$A \approx \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$$

Which can be interpreted as taking the average over f in the interval, i.e., $A \approx (b-a) < f >$, where $< f > = \frac{\sum_{i=1}^{N} f(x_i)}{N}$.