2 The Derivation of Conservation Laws

2.1 Integral and differential forms

To see how conservation laws arise from physical principles, we will begin by deriving the equation for conservation of mass in a one-dimensional gas dynamics problem, for example flow in a tube where properties of the gas such as density and velocity are assumed to be constant across each cross section of the tube. Let x represent the distance along the tube and let $\rho(x,t)$ be the density of the gas at point x and time t. This density is defined in such a way that the total mass of gas in any given section from x_1 to x_2, say, is given by the integral of the density:

$$\text{mass in } [x_1, x_2] \text{ at time } t = \int_{x_1}^{x_2} \rho(x,t)\,dx. \tag{2.1}$$

If we assume that the walls of the tube are impermeable and that mass is neither created nor destroyed, then the mass in this one section can change only because of gas flowing across the endpoints x_1 or x_2.

Now let $v(x,t)$ be the velocity of the gas at the point x at time t. Then the rate of flow, or **flux** of gas past this point is given by

$$\text{mass flux at } (x,t) = \rho(x,t)v(x,t). \tag{2.2}$$

By our comments above, the rate of change of mass in $[x_1, x_2]$ is given by the difference in fluxes at x_1 and x_2:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t). \tag{2.3}$$

This is one **integral form** of the conservation law. Another form is obtained by integrating this in time from t_1 to t_2, giving an expression for the mass in $[x_1, x_2]$ at time $t_2 > t_1$ in terms of the mass at time t_1 and the total (integrated) flux at each boundary during

this time period:

$$\begin{aligned}\int_{x_1}^{x_2} \rho(x,t_2)\,dx &= \int_{x_1}^{x_2} \rho(x,t_1)\,dx \\ &\quad + \int_{t_1}^{t_2} \rho(x_1,t)v(x_1,t)\,dt - \int_{t_1}^{t_2} \rho(x_2,t)v(x_2,t)\,dt.\end{aligned} \tag{2.4}$$

To derive the differential form of the conservation law, we must now assume that $\rho(x,t)$ and $v(x,t)$ are differentiable functions. Then using

$$\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(x,t)\,dt \tag{2.5}$$

and

$$\rho(x_2,t)v(x_2,t) - \rho(x_1,t)v(x_1,t) = \int_{x_1}^{x_2} \frac{\partial}{\partial x}(\rho(x,t)v(x,t))\,dx \tag{2.6}$$

in (2.4) gives

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \left\{\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}(\rho(x,t)v(x,t))\right\} dx\,dt = 0. \tag{2.7}$$

Since this must hold for any section $[x_1, x_2]$ and over any time interval $[t_1, t_2]$, we conclude that in fact the integrand in (2.7) must be identically zero, i.e.,

$$\rho_t + (\rho v)_x = 0 \qquad \text{conservation of mass.} \tag{2.8}$$

This is the desired **differential form** of the conservation law for the conservation of mass.

The conservation law (2.8) can be solved in isolation only if the velocity $v(x,t)$ is known a *priori* or is known as a function of $\rho(x,t)$. If it is, then ρv is a function of ρ alone, say $\rho v = f(\rho)$, and the conservation of mass equation (2.8) becomes a scalar conservation law for ρ,

$$\rho_t + f(\rho)_x = 0. \tag{2.9}$$

More typically the equation (2.8) must be solved in conjunction with equations for the conservation of momentum and energy. These equations will be derived and discussed in more detail in Chapter 5. For now we simply state them for the case of the **Euler equations** of gas dynamics:

$$(\rho v)_t + (\rho v^2 + p)_x = 0 \qquad \text{conservation of momentum} \tag{2.10}$$

$$E_t + (v(E+p))_x = 0 \qquad \text{conservation of energy} \tag{2.11}$$

Note that these equatons involve another quantity, the pressure p, which must be specified as a given function of ρ, ρv, and E in order that the fluxes are well defined functions of the conserved quantities alone. This additional equation is called the **equation of state** and depends on physical properties of the gas under study.

If we introduce the vector $u \in \mathbb{R}^3$ as

$$u(x,t) = \begin{bmatrix} \rho(x,t) \\ \rho(x,t)v(x,t) \\ E(x,t) \end{bmatrix} \equiv \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \tag{2.12}$$

then the system of equations (2.8), (2.10), (2.11) can be written simply as

$$u_t + f(u)_x = 0 \tag{2.13}$$

where

$$f(u) = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix} = \begin{bmatrix} u_2 \\ u_2^2/u_1 + p(u) \\ u_2(u_3 + p(u))/u_1 \end{bmatrix}. \tag{2.14}$$

Again, the form (2.13) is the differential form of the conservation laws, which holds in the usual sense only where u is smooth. More generally, the integral form for a system of m equations says that

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t)) \tag{2.15}$$

for all x_1, x_2, t. Equivalently, integrating from t_1 to t_2 gives

$$\begin{aligned} \int_{x_1}^{x_2} u(x,t_2)\,dx &= \int_{x_1}^{x_2} u(x,t_1)\,dx \\ &\quad + \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt \end{aligned} \tag{2.16}$$

for all x_1, x_2, t_1, and t_2. These integral forms of the conservation law will be fundamental in later analysis.

2.2 Scalar equations

Before tackling the complications of coupled systems of equations, we will first study the case of a scalar equation, where $m = 1$. An example is the conservation of mass equation (2.8) in the case where v is a known function of $\rho(x,t)$. This does not happen in gas dynamics, but can occur in other problems where the same conservation law holds. One example is a simple model of traffic flow along a highway. Here ρ is the density of vehicles and the velocity at which people drive is assumed to depend only on the local density. This example is explored in detail in Chapter 4.

Another possibility is that the velocity $v(x,t)$ is given a *priori*, completely independent of the unknown ρ. Suppose, for example, that ρ represents the density of some chemical in flowing water, a pollutant in a stream, for example. Since the total quantity of this

chemical is conserved, the same derivation as above again yields (2.8). Now it is reasonable to consider a case where v, the fluid velocity, is known from other sources and can be assumed given. Changes in concentration of the chemical will have little or no effect on the fluid dynamics and so there will be no coupling in which v depends on ρ.

Strictly speaking, this conservation law is not of the form (1.1), since the flux function now depends explicitly on x and t as well as on ρ,

$$f(\rho,x,t) = \rho v(x,t). \tag{2.17}$$

However, the conservation law is still hyperbolic and this variable coefficient linear equation is convenient to study as an introductory example. Moreover, if the velocity is constant, $u(x,t) \equiv a$, then $f(\rho) = a\rho$ and (2.8) reduces to

$$\rho_t + a\rho_x = 0. \tag{2.18}$$

This equation is known as the **linear advection equation** or sometimes the **one-way wave equation**. If this equation is solved for $t \geq 0$ with initial data

$$\rho(x,0) = \rho_0(x) \qquad -\infty < x < \infty \tag{2.19}$$

then it is easy to check (assuming ρ_0 is differentiable) that the solution is simply

$$\rho(x,t) = \rho_0(x - at). \tag{2.20}$$

Note that the initial profile simply moves downstream with velocity a, its shape unchanged. It is reasonable to say that the solution is given by (2.20) even if the initial data $\rho_0(x)$ is not smooth, as will be justified in Chapter 3.

2.3 Diffusion

In the example just given of a concentration profile moving downstream, the result that this profile remains unchanged may seem physically unreasonable. We expect mixing and diffusion to occur as well as advection. Mixing caused by turbulence in the fluid is clearly ignored in this simple one-dimensional model, but molecular diffusion should occur even in a stream moving at constant velocity, since by "velocity" we really mean the macroscopic average velocity of the molecules. Individual molecules travel with random directions and speeds distributed about this average. In particular, the molecules of our chemical undergo this random motion and hence there is a net motion of these molecules away from regions of high concentration and towards regions of lower concentration. Since the chemical is still conserved, we can incorporate this effect into the conservation law by modifying the flux function appropriately.

Consider the flux of the chemical past some point x in the tube or stream. In addition to the advective flux $a\rho$ there is also a net flux due to diffusion whenever the concentration profile is not flat at the point x. This flux is determined by "Fourier's Law of heat conduction" (heat diffuses in much the same way as the chemical concentration), which says that the diffusive flux is simply proportional to the gradient of concentration:

$$\text{diffusive flux} = -D\rho_x. \tag{2.21}$$

The diffusion coefficient $D > 0$ depends on the variance of the random component of the particle speeds. The minus sign in (2.21) is needed since the net flux is *away* from higher concentrations.

Combining this flux with the advective flux $a\rho$ gives the flux function

$$f(\rho, \rho_x) = a\rho - D\rho_x \tag{2.22}$$

and the conservation law becomes

$$\rho_t + (a\rho - D\rho_x)_x = 0 \tag{2.23}$$

or, assuming D is constant,

$$\rho_t + a\rho_x = D\rho_{xx}. \tag{2.24}$$

Equation (2.24) is called the **advection-diffusion equation** (or sometimes the convection-diffusion equation).

Note that the flux function $f(\rho, \rho_x)$ now depends on ρ_x as well as ρ. Equation (2.23), while still a conservation law, is not of the form (1.1). In this case, the dependence of f on the gradient completely changes the character of the equation and solution. The advection-diffusion equation (2.24) is a parabolic equation while (2.18) is hyperbolic. One major difference is that (2.24) always has smooth solutions for $t > 0$ even if the initial data $\rho_0(x)$ is discontinuous. This is analogous to the relation between the Euler equations and Navier-Stokes equations described in Chapter 1. We can view (2.18) as an approximation to (2.24) valid for D very small, but we may need to consider the effect of D in order to properly interpret discontinuous solutions to (2.18).

3 Scalar Conservation Laws

We begin our study of conservation laws by considering the scalar case. Many of the difficulties encountered with systems of equations are already encountered here, and a good understanding of the scalar equation is required before proceeding.

3.1 The linear advection equation

We first consider the linear advection equation, derived in Chapter 2, which we now write as

$$u_t + au_x = 0. \tag{3.1}$$

The Cauchy problem is defined by this equation on the domain $-\infty < x < \infty$, $t \geq 0$ together with initial conditions

$$u(x, 0) = u_0(x). \tag{3.2}$$

As noted previously, the solution is simply

$$u(x, t) = u_0(x - at) \tag{3.3}$$

for $t \geq 0$. As time evolves, the initial data simply propagates unchanged to the right (if $a > 0$) or left (if $a < 0$) with velocity a. The solution $u(x, t)$ is constant along each ray $x - at = x_0$, which are known as the **characteristics** of the equation. (See Fig. 3.1 for the case $a > 0$.)

Note that the characteristics are curves in the x-t plane satisfying the ordinary differential equations $x'(t) = a$, $x(0) = x_0$. If we differentiate $u(x, t)$ along one of these curves to find the rate of change of u along the characteristic, we find that

$$\begin{aligned} \frac{d}{dt}u(x(t), t) &= \frac{\partial}{\partial t}u(x(t), t) + \frac{\partial}{\partial x}u(x(t), t)\, x'(t) \\ &= u_t + au_x \\ &= 0, \end{aligned} \tag{3.4}$$

confirming that u is constant along these characteristics.

More generally, we might consider a variable coefficient advection equation of the form

$$u_t + (a(x)u)_x = 0, \tag{3.5}$$

where $a(x)$ is a smooth function. Recalling the derivation of the advection equation in Chapter 2, this models the evolution of a chemical concentration $u(x,t)$ in a stream with variable velocity $a(x)$.

We can rewrite (3.5) as

$$u_t + a(x)u_x = -a'(x)u \tag{3.6}$$

or

$$\left(\frac{\partial}{\partial t} + a(x)\frac{\partial}{\partial x}\right) u(x,t) = -a'(x)u(x,t). \tag{3.7}$$

It follows that the evolution of u along any curve $x(t)$ satisfying

$$\begin{aligned} x'(t) &= a(x(t)) \\ x(0) &= x_0 \end{aligned} \tag{3.8}$$

satisfies a simple ordinary differential equation (ODE):

$$\frac{d}{dt}u(x(t),t) = -a'(x(t))\,u(x(t),t). \tag{3.9}$$

The curves determined by (3.8) are again called characteristics. In this case the solution $u(x,t)$ is not constant along these curves, but can be easily determined by solving two sets of ODEs.

It can be shown that if $u_0(x)$ is a smooth function, say $u_0 \in C^k(-\infty,\infty)$, then the solution $u(x,t)$ is equally smooth in space and time, $u \in C^k((-\infty,\infty)\times(0,\infty))$.

3.1.1 Domain of dependence

Note that solutions to the linear advection equations (3.1) and (3.5) have the following property: the solution $u(x,t)$ at any point $(\bar{x},\bar{t})$ depends on the initial data u_0 only at a ***single*** point, namely the point $\bar{x}_0$ such that $(\bar{x},\bar{t})$ lies on the characteristic through $\bar{x}_0$. We could change the initial data at any points other than $\bar{x}_0$ without affecting the solution $u(\bar{x},\bar{t})$. The set $\mathcal{D}(\bar{x},\bar{t}) = \{\bar{x}_0\}$ is called the **domain of dependence** of the point $(\bar{x},\bar{t})$. Here this domain consists of a single point. For a system of equations this domain is typically an interval, but a fundamental fact about hyperbolic equations is that it is always a *bounded* interval. The solution at $(\bar{x},\bar{t})$ is determined by the initial data within some finite distance of the point $\bar{x}$. The size of this set usually increases with $\bar{t}$, but we have a bound of the form $\mathcal{D} \subset \{x : |x-\bar{x}| \le a_{\max}\bar{t}\}$ for some value $a_{\max}$. Conversely, initial data at any given point x_0 can influence the solution only within some

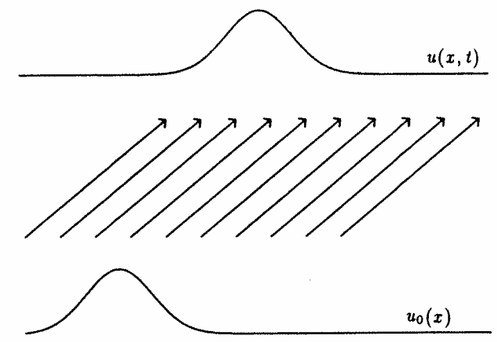


Figure 3.1. Characteristics and solution for the advection equation.

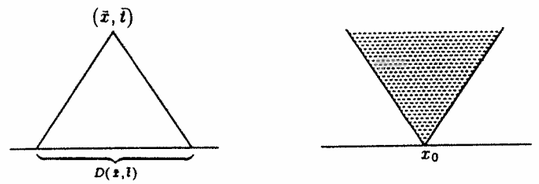


Figure 3.2. Domain of dependence and range of influence.

cone $\{x : |x-x_0| \le a_{\max}t\}$ of the x-t plane. This region is called the **range of influence** of the point x_0. See Figure 3.2 for an illustration. We summarize this by saying that hyperbolic equations have **finite propagation speed**; information can travel with speed at most $a_{\max}$. This has important consequences in developing numerical methods.

3.1.2 Nonsmooth data

In the manipulations performed above, we have assumed differentiability of $u(x,t)$. However, from our observation that the solution along a characteristic curve depends only on the one value $u_0(x_0)$, it is clear that spatial smoothness is not required for this construction of $u(x,t)$ from $u_0(x)$. We can thus define a "solution" to the PDE even if $u_0(x)$ is not a smooth function. Note that if $u_0(x)$ has a singularity at some point x_0 (a discontinuity in u_0 or some derivative), then the resulting $u(x,t)$ will have a singularity of the same order

along the characteristic curve through x_0, but will remain smooth along characteristics through smooth portions of the data. This is a fundamental property of *linear* hyperbolic equations: singularities propagate only along characteristics.

If u_0 is nondifferentiable at some point then $u(x,t)$ is no longer a classical solution of the differential equation everywhere. However, this function *does* satisfy the integral form of the conservation law, which continues to make sense for nonsmooth u. Recall that the integral form is more fundamental physically than the differential equation, which was derived from the integral form under the additional assumption of smoothness. It thus makes perfect sense to accept this generalized solution as solving the conservation law.

EXERCISE 3.1. *Let $f(u) = au$, with a constant, and let $u_0(x)$ be any integrable function. Verify that the function $u(x,t) = u_0(x-at)$ satisfies the integral form (2.16) for any x_1, x_2, t_1 and t_2.*

Other approaches can also be taken to defining this generalized solution, which extend better to the study of nonlinear equations where we can no longer simply integrate along characteristics.

One possibility is to approximate the nonsmooth data $u_0(x)$ by a sequence of smooth functions $u_0^\epsilon(x)$, with

$$\|u_0 - u_0^\epsilon\|_1 < \epsilon$$

as $\epsilon \to 0$. Here $\|\cdot\|_1$ is the 1-norm, defined by

$$\|v\|_1 = \int_{-\infty}^{\infty} |v(x)|\,dx. \tag{3.10}$$

For the linear equation we know that the PDE together with the smooth data u_0^ϵ has a smooth classical solution $u^\epsilon(x,t)$ for all $t \geq 0$. We can now define the generalized solution $u(x,t)$ by taking the limit of $u^\epsilon(x,t)$ as $\epsilon \to 0$. For example, the constant coefficient problem (3.1) has classical smooth solutions

$$u^\epsilon(x,t) = u_0^\epsilon(x-at)$$

and clearly at each time t the 1-norm limit exists and satisfies

$$u(x,t) = \lim_{\epsilon \to 0} u_0^\epsilon(x-at) = u_0(x-at)$$

as expected.

Unfortunately, this approach of smoothing the initial data will not work for nonlinear problems. As we will see, solutions to the nonlinear problem can develop singularities even if the initial data is smooth, and so there is no guarantee that classical solutions with data $u_0^\epsilon(x)$ will exist.

A better approach, which does generalize to nonlinear equations, is to leave the initial data alone but modify the PDE by adding a small diffusive term. Recall from Chapter 2

that the conservation law (3.1) should be considered as an approximation to the advection-diffusion equation

$$u_t + au_x = \epsilon u_{xx} \tag{3.11}$$

for ϵ very small. If we now let $u^\epsilon(x,t)$ denote the solution of (3.11) with data $u_0(x)$, then $u^\epsilon \in C^\infty((-\infty,\infty) \times (0,\infty))$ even if $u_0(x)$ is not smooth since (3.11) is a parabolic equation. We can again take the limit of $u^\epsilon(x,t)$ as $\epsilon \to 0$, and will obtain the same generalized solution $u(x,t)$ as before.

Note that the equation (3.11) simplifies if we make a change of variables to follow the characteristics, setting

$$v^\epsilon(x,t) = u^\epsilon(x+at,t).$$

Then it is easy to verify that v^ϵ satsifies the **heat equation**

$$v_t^\epsilon(x,t) = \epsilon v_{xx}^\epsilon(x,t). \tag{3.12}$$

Using the well-known solution to the heat equation to solve for $v(x,t)$, we have $u^\epsilon(x,t) = v^\epsilon(x-at,t)$ and so can explicitly calculate the "vanishing viscosity" solution in this case.

EXERCISE 3.2. *Show that the vanishing viscosity solution $\lim_{\epsilon \to 0} u^\epsilon(x,t)$ is equal to $u_0(x-at)$.*

3.2 Burgers' equation

Now consider the nonlinear scalar equation

$$u_t + f(u)_x = 0 \tag{3.13}$$

where $f(u)$ is a nonlinear function of u. We will assume for the most part that $f(u)$ is a convex function, $f''(u) > 0$ for all u (or, equally well, f is concave with $f''(u) < 0\ \forall u$). The convexity assumption corresponds to a "genuine nonlinearity" assumption for systems of equations that holds in many important cases, such as the Euler equations. The nonconvex case is also important in some applications (e.g. oil reservoir simulation) but is more complicated mathematically. One nonconvex example, the Buckley-Leverett equation, is discussed in the next chapter.

By far the most famous model problem in this field is **Burgers' equation**, in which $f(u) = \frac{1}{2}u^2$, so (3.13) becomes

$$u_t + uu_x = 0. \tag{3.14}$$

Actually this should be called the "inviscid Burgers' equation", since the equation studied by Burgers[5] also includes a viscous term:

$$u_t + uu_x = \epsilon u_{xx}. \tag{3.15}$$

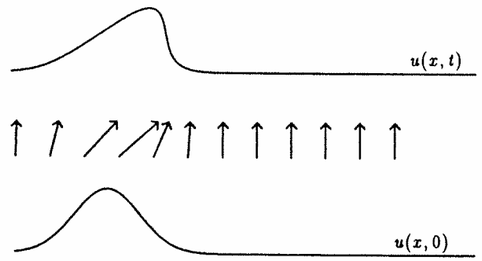


Figure 3.3. Characteristics and solution for Burgers' equation (small t).

This is about the simplest model that includes the nonlinear and viscous effects of fluid dynamics.

Around 1950, Hopf, and independently Cole, showed that the *exact* solution of the nonlinear equation (3.15) could be found using what is now called the **Cole-Hopf transformation**. This reduces (3.15) to a linear heat equation. See Chapter 4 of Whitham[97] for details.

Consider the inviscid equation (3.14) with smooth initial data. For small time, a solution can be constructed by following characteristics. Note that (3.14) looks like an advection equation, but with the advection velocity u equal to the value of the advected quantity. The characteristics satisfy

$$x'(t) = u(x(t), t) \tag{3.16}$$

and along each characteristic u is constant, since

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \frac{\partial}{\partial t}u(x(t),t) + \frac{\partial}{\partial x}u(x(t),t)\,x'(t) \\ &= u_t + uu_x \\ &= 0. \end{aligned} \tag{3.17}$$

Moreover, since u is constant on each characteristic, the slope $x'(t)$ is constant by (3.16) and so the characteristics are straight lines, determined by the initial data (see Fig. 3.3).

If the initial data is smooth, then this can be used to determine the solution $u(x,t)$ for small enough t that characteristics do not cross: For each (x,t) we can solve the equation

$$x = \xi + u(\xi, 0)t \tag{3.18}$$

for ξ and then

$$u(x,t) = u(\xi, 0). \tag{3.19}$$

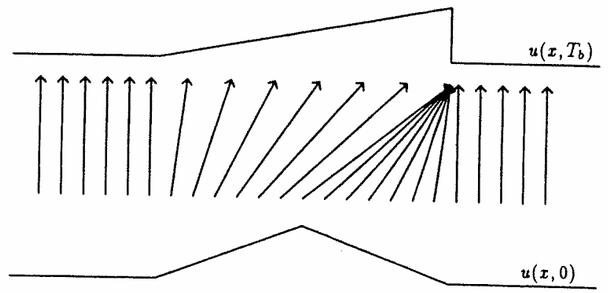


Figure 3.4. Shock formation in Burgers' equation.

3.3 Shock formation

For larger t the equation (3.18) may not have a unique solution. This happens when the characteristics cross, as will eventually happen if $u_x(x,0)$ is negative at any point. At the time T_b where the characteristics first cross, the function $u(x,t)$ has an infinite slope — the wave "breaks" and a shock forms. Beyond this point there is no classical solution of the PDE, and the weak solution we wish to determine becomes discontinuous.

Figure 3.4 shows an extreme example where the initial data is piecewise linear and many characteristics come together at once. More generally an infinite slope in the solution may appear first at just one point x, corresponding via (3.18) to the point ξ where the slope of the initial data is most negative. At this point the wave is said to "break", by analogy with waves on a beach.

EXERCISE 3.3. *Show that if we solve (3.14) with smooth initial data $u_0(x)$ for which $u_0'(x)$ is somewhere negative, then the wave will break at time*

$$T_b = \frac{-1}{\min u_0'(x)}. \tag{3.20}$$

Generalize this to arbitrary convex scalar equations.

For times $t > T_b$ some of the characteristics have crossed and so there are points x where there are three characteristics leading back to $t = 0$. One can view the "solution" u at such a time as a triple-valued function (see Fig. 3.5).

This sort of solution makes sense in some contexts, for example a breaking wave on a sloping beach can be modeled by the shallow water equations and, for a while at least, does behave as seen in Fig. 3.5, with fluid depth a triple-valued function.

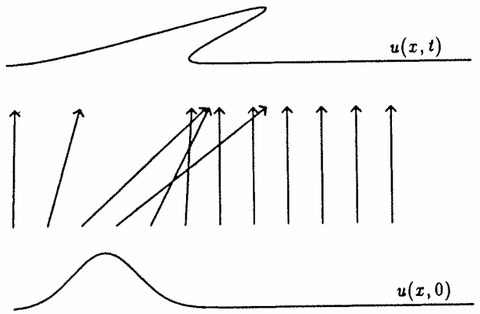


Figure 3.5. Triple-valued solution to Burgers' equation at time $t > T_b$.

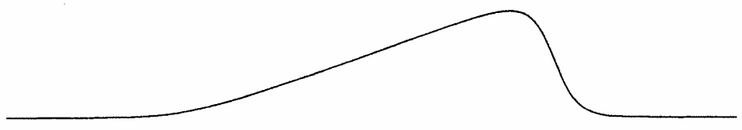


Figure 3.6. Solution to the viscous Burgers' equation at time T_b for the data shown in Figure 3.4.

However, in most physical situations this does not make sense. For example, the density of a gas cannot possibly be triple valued at a point. What happens instead at time T_b?

We can determine the correct physical behavior by adopting the vanishing viscosity approach. The equation (3.14) is a model of (3.15) valid only for small ϵ and smooth u. When it breaks down, we must return to (3.15). If the initial data is smooth and ϵ very small, then before the wave begins to break the ϵu_{xx} term is negligible compared to the other terms and the solutions to both PDEs look nearly identical. Figure 3.3, for example, would be essentially unchanged if we solved (3.15) with small ϵ rather than (3.14). However, as the wave begins to break, the second derivative term u_{xx} grows much faster than u_x, and at some point the ϵu_{xx} term is comparable to the other terms and begins to play a role. This term keeps the solution smooth for all time, preventing the breakdown of solutions that occurs for the hyperbolic problem.

For very small values of ϵ, the discontinuous solution at T_b shown in Figure 3.4 would

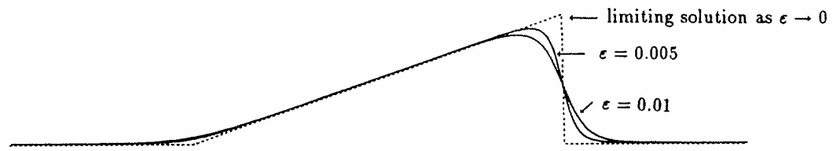


Figure 3.7. Solution to the viscous Burgers' equation for two different values of ϵ.

be replaced by a smooth continuous function as in Figure 3.6. As $\epsilon \to 0$ this becomes sharper and approaches the discontinuous solution of Figure 3.4.

For times $t > T_b$, such as was shown in Figure 3.5, the viscous solution for $\epsilon > 0$ would continue to be smooth and single valued, with a shape similar to that shown in Figure 3.6. The behavior as $\epsilon \to 0$ is indicated in Figure 3.7.

It is this vanishing viscosity solution that we hope to capture by solving the inviscid equation.

3.4 Weak solutions

A natural way to define a generalized solution of the inviscid equation that does not require differentiability is to go back to the integral form of the conservation law, and say that $u(x,t)$ is a generalized solution if (2.7) is satisfied for all x_1, x_2, t_1, t_2.

There is another approach that results in a different integral formulation that is often more convenient to work with. This is a mathematical technique that can be applied more generally to rewrite a differential equation in a form where less smoothness is required to define a "solution". The basic idea is to take the PDE, multiply by a smooth "test function", integrate one or more times over some domain, and then use integration by parts to move derivatives off the function u and onto the smooth test function. The result is an equation involving fewer derivatives on u, and hence requiring less smoothness.

In our case we will use test functions $\phi \in C_0^1(\mathbb{R} \times \mathbb{R})$. Here C_0^1 is the space of function that are continuously differentiable with "compact support". The latter requirement means that $\phi(x,t)$ is identically zero outside of some bounded set, and so the support of the function lies in a compact set.

If we multiply $u_t + f_x = 0$ by $\phi(x,t)$ and then integrate over space and time, we obtain

$$\int_0^\infty \int_{-\infty}^{+\infty} [\phi u_t + \phi f(u)_x]\, dx\, dt = 0. \tag{3.21}$$

Now integrate by parts, yielding

$$\int_0^\infty \int_{-\infty}^{+\infty} [\phi_t u + \phi_x f(u)]\, dx\, dt = -\int_{-\infty}^{\infty} \phi(x,0)u(x,0)\, dx. \tag{3.22}$$

Note that nearly all the boundary terms which normally arise through integration by parts drop out due to the requirement that ϕ have compact support, and hence vanishes at infinity. The remaining boundary term brings in the initial conditions of the PDE, which must still play a role in our weak formulation.

DEFINITION 3.1. *The function $u(x,t)$ is called a weak solution of the conservation law if (3.22) holds for all functions $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$.*

The advantage of this formulation over the original integral form (2.16) is that the integration in (3.22) is over a fixed domain, all of $\mathbb{R} \times \mathbb{R}^+$. The integral form (2.16) is over an arbitrary rectangle, and to check that $u(x,t)$ is a solution we must verify that this holds for all choices of x_1, x_2, t_1 and t_2. Of course, our new form (3.22) has a similar feature, we must check that it holds for all $\phi \in C_0^1$, but this turns out to be an easier task.

Mathematically the two integral forms are equivalent and we should rightly expect a more direct connection between the two that does not involve the differential equation. This can be achieved by considering special test functions $\phi(x,t)$ with the property that

$$\phi(x,t) = \begin{cases} 1 & \text{for } (x,t) \in [x_1,x_2] \times [t_1,t_2] \\ 0 & \text{for } (x,t) \notin [x_1-\epsilon, x_2+\epsilon] \times [t_1-\epsilon, t_2+\epsilon] \end{cases} \tag{3.23}$$

and with ϕ smooth in the intermediate strip of width ϵ. Then $\phi_x = \phi_t = 0$ except in this strip and so the integral (3.22) reduces to an integral over this strip. As $\epsilon \to 0$, ϕ_x and ϕ_t approach delta functions that sample u or $f(u)$ along the boundaries of the rectangle $[x_1,x_2] \times [t_1,t_2]$, so that (3.22) approaches the integral form (2.16). By making this rigorous, we can show that any weak solution satisfies the original integral conservation law.

The vanishing viscosity generalized solution we defined above is a weak solution in the sense of (3.22), and so this definition includes the solution we are looking for. Unfortunately, weak solutions are often not unique, and so an additional problem is often to identify *which* weak solution is the physically correct vanishing viscosity solution. Again, one would like to avoid working with the viscous equation directly, but it turns out that there are other conditions one can impose on weak solutions that are easier to check and will also pick out the correct solution. As noted in Chapter 1, these are usually called *entropy conditions* by analogy with the gas dynamics case. The vanishing viscosity solution is also called the **entropy solution** because of this.

3.5 The Riemann Problem

The conservation law together with piecewise constant data having a single discontinuity is known as the Riemann problem. As an example, consider Burgers' equation $u_t + uu_x = 0$

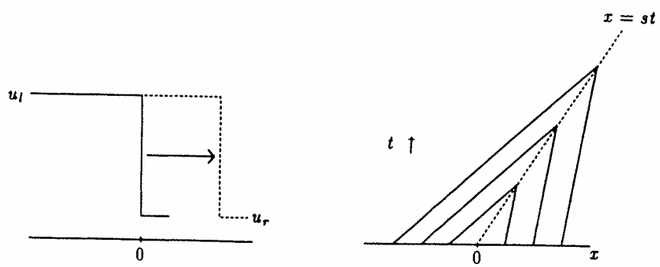


Figure 3.8. Shock wave.

with piecewise constant initial data

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases} \tag{3.24}$$

The form of the solution depends on the relation between u_l and u_r.

Case I. $u_l > u_r$.

In this case there is a unique weak solution,

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases} \tag{3.25}$$

where

$$s = (u_l + u_r)/2 \tag{3.26}$$

is the **shock speed**, the speed at which the discontinuity travels. A general expression for the shock speed will be derived below. Note that characteristics in each of the regions where u is constant go *into* the shock (see Fig. 3.8) as time advances.

EXERCISE 3.4. *Verify that (3.25) is a weak solution to Burgers' equation by showing that (3.22) is satisfied for all $\phi \in C_0^1$.*

EXERCISE 3.5. *Show that the viscous equation (3.15) has a travelling wave solution of the form $u^\epsilon(x,t) = w(x-st)$ by deriving an ODE for w and verifying that this ODE has solutions of the form*

$$w(y) = u_r + \frac{1}{2}(u_l - u_r)\left[1 - \tanh((u_l - u_r)y/4\epsilon)\right] \tag{3.27}$$

with s again given by (3.26). Note that $w(y) \to u_l$ as $y \to -\infty$ and $w(y) \to u_r$ as $y \to +\infty$. Sketch this solution and indicate how it varies as $\epsilon \to 0$.

The smooth solution $u^\epsilon(x,t)$ found in Exercise 3.5 converges to the shock solution (3.25) as $\epsilon \to 0$, showing that our shock solution is the desired vanishing viscosity solution. The shape of $u^\epsilon(x,t)$ is often called the "viscous profile" for the shock wave.

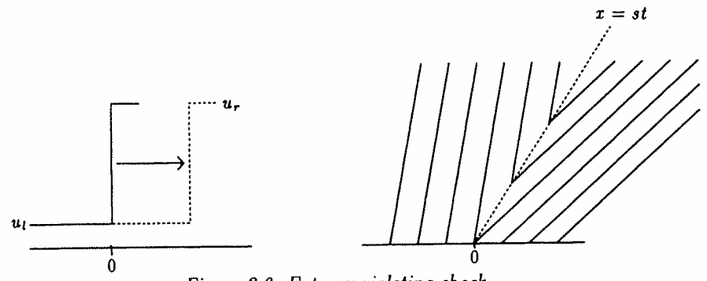


Figure 3.9. Entropy-violating shock.

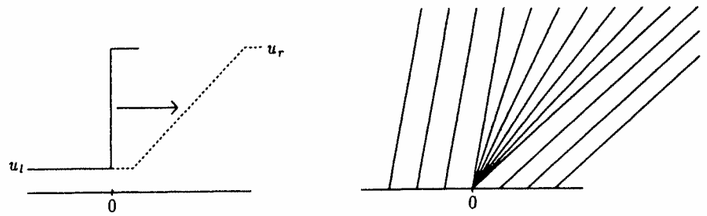


Figure 3.10. Rarefaction wave.

Case II. $u_l < u_r$.

In this case there are infinitely many weak solutions. One of these is again (3.25), (3.26) in which the discontinuity propagates with speed s. Note that characteristics now go *out* of the shock (Fig. 3.9) and that this solution is not stable to perturbations. If the data is smeared out slightly, or if a small amount of viscosity is added to the equation, the solution changes completely.

Another weak solution is the **rarefaction wave**

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ x/t & u_l t \le x \le u_r t \\ u_r & x > u_r t \end{cases} \tag{3.28}$$

This solution is stable to perturbations and is in fact the vanishing viscosity generalized solution (Fig. 3.10).

EXERCISE 3.6. *There are infinitely many other weak solutions to (3.14) when $u_l < u_r$. Show, for example, that*

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t \le x \le u_m t \\ x/t & u_m t \le x \le u_r t \\ u_r & x > u_r t \end{cases}$$

is a weak solution for any u_m with $u_l \le u_m \le u_r$ and $s_m = (u_l + u_m)/2$. Sketch the characteristics for this solution. Find a class of weak solutions with three discontinuities.

EXERCISE 3.7. *Show that for a general convex scalar problem (3.13) with data (3.24) and $u_l < u_r$, the rarefaction wave solution is given by*

$$u(x,t) = \begin{cases} u_l & x < f'(u_l)t \\ v(x/t) & f'(u_l)t \le x \le f'(u_r)t \\ u_r & x > f'(u_r)t \end{cases} \tag{3.29}$$

where $v(\xi)$ is the solution to $f'(v(\xi)) = \xi$.

3.6 Shock speed

The propagating shock solution (3.25) is a weak solution to Burgers' equation only if the speed of propagation is given by (3.26). The same discontinuity propagating at a different speed would not be a weak solution.

The speed of propagation can be determined by conservation. If M is large compared to st then by (2.15), $\int_{-M}^{M} u(x,t)\,dx$ must increase at the rate

$$\begin{aligned} \frac{d}{dt}\int_{-M}^{M} u(x,t)\,dx &= f(u_l) - f(u_r) \\ &= \frac{1}{2}(u_l + u_r)(u_l - u_r) \end{aligned} \tag{3.30}$$

for Burgers' equation. On the other hand, the solution (3.25) clearly has

$$\int_{-M}^{M} u(x,t)\,dx = (M + st)u_l + (M - st)u_r \tag{3.31}$$

so that

$$\frac{d}{dt}\int_{-M}^{M} u(x,t)\,dx = s(u_l - u_r). \tag{3.32}$$

Comparing (3.30) and (3.32) gives (3.26).

More generally, for arbitrary flux function $f(u)$ this same argument gives the following relation between the shock speed s and the states u_l and u_r, called the **Rankine-Hugoniot jump condition**:

$$f(u_l) - f(u_r) = s(u_l - u_r). \tag{3.33}$$

For scalar problems this gives simply

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{[f]}{[u]} \tag{3.34}$$

where $[\cdot]$ indicates the jump in some quantity across the discontinuity. Note that any jump is allowed, provided the speed is related via (3.34).

For systems of equations, $u_l - u_r$ and $f(u_r) - f(u_l)$ are both vectors while s is still a scalar. Now we cannot always solve for s to make (3.33) hold. Instead, only certain jumps $u_l - u_r$ are allowed, namely those for which the vectors $f(u_l) - f(u_r)$ and $u_l - u_r$ are linearly dependent.

EXAMPLE 3.1. For a linear system with $f(u) = Au$, (3.33) gives

$$A(u_l - u_r) = s(u_l - u_r), \tag{3.35}$$

i.e., $u_l - u_r$ must be an eigenvector of the matrix A and s is the associated eigenvalue. For a linear system, these eigenvalues are the characteristic speeds of the system. Thus discontinuities can propagate only along characteristics, a fact that we have already seen for the scalar case.

So far we have considered only piecewise constant initial data and shock solutions consisting of a single discontinuity propagating at constant speed. More typically, solutions have both smooth regions, where the PDEs are satisfied in the classical sense, and propagating discontinuities whose strength and speed vary as they interact with the smooth flow or collide with other shocks.

The Rankine-Hugoniot (R-H) conditions (3.33) hold more generally across any propagating shock, where now u_l and u_r denote the values immediately to the left and right of the discontinuity and s is the corresponding instantaneous speed, which varies along with u_l and u_r.

EXAMPLE 3.2. As an example, the following "N wave" is a solution to Burgers' equation:

$$u(x,t) = \begin{cases} x/t & -\sqrt{t} < x < \sqrt{t} \\ 0 & \text{otherwise} \end{cases} \tag{3.36}$$

This solution has two shocks propagating with speeds $\pm\frac{1}{2\sqrt{t}}$. The right-going shock has left and right states $u_l = \sqrt{t}/t = 1/\sqrt{t}$, $u_r = 0$ and so the R-H condition is satisfied, and similarly for the left-going shock. See Figure 3.11.

To verify that the R-H condition must be instantaneously satisfied when u_l and u_r vary, we apply the same conservation argument as before but now to a small rectangle as shown in Figure 3.12, with $x_2 = x_1 + \Delta x$ and $t_2 = t_1 + \Delta t$. Assuming that u is smoothly varying on each side of the shock, and that the shock speed $s(t)$ is consequently also smoothly varying, we have the following relation between Δx and Δt:

$$\Delta x = s(t_1)\Delta t + O(\Delta t^2). \tag{3.37}$$

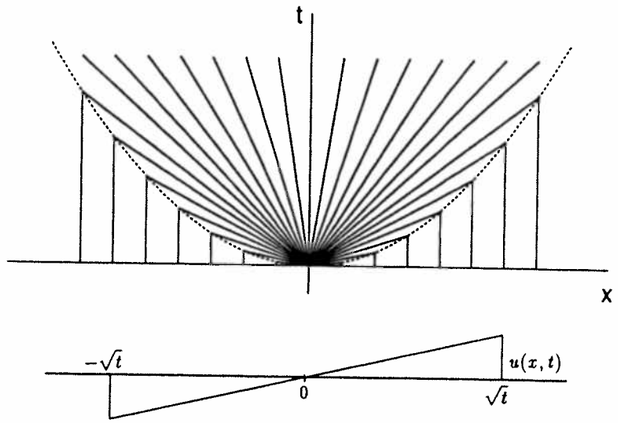


Figure 3.11. N wave solution to Burgers' equation.

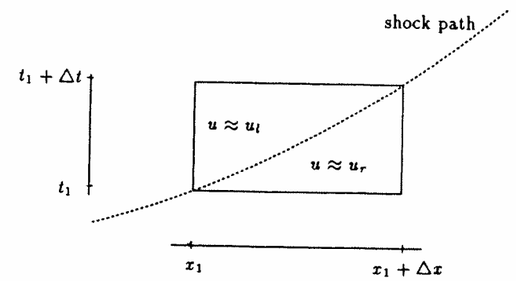


Figure 3.12. Region of integration for shock speed calculation.

From the integral form of the conservation law, we have

$$\int_{x_1}^{x_1+\Delta x} u(x,t_1+\Delta t)\,dx = \int_{x_1}^{x_1+\Delta x} u(x,t_1)\,dx + \int_{t_1}^{t_1+\Delta t} f(u(x_1,t))\,dt - \int_{t_1}^{t_1+\Delta t} f(u(x_1+\Delta x,t))\,dt. \tag{3.38}$$

In the triangular portion of the infinitesimal rectangle that lies to the left of the shock, $u(x,t) = u_l(t_1) + O(\Delta t)$, while in the complementary triangle, $u(x,t) = u_r(t_1) + O(\Delta t)$. Using this in (3.38) gives

$$\Delta x\, u_l = \Delta x\, u_r + \Delta t\, f(u_l) - \Delta t\, f(u_r) + O(\Delta t^2).$$

Using the relation (3.37) in the above equation and then dividing by Δt gives

$$s\Delta t(u_l - u_r) = \Delta t(f(u_l) - f(u_r)) + O(\Delta t)$$

where s, u_l, and u_r are all evaluated at t_1. Letting $\Delta t \to 0$ gives the R-H condition (3.33).

EXERCISE 3.8. *Solve Burgers' equation with initial data*

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 < x < 2 \\ 0 & x > 2. \end{cases} \tag{3.39}$$

Sketch the characteristics and shock paths in the x-t plane. Hint: The two shocks merge into one shock at some point.

The equal area rule. One technique that is useful for determining weak solutions by hand is to start with the "solution" constructed using characteristics (which may be multi-valued if characteristics cross) and then eliminate the multi-valued parts by inserting shocks. The shock location can be determined by the "equal area rule", which is best understood by looking at Figure 3.13. The shock is located such that the shaded regions cut off on either side have equal areas, as in Figure 3.13b. This is a consequence of conservation — the integral of the discontinuous weak solution (shaded area in Figure 3.13c) must be the same as the area "under" the multi-valued solution (shaded area in 3.13a), since both "solve" the same conservation law.

3.7 Manipulating conservation laws

One danger to observe in dealing with conservation laws is that transforming the differential form into what appears to be an equivalent differential equation may not give an equivalent equation in the context of weak solutions.

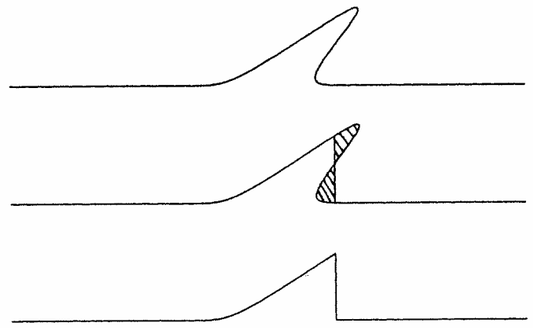


Figure 3.13. Equal area rule for shock location.

EXAMPLE 3.3. If we multiply Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \tag{3.40}$$

by $2u$, we obtain $2uu_t + 2u^2u_x = 0$, which can be rewritten as

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0. \tag{3.41}$$

This is again a conservation law, now for u^2 rather than u itself, with flux function $f(u^2) = \frac{2}{3}(u^2)^{3/2}$. The differential equations (3.40) and (3.41) have precisely the same smooth solutions. However, they have different weak solutions, as we can see by considering the Riemann problem with $u_l > u_r$. The unique weak solution of (3.40) is a shock traveling at speed

$$s_1 = \frac{\left[\frac{1}{2}u^2\right]}{[u]} = \frac{1}{2}(u_l + u_r), \tag{3.42}$$

whereas the unique weak solution to (3.41) is a shock traveling at speed

$$s_2 = \frac{\left[\frac{2}{3}u^3\right]}{[u^2]} = \frac{2}{3}\left(\frac{u_r^3 - u_l^3}{u_r^2 - u_l^2}\right). \tag{3.43}$$

It is easy to check that

$$s_2 - s_1 = \frac{1}{6}\frac{(u_l - u_r)^2}{(u_l + u_r)} \tag{3.44}$$

and so $s_2 \neq s_1$ when $u_l \neq u_r$, and the two equations have different weak solutions. The derivation of (3.41) from (3.40) requires manipulating derivatives in a manner that is valid only when u is smooth.

3.8 Entropy conditions

As demonstrated above, there are situations in which the weak solution is not unique and an additional condition is required to pick out the physically relevant vanishing viscosity solution. The condition which defines this solution is that it should be the limiting solution of the viscous equation as $\epsilon \to 0$, but this is not easy to work with. We want to find simpler conditions.

For scalar equations there is an obvious condition suggested by Figures 3.8 and 3.10. A shock should have characteristics going into the shock, as time advances. A propagating discontinuity with characteristics coming *out* of it, as in Figure 3.9, is unstable to perturbations. Either smearing out the initial profile a little, or adding some viscosity to the system, will cause this to be replaced by a rarefaction fan of characteristics, as in Figure 3.10. This gives our first version of the entropy condition:

ENTROPY CONDITION (VERSION I): *A discontinuity propagating with speed s given by (3.33) satisfies the entropy condition if*

$$f'(u_l) > s > f'(u_r). \tag{3.45}$$

Note that $f'(u)$ is the characteristic speed. For convex f, the Rankine-Hugoniot speed s from (3.34) must lie between $f'(u_l)$ and $f'(u_r)$, so (3.45) reduces to simply the requirement that $f'(u_l) > f'(u_r)$, which again by convexity requires $u_l > u_r$.

A more general form of this condition, due to Oleinik, applies also to nonconvex scalar flux functions f:

ENTROPY CONDITION (VERSION II): *$u(x,t)$ is the entropy solution if all discontinuities have the property that*

$$\frac{f(u) - f(u_l)}{u - u_l} \geq s \geq \frac{f(u) - f(u_r)}{u - u_r} \tag{3.46}$$

for all u between u_l and u_r.

For convex f, this requirement reduces to (3.45).

Another form of the entropy condition is based on the spreading of characteristics in a rarefaction fan. If $u(x,t)$ is an increasing function of x in some region, then characteristics spread out if $f'' > 0$. The rate of spreading can be quantified, and gives the following condition, also due to Oleinik[57].

ENTROPY CONDITION (VERSION III): *$u(x,t)$ is the entropy solution if there is a constant $E > 0$ such that for all $a > 0$, $t > 0$ and $x \in \mathbb{R}$,*

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t}. \tag{3.47}$$

Note that for a discontinuity propagating with constant left and right states u_l and u_r, this can be satisfied only if $u_r - u_l \leq 0$, so this agrees with (3.45). The form of (3.47) may seem unnecessarily complicated, but it turns out to be easier to apply in some contexts. In particular, this formulation has advantages in studying numerical methods. One problem we face in developing numerical methods is guaranteeing that they converge to the correct solution. Some numerical methods are known to converge to the *wrong* weak solution in some instances. The criterion (3.45) is hard to apply to discrete solutions — a discrete approximation defined only at grid points is in some sense discontinuous everywhere. If $U_j < U_{j+1}$ at some grid point, how do we determine whether this is a jump that violates the entropy condition, or is merely part of a smooth approximation of a rarefaction wave? Intuitively, we know the answer: it's part of a smooth approximation, and therefore acceptable, if the size of this jump is $O(\Delta x)$ as we refine the grid and $\Delta x \to 0$. To turn this into a proof that some method converges to the correct solution, we must quantify this requirement and (3.47) gives what we need. Taking $a = \Delta x$, we must ensure that there is a constant $E > 0$ such that

$$U_{j+1}(t) - U_j(t) < \left(\frac{E}{t}\right)\Delta x \tag{3.48}$$

for all $t > 0$ as $\Delta x \to 0$. This inequality can often be established for discrete methods.

In fact, Oleinik's original proof that an entropy solution to (3.13) satisfying (3.47) always exists proceeds by defining such a discrete approximation and then taking limits. This is also presented in Theorem 16.1 of Smoller[77].

3.8.1 Entropy functions

Yet another approach to the entropy condition is to define an entropy function $\eta(u)$ for which an additional conservation law holds for smooth solutions that becomes an inequality for discontinuous solutions. In gas dynamics, there exists a physical quantity called the entropy that is known to be constant along particle paths in smooth flow and to jump to a higher value as the gas crosses a shock. It can never jump to a lower value, and this gives the physical entropy condition that picks out the correct weak solution in gas dynamics.

Suppose some function $\eta(u)$ satisfies a conservation law of the form

$$\eta(u)_t + \psi(u)_x = 0 \tag{3.49}$$

for some **entropy flux** $\psi(u)$. Then we can obtain from this, for smooth u,

$$\eta'(u)u_t + \psi'(u)u_x = 0. \tag{3.50}$$

Recall that the conservation law (3.13) can be written as $u_t + f'(u)u_x = 0$. Multiply this by $\eta'(u)$ and compare with (3.50) to obtain

$$\psi'(u) = \eta'(u)f'(u). \tag{3.51}$$

For a scalar conservation law this equation admits many solutions $\eta(u)$, $\psi(u)$. For a system of equations η and ψ are still *scalar* functions, but now (3.51) reads $\nabla\psi(u) = f'(u)\nabla\eta(u)$, which is a system of m equations for the two variables η and ψ. If $m > 2$ this may have no solutions.

An additional condition we place on the entropy function is that it be *convex*, $\eta''(u) > 0$, for reasons that will be seen below.

The entropy $\eta(u)$ is conserved for *smooth* flows by its definition. For discontinuous solutions, however, the manipulations performed above are not valid. Since we are particularly interested in how the entropy behaves for the vanishing viscosity weak solution, we look at the related viscous problem and will then let the viscosity tend to zero. The viscous equation is

$$u_t + f(u)_x = \epsilon u_{xx}. \tag{3.52}$$

Since solutions to this equation are always smooth, we can derive the corresponding evolution equation for the entropy following the same manipulations we used for smooth solutions of the inviscid equation, multiplying (3.52) by $\eta'(u)$ to obtain

$$\eta(u)_t + \psi(u)_x = \epsilon\eta'(u)u_{xx}. \tag{3.53}$$

We can now rewrite the right hand side to obtain

$$\eta(u)_t + \psi(u)_x = \epsilon(\eta'(u)u_x)_x - \epsilon\eta''(u)u_x^2. \tag{3.54}$$

Integrating this equation over the rectangle $[x_1, x_2] \times [t_1, t_2]$ gives

$$\begin{aligned}\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x\,dx\,dt &= \epsilon\int_{t_1}^{t_2} [\eta'(u(x_2,t))u_x(x_2,t) - \eta'(u(x_1,t))u_x(x_1,t)]\,dt \\ &\quad -\epsilon\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta''(u)u_x^2\,dx\,dt.\end{aligned}$$

As $\epsilon \to 0$, the first term on the right hand side vanishes. (This is clearly true if u is smooth at x_1 and x_2, and can be shown more generally.) The other term, however, involves integrating u_x^2 over the $[x_1, x_2] \times [t_1, t_2]$. If the limiting weak solution is discontinuous along a curve in this rectangle, then this term will not vanish in the limit. However, since $\epsilon > 0$, $u_x^2 > 0$ and $\eta'' > 0$ (by our convexity assumption), we can conclude that the right hand side is nonpositive in the limit and hence the vanishing viscosity weak solution satisfies

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x\,dx\,dt \le 0 \tag{3.55}$$

for all x_1, x_2, t_1 and t_2. Alternatively, in integral form,

$$\int_{x_1}^{x_2} \eta(u(x,t))\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \psi(u(x,t))\,dt\Big|_{x_1}^{x_2} \le 0, \tag{3.56}$$

i.e.,

$$\begin{aligned}\int_{x_1}^{x_2} \eta(u(x,t_2))\,dx \;\le\; &\int_{x_1}^{x_2} \eta(u(x,t_1))\,dx \\ &- \left(\int_{t_1}^{t_2} \psi(u(x_2,t))\,dt - \int_{t_1}^{t_2} \psi(u(x_1,t))\,dt\right).\end{aligned} \tag{3.57}$$

Consequently, the total integral of η is not necessarily conserved, but can only *decrease*. (Note that our mathematical assumption of convexity leads to an "entropy function" that decreases, whereas the physical entropy in gas dynamics increases.) The fact that (3.55) holds for all x_1, x_2, t_1 and t_2 is summarized by saying that $\eta(u)_t + \psi(u)_x \le 0$ in the weak sense. This gives our final form of the entropy condition, called the **entropy inequality**.

ENTROPY CONDITION (VERSION IV): *The function $u(x,t)$ is the entropy solution of (3.13) if, for all convex entropy functions and corresponding entropy fluxes, the inequality*

$$\eta(u)_t + \psi(u)_x \le 0 \tag{3.58}$$

is satisfied in the weak sense.

This formulation is also useful in analyzing numerical methods. If a discrete form of this entropy inequality is known to hold for some numerical method, then it can be shown that the method converges to the entropy solution.

Just as for the conservation law, an alternative weak form of the entropy condition can be formulated by integrating against smooth test functions ϕ, now required to be nonnegative since the entropy condition involves an inequality. The **weak form of the entropy inequality** is

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty \phi_t(x,t)\eta(u(x,t)) + \phi_x(x,t)\psi(u(x,t))\,dx\,dt \\ &\qquad \le -\int_{-\infty}^\infty \phi(x,0)\eta(u(x,0))\,dx\end{aligned} \tag{3.59}$$

for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R})$ with $\phi(x,t) \ge 0$ for all x, t.

EXAMPLE 3.4. Consider Burgers' equation with $f(u) = \frac{1}{2}u^2$ and take $\eta(u) = u^2$. Then (3.51) gives $\psi'(u) = 2u^2$ and hence $\psi(u) = \frac{2}{3}u^3$. Then entropy condition (3.58) reads

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x \le 0. \tag{3.60}$$

For smooth solutions of Burgers' equation this should hold with equality, as we have already seen in Example 3.3. If a discontinuity is present, then integrating $(u^2)_t + (\frac{2}{3}u^3)_x$ over an infinitesmal rectangle as in Figure 3.12 gives

$$\begin{aligned}\int_{x_1}^{x_2} u^2(x,t)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{2}{3}u^3(x,t)\,dt\Big|_{x_1}^{x_2} &= s_1\Delta t(u_l^2 - u_r^2) + \frac{2}{3}\Delta t(u_r^3 - u_l^3) + O(\Delta t^2) \\ &= \Delta t(u_l^2 - u_r^2)(s_1 - s_2) + O(\Delta t^2) \\ &= -\frac{1}{6}(u_l - u_r)^3\Delta t + O(\Delta t^2)\end{aligned}$$

where s_1 and s_2 are given by (3.42) and (3.43) and we have used (3.44). For small $\Delta t > 0$, the $O(\Delta t^2)$ term will not affect the sign of this quantity and so the weak form (3.56) is satisfied if and only if $(u_l - u_r)^3 > 0$, and hence the only allowable discontinuities have $u_l > u_r$, as expected.

4 Some Scalar Examples

In this chapter we will look at a couple of examples of scalar conservation laws with some physical meaning, and apply the theory developed in the previous chapter. The first of these examples (traffic flow) should also help develop some physical intuition that is applicable to the more complicated case of gas dynamics, with gas molecules taking the place of cars. This application is discussed in much more detail in Chapter 3 of Whitham[97]. The second example (two phase flow) shows what can happen when f is not convex.

4.1 Traffic flow

Consider the flow of cars on a highway. Let ρ denote the density of cars (in vehicles per mile, say) and u the velocity. In this application ρ is restricted to a certain range, $0 \le \rho \le \rho_{\max}$, where $\rho_{\max}$ is the value at which cars are bumper to bumper.

Since cars are conserved, the density and velocity must be related by the continuity equation derived in Section 1,

$$\rho_t + (\rho u)_x = 0. \tag{4.1}$$

In order to obtain a scalar conservation law for ρ alone, we now assume that u is a given function of ρ. This makes sense: on a highway we would optimally like to drive at some speed $u_{\max}$ (the speed limit perhaps), but in heavy traffic we slow down, with velocity decreasing as density increases. The simplest model is the linear relation

$$u(\rho) = u_{\max}(1 - \rho/\rho_{\max}). \tag{4.2}$$

At zero density (empty road) the speed is $u_{\max}$, but decreases to zero as ρ approaches $\rho_{\max}$. Using this in (4.1) gives

$$\rho_t + f(\rho)_x = 0 \tag{4.3}$$

where

$$f(\rho) = \rho u_{\max}(1 - \rho/\rho_{\max}). \tag{4.4}$$

Whitham notes that a good fit to data for the Lincoln tunnel was found by Greenberg in 1959 by

$$f(\rho) = a\rho \log(\rho_{\max}/\rho),$$

a function shaped similar to (4.4).

The characteristic speeds for (4.3) with flux (4.4) are

$$f'(\rho) = u_{\max}(1 - 2\rho/\rho_{\max}), \tag{4.5}$$

while the shock speed for a jump from ρ_l to ρ_r is

$$s = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} = u_{\max}(1 - (\rho_l + \rho_r)/\rho_{\max}). \tag{4.6}$$

The entropy condition (3.45) says that a propagating shock must satisfy $f'(\rho_l) > f'(\rho_r)$ which requires $\rho_l < \rho_r$. Note this is the opposite inequality as in Burgers' equation since here f is concave rather than convex.

We now consider a few examples of solutions to this equation and their physical interpretation.

EXAMPLE 4.1. Take initial data

$$\rho(x,0) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases} \tag{4.7}$$

where $0 < \rho_l < \rho_r < \rho_{\max}$. Then the solution is a shock wave traveling with speed s given by (4.6). Note that although $u(\rho) \geq 0$ the shock speed s can be either positive or negative depending on ρ_l and ρ_r.

Consider the case $\rho_r = \rho_{\max}$ and $\rho_l < \rho_{\max}$. Then $s < 0$ and the shock propagates to the left. This models the situation in which cars moving at speed $u_l > 0$ unexpectedly encounter a bumper-to-bumper traffic jam and slam on their brakes, instantaneously reducing their velocity to 0 while the density jumps from ρ_l to $\rho_{\max}$. This discontinuity occurs at the shock, and clearly the shock location moves to the left as more cars join the traffic jam. This is illustrated in Figure 4.1, where the vehicle trajectories ("particle paths") are sketched. Note that the distance between vehicles is inversely proportional to density. (In gas dynamics, $1/\rho$ is called the *specific volume*.)

The particle paths should not be confused with the characteristics, which are shown in Figure 4.2 for the case $\rho_l = \frac{1}{2}\rho_{\max}$ (so $u_l = \frac{1}{2}u_{\max}$), as is the case in Figure 4.1 also. In this case, $f'(\rho_l) = 0$. If $\rho_l > \frac{1}{2}\rho_{\max}$ then $f'(\rho_l) < 0$ and all characteristics go to the left, while if $\rho_l < \frac{1}{2}\rho_{\max}$ then $f'(\rho_l) > 0$ and characteristics to the left of the shock are rightward going.

EXERCISE 4.1. *Sketch the particle paths and characteristics for a case with* $\rho_l + \rho_r < \rho_{\max}$.

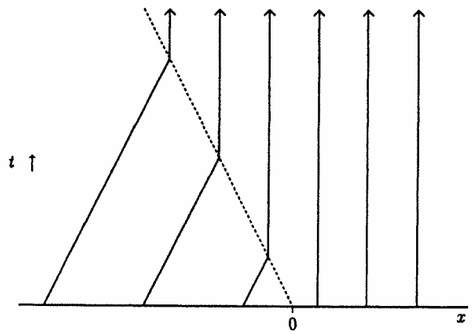


Figure 4.1. Traffic jam shock wave (vehicle trajectories), with data $\rho_l = \frac{1}{2}\rho_{\max}$, $\rho_r = \rho_{\max}$.

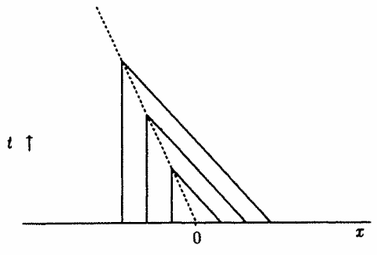


Figure 4.2. Characteristics.

EXAMPLE 4.2. Again consider a Riemann problem with data of the form (4.7), but now take $0 < \rho_r < \rho_l < \rho_{\max}$ so that the solution is a rarefaction wave. Figure 4.3 shows the case where $\rho_l = \rho_{\max}$ and $\rho_r = \frac{1}{2}\rho_{\max}$. This might model the startup of cars after a light turns green. The cars to the left are initially stationary but can begin to accelerate once the cars in front of them begin to move. Since the velocity is related to the density by (4.2), each driver can speed up only by allowing the distance between her and the previous car to increase, and so we see a gradual acceleration and spreading out of cars.

As cars go through the rarefaction wave, the density decreases. Cars spread out or become "rarefied" in the terminology used for gas molecules.

Of course in this case there is another weak solution to (4.3), the entropy-violating shock. This would correspond to drivers accelerating instantaneously from $u_l = 0$ to $u_r > 0$ as the preceding car moves out of the way. This behavior is not usually seen in practice except perhaps in high school parking lots.

The reason is that in practice there is "viscosity", which here takes the form of slow response of drivers and automobiles. In the shock wave example above, the instantaneous jump from $u_l > 0$ to $u_r = 0$ as drivers slam on their brakes is obviously a mathematical idealization. However, in terms of modeling the big picture — how the traffic jam evolves — the detailed structure of $u(x)$ in the shock is unimportant.

EXERCISE 4.2. *For cars starting at a green light with open road ahead of them, the initial conditions would really be (4.7) with $\rho_l = \rho_{\max}$ and $\rho_r = 0$. Solve this Riemann problem and sketch the particle paths and characteristics.*

EXERCISE 4.3. *Sketch the distribution of ρ and u at some fixed time $t > 0$ for the solution of Exercise 4.2.*

EXERCISE 4.4. *Determine the manner in which a given car accelerates in the solution to Exercise 4.2, i.e. determine $v(t)$ where v represents the velocity along some particular particle path as time evolves.*

4.1.1 Characteristics and "sound speed"

For a scalar conservation law, information always travels with speed $f'(\rho)$ as long as the solution is smooth. If fact, the solution is constant along characteristics since

$$\rho_t + f'(\rho)\rho_x = 0. \tag{4.8}$$

We can obtain another interpretation of this if we consider the special case of nearly constant initial data, say

$$\rho(x,0) = \rho_0 + \epsilon\rho_1(x,0). \tag{4.9}$$

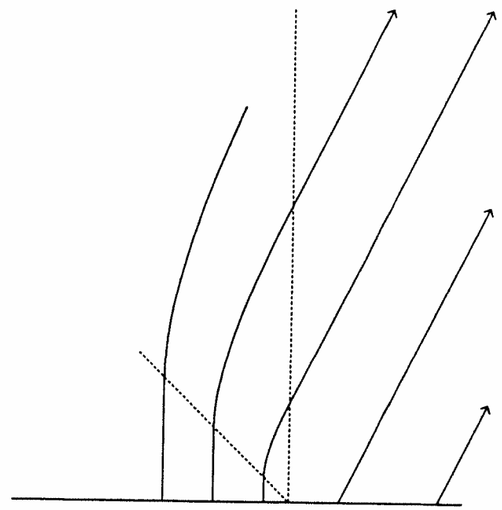


Figure 4.3. Rarefaction wave (vehicle trajectories), with data $\rho_l = \rho_{\max}$, $\rho_r = \frac{1}{2}\rho_{\max}$.

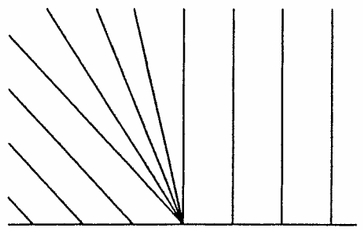


Figure 4.4. Characteristics.

Then we can approximate our nonlinear equation by a linear equation. Assuming

$$\rho(x,t) = \rho_0 + \epsilon\rho_1(x,t) \tag{4.10}$$

remains valid with $\rho_1 = O(1)$, we find that

$$\begin{aligned} \rho_t &= \epsilon\rho_{1t} \\ \rho_x &= \epsilon\rho_{1x} \\ f'(\rho) &= f'(\rho_0) + \epsilon\rho_1 f''(\rho_0) + O(\epsilon^2). \end{aligned}$$

Using these in (4.8) and dividing by ϵ gives

$$\rho_{1t} + f'(\rho_0)\rho_{1x} = -\epsilon f''(\rho_0)\rho_1\rho_{1x} + O(\epsilon^2). \tag{4.11}$$

For small ϵ the behavior, at least for times $t \ll 1/\epsilon$, is governed by the equation obtained by ignoring the right hand side. This gives a constant coefficient linear advection equation for $\rho_1(x,t)$:

$$\rho_{1t} + f'(\rho_0)\rho_{1x} = 0. \tag{4.12}$$

The initial data simply propagates unchanged with velocity $f'(\rho_0)$.

In the traffic flow model this corresponds to a situation where cars are nearly evenly spaced with small variation in the density. These variations will propagate with velocity roughly $f'(\rho_0)$.

As a specific example, suppose the data is constant except for a small rise in density at some point, a crowding on the highway. The cars in this disturbance are going slower than the cars either ahead or behind, with two effects. First, since they are going slower than the cars behind, the cars behind will start to catch up, seeing a rise in their local density and therefore be forced to slow down. Second, since the cars in the disturbance are going slower than the cars ahead of them, they will start to fall behind, leading to a decrease in their local density and an increase in speed. The consequence is that the disturbance will propagate "backwards" through the line of cars. Here by "backwards" I mean from the standpoint of any given driver. He slows down because the cars in front of him have, and his behavior in turn affects the drivers behind him.

Note that in spite of this, the speed at which the disturbance propagates could be *positive*, if $f'(\rho_0) > 0$, which happens if $\rho_0 < \frac{1}{2}\rho_{\max}$. This is illustrated in Figure 4.5. Here the vehicle trajectories are sketched. The jog in each trajectory is the effect of the car slowing down as the disturbance passes.

The nearly linear behavior of small amplitude disturbances is also seen in gas dynamics. In fact, this is precisely how sound waves propagate. If the gas is at rest, $v_0 = 0$ in the linearization, then sound propagates at velocities $\pm c$, where the **sound speed** c depends on the equation of state (we will see that c^2 is the derivative of pressure with respect to density at constant entropy). If we add some uniform motion to the gas, so $v_0 \neq 0$, then

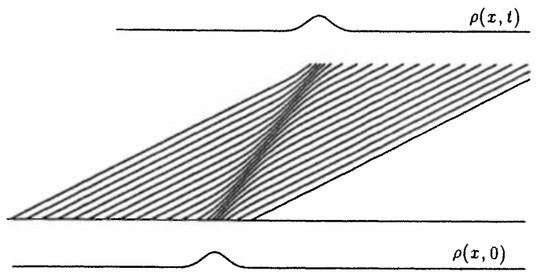


Figure 4.5. Vehicle trajectories and propagation of a small disturbance.

sound waves propagate at speeds $v_0 \pm c$. This simple shift arises from the fact that you can add a uniform velocity to a solution of the Euler equations and it remains a solution.

This is not true for the traffic flow model, since u is assumed to be a given function of ρ. However, it should be be clear that the velocity which corresponds most naturally to the sound speed in gas dynamics is

$$c = f'(\rho_0) - u(\rho_0), \tag{4.13}$$

so that disturbances propagate at speed $f'(\rho_0) = u(\rho_0) + c$, or at speed c relative to the traffic flow. Using (4.2) and (4.5), this becomes

$$c = -u_{\max}\,\rho_0/\rho_{\max}. \tag{4.14}$$

EXERCISE 4.5. *What is the physical significance of the fact that $c < 0$?*

In gas dynamics the case $v < c$ is called **subsonic flow**, while if $v > c$ the flow is **supersonic**. By analogy, the value $\rho_0 = \frac{1}{2}\rho_{\max}$ at which $f'(\rho_0) = 0$ is called the **sonic point**, since this is the value for which $u(\rho_0) = c$. For $\rho < \frac{1}{2}\rho_{\max}$, the cars are moving faster than disturbances propagate backwards through the traffic, giving the situation already illustrated in Figure 4.5.

EXERCISE 4.6. *Sketch particle paths similar to Figure 4.5 for the case $\rho_0 = \frac{1}{2}\rho_{\max}$.*

EXERCISE 4.7. *Consider a shock wave with left and right states ρ_l and ρ_r, and let the shock strength approach zero, by letting $\rho_l \to \rho_r$. Show that the shock speed for these weak shocks approaches the linearized propagation speed $f'(\rho_r)$.*

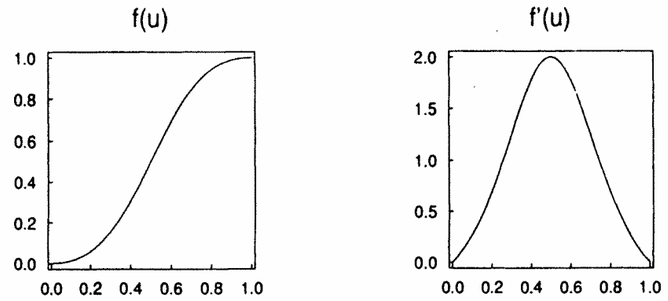


Figure 4.6. Flux function for Buckley-Leverett equation.

4.2 Two phase flow

When f is convex, the solution to the Riemann problem is always either a shock or a rarefaction wave. When f is not convex, the entropy solution might involve both. To illustrate this, we will look at the **Buckley-Leverett equations**, a simple model for two phase fluid flow in a porous medium. One application is to oil reservoir simulation. When an underground source of oil is tapped, a certain amount of oil flows out on its own due to high pressure in the reservoir. After the flow stops, there is typically a large amount of oil still in the ground. One standard method of "secondary recovery" is to pump water into the oil field through some wells, forcing oil out through others. In this case the two phases are oil and water, and the flow takes place in a porous medium of rock or sand.

The Buckley-Leverett equations are a particularly simple scalar model that captures some features of this flow. In one space dimension the equation has the standard conservation law form (3.13) with

$$f(u) = \frac{u^2}{u^2 + a(1-u)^2} \tag{4.15}$$

where a is a constant. Figure 4.6 shows $f(u)$ when $a = 1/2$. Here u represents the saturation of water and so lies between 0 and 1.

Now consider the Riemann problem with initial states $u_l = 1$ and $u_r = 0$, modeling the flow of pure water ($u = 1$) into pure oil ($u = 0$). By following characteristics, we can construct the triple-valued solution shown in Figure 4.7a. Note that the characteristic velocities are $f'(u)$ so that the profile of this bulge seen here at time t is simply the graph of $tf'(u)$ turned sideways.

We can now use the equal area rule to replace this triple-valued solution by a correct shock. The resulting weak solution is shown in Figure 4.7b, along with the characteristics in Figure 4.7c.

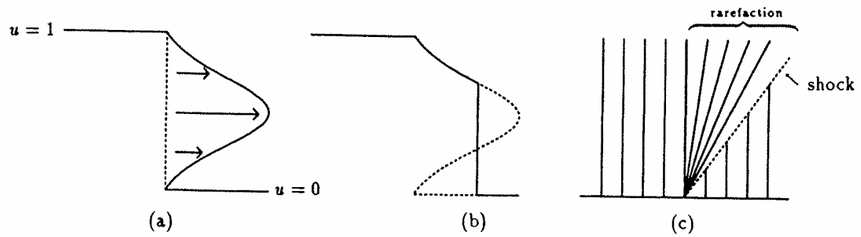


Figure 4.7. Riemann solution for Buckley-Leverett equation.

EXERCISE 4.8. *Use the equal area rule to find an expression for the shock location as a function of t and verify that the Rankine-Hugoniot condition is always satisfied.*

If you do the above exercise, you will find that the shock location moves at a constant speed, and the post-shock value u^* is also constant. This might be surprising, unless you are familiar with self-similarity of solutions to the Riemann problem, in which case you should have expected this. This will be discussed later.

Note the physical interpretation of the solution shown in Figure 4.7. As the water moves in, it displaces a certain fraction u^* of the oil immediately. Behind the shock, there is a mixture of oil and water, with less and less oil as time goes on. At a production well (at the point $x = 1$, say), one obtains pure oil until the shock arrives, followed by a mixture of oil and water with diminishing returns as time goes on. It is impossible to recover all of the oil in finite time by this technique.

Note that the Riemann solution involves both a shock and a rarefaction wave. If $f(u)$ had more inflection points, the solution might involve several shocks and rarefactions.

EXERCISE 4.9. *Explain why it is impossible to have a Riemann solution involving both a shock and a rarefaction when f is convex or concave.*

It turns out that the solution to the Riemann problem can be determined from the graph of f in a simple manner. If $u_r < u_l$ then construct the **convex hull** of the set $\{(x, y) : u_r \le x \le u_l \text{ and } y \le f(x)\}$. The convex hull is the smallest convex set containing the original set. This is shown in Figure 4.8 for the case $u_l = 1,\ u_r = 0$.

If we look at the upper boundary of this set, we see that it is composed of a straight line segment from $(0, 0)$ to $(u^*, f(u^*))$ and then follows $y = f(x)$ up to $(1, 1)$. The point of tangency u^* is precisely the post-shock value. The straight line represents a shock jumping from $u = 0$ to $u = u^*$ and the segment where the boundary follows $f(x)$ is the

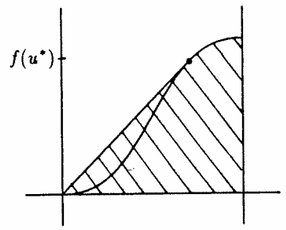


Figure 4.8. Convex hull showing shock and rarefaction.

rarefaction wave. This works more generally for any two states (provided $u_l > u_r$) and for any f.

Note that the slope of the line segment is $s^* = [f(u^*) - f(u_r)] \,/\, [u^* - u_r]$, which is precisely the shock speed. The fact that this line is tangent to the curve $f(x)$ at u^* means that $s^* = f'(u^*)$, the shock moves at the same speed as the characteristics at this edge of the rarefaction fan, as seen in Figure 4.7c.

If the shock were connected to some point $\hat{u} < u^*$, then the shock speed $[f(\hat{u}) - f(u_r)] \,/\, [\hat{u} - u_r]$ would be less than $f'(\hat{u})$, leading to a triple-valued solution. On the other hand, if the shock were connected to some point above u^* then the entropy condition (3.46) would be violated. This explains the tangency requirement, which comes out naturally from the convex hull construction.

EXERCISE 4.10. *Show that (3.46) is violated if the shock goes above u^*.*

If $u_l < u_r$ then the same idea works but we look instead at the convex hull of the set of points *above* the graph, $\{(x, y) : u_l \le x \le u_r \text{ and } y \ge f(x)\}$.

Note that if f is convex, then the convex hull construction gives either a single line segment (single shock) if $u_l > u_r$ or the function f itself (single rarefaction) if $u_l < u_r$.

5 Some Nonlinear Systems

Before developing the theory for systems of conservation laws, it is useful to have some specific examples in mind. In this chapter we will derive some systems of conservation laws.

5.1 The Euler equations

The Euler equations of gas dynamics are a particularly important example. The continuity equation for conservation of mass was derived in Chapter 2. Here we will consider the momentum and energy equations in more detail, as well as the equation of state and a few other quantities of physical (and mathematical) significance, such as the entropy. We will also look at some simplifications, the isentropic and isothermal cases, where systems of two equations are obtained. These provide very good examples which will be used in the coming chapters to illustrate the nonlinear theory.

The derivations here will be very brief, with an emphasis on the main ideas without a detailed description of the physics. A more thorough introduction can be found in Whitham[97], Courant-Friedrichs[11], or any good book on gas dynamics, e.g. [51],[71],[94].

Recall that ρ is the density, v the velocity, E the total energy, and p the pressure of the gas. The continuity equation is

$$\rho_t + (\rho v)_x = 0. \tag{5.1}$$

The mass flux is given by ρv. More generally, for any quantity z that is advected with the flow there will be a contribution to the flux for z of the form zv. Thus, the momentum equation has a contribution of the form $(\rho v)v = \rho v^2$ and the energy equation has a flux contribution Ev.

In addition to advection, there are forces on the fluid that cause acceleration due to Newton's laws, and hence changes in momentum. If there are no outside forces, then the only force is due to variations in the fluid itself, and is proportional to the pressure gradient which is simply p_x in one dimension. Combining this with the advective flux

gives the momentum equation

$$(\rho v)_t + (\rho v^2 + p)_x = 0. \tag{5.2}$$

The total energy E is often decomposed as

$$E = \frac{1}{2}\rho v^2 + \rho e. \tag{5.3}$$

The first term here is the kinetic energy, while ρe is the internal energy. The variable e, internal energy per unit mass, is called the **specific internal energy**. (In general "specific" means "per unit mass"). Internal energy includes rotational and vibrational energy and possibly other forms of energy in more complicated situations. In the Euler equations we assume that the gas is in chemical and thermodynamic equilibrium and that the internal energy is a known function of pressure and density:

$$e = e(p, \rho). \tag{5.4}$$

This is the "equation of state" for the gas, which depends on the particular gas under study.

The total energy advects with the flow, but is also modified due to work done on the system. In the absence of outside forces, work is done only by the pressure forces and is proportional to the gradient of vp. The conservation law for total energy thus takes the form

$$E_t + [v(E + p)]_x = 0 \tag{5.5}$$

in one dimension.

Putting these equations together gives the system of Euler equations

$$\begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}_t + \begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E + p) \end{bmatrix}_x = 0. \tag{5.6}$$

In two space dimensions the Euler equations take the form

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{bmatrix}_x + \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{bmatrix}_y = 0. \tag{5.7}$$

where (u, v) is the 2D fluid velocity.

The equation of state. We still need to specify the equation of state relating the internal energy to pressure and density.

5.1.1 Ideal gas

For an ideal gas, internal energy is a function of temperature alone, $e = e(T)$, and T is related to p and ρ by the **ideal gas law**,

$$p = \mathcal{R}\rho T \tag{5.8}$$

where $\mathcal{R}$ is a constant. To good approximation, the internal energy is simply proportional to the temperature,

$$e = c_v T, \tag{5.9}$$

where c_v is a constant known as the **specific heat at constant volume**. Such gases are called **polytropic**. If energy is added to a fixed quantity of gas, and the volume is held constant, then the change in energy and change in temperature are related via

$$de = c_v dT. \tag{5.10}$$

On the other hand, if the gas is allowed to expand while the energy is added, and pressure is held constant instead, not all of the energy goes into increasing the internal energy. The work done in expanding the volume $1/\rho$ by $d(1/\rho)$ is $pd(1/\rho)$ and we obtain another relation

$$de + pd(1/\rho) = c_p dT \tag{5.11}$$

or

$$d(e + p/\rho) = c_p dT \tag{5.12}$$

where c_p is the **specific heat at constant pressure**. The quantity

$$h = e + p/\rho \tag{5.13}$$

is called the **enthalpy**. For a polytropic gas, c_p is also assumed to be constant so that (5.12) yields

$$h = c_p T. \tag{5.14}$$

Note that by the ideal gas law,

$$c_p - c_v = \mathcal{R}. \tag{5.15}$$

The equation of state for an polytropic gas turns out to depend only on the **ratio of specific heats**, usually denoted by

$$\gamma = c_p/c_v. \tag{5.16}$$

Internal energy in a molecule is typically split up between various degrees of freedom (translational, rotational, vibrational, etc.). How many degrees of freedom exist depends on the nature of the gas. The general *principle of equipartition of energy* says that the

average energy in each of these is the same. Each degree of freedom contributes an average energy of $\frac{1}{2}kT$ per molecule, where k is **Boltzmann's constant**. This gives a total contribution of $\frac{\alpha}{2}kT$ per molecule if there are α degrees of freedom. Multiplying this by n, the number of molecules per unit mass (which depends on the gas), gives

$$e = \frac{\alpha}{2}nkT. \tag{5.17}$$

The product nk is precisely the gas constant $\mathcal{R}$, so comparing this to (5.9) gives

$$c_v = \frac{\alpha}{2}\mathcal{R}. \tag{5.18}$$

From (5.15) we obtain

$$c_p = \left(1 + \frac{\alpha}{2}\right)\mathcal{R}, \tag{5.19}$$

and so

$$\gamma = c_p/c_v = \frac{\alpha + 2}{\alpha}. \tag{5.20}$$

For a monatomic gas the only degrees of freedom are the three translational degrees, so $\alpha = 3$ and $\gamma = 5/3$. For a diatomic gas (such as air, which is composed primarily of N_2 and O_2), there are also two rotational degrees of freedom and $\alpha = 5$, so that $\gamma = 7/5 = 1.4$.

The equation of state for a polytropic gas. Note that $T = p/\mathcal{R}\rho$ so that

$$e = c_v T = \left(\frac{c_v}{\mathcal{R}}\right)\frac{p}{\rho} = \frac{p}{(\gamma - 1)\rho} \tag{5.21}$$

by (5.15) and (5.16). Using this in (5.3) gives the common form of the equation of state for a polytropic gas:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2. \tag{5.22}$$

5.1.2 Entropy

Another important thermodynamic quantity is the entropy. Roughly speaking, this measures the disorder in the system. The entropy S is defined up to an additive constant by

$$S = c_v \log(p/\rho^\gamma) + \text{constant}. \tag{5.23}$$

This can be solved for p to give

$$p = \kappa e^{S/c_v}\rho^\gamma, \tag{5.24}$$

where κ is a constant.

From the Euler equations we can derive the relation

$$S_t + uS_x = 0 \tag{5.25}$$

which says that entropy is constant along particle paths in regions of smooth flow. In fact, (5.25) can be derived from fundamental principles and this equation, together with the conservation of mass and momentum equations, gives an alternative formulation of the Euler equations (though not in conservation form):

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x &= 0 \\ S_t + uS_x &= 0 \end{aligned} \tag{5.26}$$

It turns out that the equation of state then gives p as a function of ρ and S alone, e.g. (5.24) for a polytropic gas. In this form the partial derivative of p with respect to ρ (holding S fixed) plays a fundamental role: its square root c is the local **speed of sound** in the gas. For a polytropic gas we have

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_{S=\text{constant}} = \gamma\kappa e^{S/c_v}\rho^{\gamma-1} = \gamma p/\rho \tag{5.27}$$

and so

$$c = \sqrt{\gamma p/\rho}. \tag{5.28}$$

From our standpoint the most important property of entropy is the fact that in smooth flow entropy remains constant on each particle path, while if a particle crosses a shock then the entropy may jump, but must *increase*. This is the physical entropy condition for shocks.

Note that along a particle path in smooth flow, since S is constant we find by (5.24) that

$$p = \hat{\kappa}\rho^\gamma \tag{5.29}$$

where $\hat{\kappa} = \kappa e^{S/c_v}$ is a constant which depends only on the initial entropy of the particle. This explicit relation between density and pressure along particle paths is sometimes useful. Of course, if the initial entropy varies in space then $\hat{\kappa}$ will be different along different particle paths.

5.2 Isentropic flow

If the entropy is constant everywhere then (5.29) holds with the same value of $\hat{\kappa}$ everywhere and the Euler equations simplify. This is the case, for example, if we consider fluid flow that starts at a uniform rest state (so S is constant) and remains smooth (so S remains constant). Then using (5.29), the equation of state (5.22) reduces to an explicit expression for E in terms of ρ and ρu. The energy equation then becomes redundant and the Euler equations reduce to a system of two equations, the **isentropic equations**,

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + \hat{\kappa}\rho^\gamma \end{bmatrix}_x = 0. \tag{5.30}$$

5.3 Isothermal flow

Taking $\gamma = 1$ in the isentropic equations (5.30) gives a particularly simple set of equations. This will prove to be a useful example for illustrating the theory presented later since the algebra is relatively simple and yet the behavior is analogous to what is seen for the full Euler equations. By the above discussion, the case $\gamma = 1$ is not physically relevant. However, these same equations can be derived by considering a different physical setup, the case of "isothermal" flow.

Here we consider the flow of gas in a tube that is immersed in a bath at a constant temperature $\bar{T}$, and assume that this bath maintains a constant temperature within the gas. Then the ideal gas law (5.8) reduces to

$$p = a^2 \rho \tag{5.31}$$

where $a^2 \equiv \mathcal{R}\bar{T}$ is a constant and a is the sound speed. Note that maintaining this constant temperature requires heat flux through the wall of the tube, and so energy is no longer conserved in the tube. But mass and momentum are still conserved and these equations, together with the equation of state (5.31), lead to the **isothermal equations**,

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix}_x = 0. \tag{5.32}$$

5.4 The shallow water equations

The study of wave motion in shallow water leads to a system of conservation laws with a similar structure. To derive the one-dimensional equations, we consider fluid in a channel and assume that the vertical velocity of the fluid is negligible and the horizontal velocity $u(x,t)$ is roughly constant through any vertical cross section. This is true if we consider small amplitude waves in a fluid that is shallow relative to the wave length.

We now assume the fluid is incompressible, so the density $\bar{\rho}$ is constant. Instead the height $h(x,t)$ varies, and so the total mass in $[x_1, x_2]$ at time t is

$$\text{total mass in } [x_1, x_2] = \int_{x_1}^{x_2} \bar{\rho} h(x,t)\, dx.$$

The momentum at each point is $\bar{\rho} u(x,t)$ and integrating this vertically gives the mass flux to be $\bar{\rho} u(x,t) h(x,t)$. The constant $\bar{\rho}$ drops out of the conservation of mass equation, which then takes the familiar form

$$h_t + (uh)_x = 0. \tag{5.33}$$

The conservation of momentum equation also takes the same form as in the Euler equations,

$$(\bar{\rho} hu)_t + (\bar{\rho} hu^2 + p)_x = 0, \tag{5.34}$$

but now the pressure p is determined from a hydrostatic law, stating that the pressure at depth y is $\bar{\rho} g y$, where g is the gravitational constant. Integrating this vertically from $y = 0$ to $y = h(x,t)$ gives the total pressure felt at (x,t), the proper pressure term in the momentum flux:

$$p = \frac{1}{2} \bar{\rho} g h^2. \tag{5.35}$$

Using this in (5.34) and cancelling $\bar{\rho}$ gives

$$(hu)_t + \left(hu^2 + \frac{1}{2} g h^2 \right)_x = 0. \tag{5.36}$$

Note that the system (5.33), (5.36) is equivalent to the isentropic equation (5.30) in the case $\gamma = 2$.

Equation (5.36) can be simplified by expanding the derivatives and using (5.33) to replace the h_t term. Then several terms drop out and (5.36) is reduced to

$$u_t + \left(\frac{1}{2} u^2 + gh \right)_x = 0. \tag{5.37}$$

Finally, the explicit dependence on g can be eliminated by introducing the variable $\varphi = gh$ into (5.33) and (5.37). The system for the **one-dimensional shallow water equations** then becomes

$$\begin{bmatrix} u \\ \varphi \end{bmatrix}_t + \begin{bmatrix} u^2/2 + \varphi \\ u\varphi \end{bmatrix}_x = 0. \tag{5.38}$$

6 Linear Hyperbolic Systems

In this chapter we begin the study of systems of conservation laws by reviewing the theory of a constant coefficient linear hyperbolic system. Here we can solve the equations explicitly by transforming to characteristic variables. We will also obtain explicit solutions of the Riemann problem and introduce a "phase space" interpretation that will be very useful in our study of nonlinear systems.

Consider the linear system

$$\begin{aligned} u_t + Au_x &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{6.1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times m}$ is a constant matrix. This is a system of conservation laws with the flux function $f(u) = Au$. This system is called **hyperbolic** if A is diagonalizable with real eigenvalues, so that we can decompose

$$A = R\Lambda R^{-1} \tag{6.2}$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a diagonal matrix of eigenvalues and $R = [r_1|r_2|\cdots|r_m]$ is the matrix of right eigenvectors. Note that $AR = R\Lambda$, i.e.,

$$Ar_p = \lambda_p r_p \quad \text{for } p = 1,\ 2,\ \ldots,\ m. \tag{6.3}$$

The system is called **strictly hyperbolic** if the eigenvalues are distinct. We will always make this assumption as well. (For a demonstration that the behavior can change dramatically in the nonstrictly hyperbolic case, see Exercise 7.4.)

6.1 Characteristic variables

We can solve (6.1) by changing to the "characteristic variables"

$$v = R^{-1}u. \tag{6.4}$$

Multiplying (6.1) by R^{-1} and using (6.2) gives

$$R^{-1}u_t + \Lambda R^{-1}u_x = 0 \tag{6.5}$$

or, since R^{-1} is constant,

$$v_t + \Lambda v_x = 0. \tag{6.6}$$

Since Λ is diagonal, this decouples into m independent scalar equations

$$(v_p)_t + \lambda_p (v_p)_x = 0, \qquad p = 1,\ 2,\ \ldots,\ m. \tag{6.7}$$

Each of these is a constant coefficient linear advection equation, with solution

$$v_p(x,t) = v_p(x - \lambda_p t, 0). \tag{6.8}$$

Since $v = R^{-1}u$, the initial data for v_p is simply the pth component of the vector

$$v(x,0) = R^{-1}u_0(x). \tag{6.9}$$

The solution to the original system is finally recovered via (6.4):

$$u(x,t) = Rv(x,t). \tag{6.10}$$

Note that the value $v_p(x,t)$ is the coefficient of r_p in an eigenvector expansion of the vector $u(x,t)$, i.e., (6.10) can be written out as

$$u(x,t) = \sum_{p=1}^{m} v_p(x,t) r_p. \tag{6.11}$$

Combining this with the solutions (6.8) of the decoupled scalar equations gives

$$u(x,t) = \sum_{p=1}^{m} v_p(x - \lambda_p t, 0) r_p. \tag{6.12}$$

Note that $u(x,t)$ depends only on the initial data at the m points $x - \lambda_p t$, so the domain of dependence is given by $\mathcal{D}(\bar{x},\bar{t}) = \{x = \bar{x} - \lambda_p \bar{t},\ p = 1,\ 2,\ \ldots,\ m\}$.

The curves $x = x_0 + \lambda_p t$ satisfying $x'(t) = \lambda_p$ are the "characteristics of the pth family", or simply "p-characteristics". These are straight lines in the case of a constant coefficient system. Note that for a strictly hyperbolic system, m distinct characteristic curves pass through each point in the x-t plane. The coefficient $v_p(x,t)$ of the eigenvector r_p in the eigenvector expansion (6.11) of $u(x,t)$ is constant along any p-characteristic.

6.2 Simple waves

We can view the solution as being the superposition of m waves, each of which is advected independently with no change in shape. The pth wave has shape $v_p(x,0)r_p$ and propagates with speed λ_p. This solution has a particularly simple form if $v_p(x,0)$ is constant in x for all but one value of p, say $v_p(x,0) \equiv c_p$ for $p \neq i$. Then the solution has the form

$$\begin{aligned} u(x,t) &= \sum_{p \neq i} c_p r_p + v_i(x - \lambda_i t, 0) r_i \\ &= u_0(x - \lambda_i t) \end{aligned} \tag{6.13}$$

and the initial data simply propagates with speed λ_i. Since $m-1$ of the characteristic variables are constant, the equation essentially reduces to $u_t + \lambda_i u_x = 0$ which governs the behavior of the ith family. Nonlinear equations have analogous solutions, called "simple waves", in which variations occur only in one characteristic family (see [97]).

6.3 The wave equation

The canonical example of a hyperbolic PDE is the second order scalar wave equation,

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty \tag{6.14}$$

with initial data

$$\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= u_1(x). \end{aligned}$$

This can be rewritten as a first order system of conservation laws by introducing

$$v = u_x \quad \text{and} \quad w = u_t$$

Then $v_t = w_x$ by equality of mixed partials, and (6.14) becomes $w_t = c^2 v_x$, so we obtain the system

$$\begin{bmatrix} v \\ w \end{bmatrix}_t + \begin{bmatrix} -w \\ -c^2 v \end{bmatrix}_x = 0, \tag{6.15}$$

which is of the form (6.1) with

$$A = \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix}. \tag{6.16}$$

The initial conditions become

$$\begin{aligned} v(x,0) &= u_0'(x) \\ w(x,0) &= u_1(x). \end{aligned} \tag{6.17}$$

Note that this system is hyperbolic since the eigenvalues of A are $\pm c$, which are the wave speeds.

EXERCISE 6.1. *Compute the eigenvectors of A in (6.16) and use these to decouple (6.15) into a pair of scalar equations. Solve these equations to find that*

$$v(x,t) = \frac{1}{2}\left[u_0'(x_1) + \frac{1}{c}u_1(x_1) + u_0'(x_2) - \frac{1}{c}u_1(x_2)\right] \tag{6.18}$$

$$w(x,t) = \frac{1}{2}\left[cu_0'(x_1) + u_1(x_1) - cu_0'(x_2) + u_1(x_2)\right] \tag{6.19}$$

where $x_1 = x + ct$, $x_2 = x - ct$. Use this to compute $u(x,t)$ in the original wave equation (6.14).

EXERCISE 6.2. *Rewrite the 2D wave equation*

$$u_{tt} = c^2(u_{xx} + u_{yy}) \tag{6.20}$$

as a first order system of the form $q_t + Aq_x + Bq_y = 0$ where $q = [v,\ w,\ \varphi]$ by introducing $v = u_x$, $w = u_y$, $\varphi = u_t$.

EXERCISE 6.3. *Solve the "linearized shallow water equations",*

$$\begin{bmatrix} u \\ \varphi \end{bmatrix}_t + \begin{bmatrix} \bar{u} & 1 \\ \bar{\varphi} & \bar{u} \end{bmatrix} \begin{bmatrix} u \\ \varphi \end{bmatrix}_x = 0 \tag{6.21}$$

where $\bar{u}$ and $\bar{\varphi} > 0$ are constants, with some given initial conditions for u and φ.

Propagation of singularities. If we consider a point $(\bar{x}, \bar{t})$ for which the initial data is smooth at each point $x_0 = \bar{x} - \lambda_p \bar{t}$, $p = 1, 2, \ldots, m$, then $v(x - \lambda_p t, 0) = R^{-1}u(x - \lambda_p t, 0)$ is also a smooth function at each of these points, and hence so is $u(x,t)$. It follows that any singularities in the initial data can propagate only along characteristics, just as in the scalar linear case. Moreover, smooth initial data gives smooth solutions.

6.4 Linearization of nonlinear systems

Now consider a nonlinear system of conservation laws

$$u_t + f(u)_x = 0, \tag{6.22}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$. This can be written in the quasilinear form

$$u_t + A(u)u_x = 0 \tag{6.23}$$

where $A(u) = f'(u)$ is the $m \times m$ Jacobian matrix. Again the system is **hyperbolic** if $A(u)$ is diagonalizable with real eigenvalues for all values of u, at least in some range

where the solution is known to lie, and strictly hyperbolic if the eigenvalues are distinct for all u.

We can define characteristics as before by integrating the eigenvalues of $A(u)$. There are m characteristic curves through each point. The curves $x(t)$ in the pth family satisfy

$$x'(t) = \lambda_p(u(x(t),t)),$$
$$x(0) = x_0,$$

for some x_0. Note that the λ_p now depend on u, the solution of the problem, and so we can no longer solve the problem by first determining the characteristics and then solving a system of ODEs along the characteristics. Instead a more complicated coupled system is obtained, reducing the effectiveness of this approach. Characteristics do yield valuable information about what happens locally for smooth data, however. In particular, if we linearize the problem about a constant state $\bar{u}$ we obtain a constant coefficient linear system, with the Jacobian frozen at $A(\bar{u})$. This is relevant if we consider the propagation of small disturbances as we did for the scalar traffic flow model in Chapter 4. Assume an expansion of the solution with the form

$$u(x,t) = \bar{u} + \epsilon u^{(1)}(x,t) + \epsilon^2 u^{(2)}(x,t) + \cdots, \tag{6.24}$$

where $\bar{u}$ is constant and ϵ is small, then by the same derivation as equation (4.12) we find that $u^{(1)}(x,t)$ satisfies

$$u_t^{(1)}(x,t) + A(\bar{u})u_x^{(1)}(x,t) = 0. \tag{6.25}$$

Small disturbances thus propagate (approximately) along characteristic curves of the form $x_p(t) = x_0 + \lambda_p(\bar{u})t$. Higher order corrections for nonlinear problems can be obtained by retaining more terms in the expansion. The propagation of discontinuities and other singularities can also be studied through the use of similar expansions. This "geometrical optics" approach to studying weakly nonlinear phenomenon is a powerful tool, but one we will not pursue here. (See Whitham[97], for example.)

Weak shocks. Recall that for a linear system singularities propagate only along characteristics. For nonlinear problems this is not the case, as we have already seen for nonlinear scalar equations in Chapter 3. Instead, the Rankine-Hugoniot jump condition (3.33) must be satisfied for a propagating discontinuity,

$$f(u_l) - f(u_r) = s(u_l - u_r) \tag{6.26}$$

where s is the propagation speed. However, for very weak discontinuities this relates nicely to the linear theory. Suppose that

$$\|u_r - u_l\| \equiv \epsilon \ll 1. \tag{6.27}$$

In this case expanding $f(u_l)$ about u_r gives

$$f(u_l) = f(u_r) + f'(u_r)(u_l - u_r) + O(\epsilon^2) \tag{6.28}$$

so that (6.26) yields

$$f'(u_r)(u_l - u_r) = s(u_l - u_r) + O(\epsilon^2). \tag{6.29}$$

In the limit as $\epsilon \to 0$, the normalized vector $(u_l - u_r)/\epsilon$ must approach an eigenvector of $f'(u_r) = A(u_r)$ with s approaching the corresponding eigenvalue. This observation will be very important in studying the structure of nonlinear solutions more generally.

6.4.1 Sound waves

A familiar example of small disturbances in gas dynamics is the propagation of sound waves in the air. If we consider the Euler equations for a polytropic gas, in which the flux is (2.14) with equation of state (5.22), then the flux Jacobian matrix is

$$f'(u) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)v^2 & (3-\gamma)v & (\gamma-1) \\ \frac{1}{2}(\gamma-1)v^3 - v(E+p)/\rho & (E+p)/\rho - (\gamma-1)v^2 & \gamma v \end{bmatrix}. \tag{6.30}$$

A tedious calculation confirms that the eigenvalues are given by $\lambda_1 = v - c$, $\lambda_2 = v$, and $\lambda_3 = v + c$, where

$$c = \sqrt{\gamma p/\rho}. \tag{6.31}$$

This is the local **speed of sound** in the gas. If we linearize these equations about some state $\bar{u}$, we see that small disturbances propagate at speeds $\bar{v}$, $\bar{v} \pm \bar{c}$, *i.e.* at speeds 0, $\pm\bar{c}$ relative to the background velocity $\bar{v}$. The waves traveling at speeds $\pm\bar{c}$ are sound waves; our ears are sensitive to the small variations in pressure in these waves.

The waves that travel with the velocity of the gas turn out to be simply density variations which advect with the fluid just as in the scalar linear advection equation. Even for the full nonlinear equations it is easy to check that there are solutions of the form

$$\begin{aligned} \rho(x,t) &= \hat{\rho}(x - \bar{v}t) \\ v(x,t) &= \bar{v} \\ p(x,t) &= \bar{p} \end{aligned} \tag{6.32}$$

where $\hat{\rho}(x)$ is any density distribution.

EXERCISE 6.4. *Verify that (6.32) satisfies the nonlinear Euler equations. Is this solution isentropic?*

EXERCISE 6.5. *The sound speed for the Euler equations is most easily calculated using the equations in the form (5.26), where the entropy S is one of the variables and the equation of state gives $p = p(\rho, S)$. Linearize these equations about a constant state and compute the eigenvalues of the resulting matrix. This verifies that more generally the sound speed is given by $\sqrt{\partial p/\partial \rho}$, which for the polytropic case reduces to (6.31).*

EXERCISE 6.6. *Show that the sound speed in the isentropic equations (5.30) is the same as in the full Euler equations. For this system of two equations, small disturbance waves travel with speeds $\bar{v} \pm \bar{c}$. Waves of the form (6.32) traveling with speed $\bar{v}$ are not isentropic and no longer appear.*

EXERCISE 6.7. *Verify that linearizing the shallow water equations (5.38) gives the system (6.21). What is the "sound speed" for this system?*

6.5 The Riemann Problem

For the constant coefficient linear system, the Riemann problem can be explicitly solved. We will see shortly that the solution to a nonlinear Riemann problem has a simple structure which is quite similar to the structure of this linear solution, and so it is worthwhile studying the linear case in some detail.

The Riemann problem consists of the equation $u_t + Au_x = 0$ together with piecewise constant initial data of the form

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \tag{6.33}$$

We will assume that this system is **strictly hyperbolic**, which means that the eigenvalues of A are real and distinct, and order them

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m. \tag{6.34}$$

Recall that the general solution to the linear problem is given by (6.12). For the Riemann problem we can simplify the notation if we decompose u_l and u_r as

$$u_l = \sum_{p=1}^{m} \alpha_p r_p \qquad u_r = \sum_{p=1}^{m} \beta_p r_p. \tag{6.35}$$

Then

$$v_p(x,0) = \begin{cases} \alpha_p & x < 0 \\ \beta_p & x > 0 \end{cases} \tag{6.36}$$

and so

$$v_p(x,t) = \begin{cases} \alpha_p & \text{if } x - \lambda_p t < 0 \\ \beta_p & \text{if } x - \lambda_p t > 0. \end{cases} \tag{6.37}$$

If we let $P(x,t)$ be the maximum value of p for which $x - \lambda_p t > 0$, then

$$u(x,t) = \sum_{p=1}^{P(x,t)} \beta_p r_p + \sum_{p=P(x,t)+1}^{m} \alpha_p r_p. \tag{6.38}$$

The determination of $u(x,t)$ at a given point is illustrated in Figure 6.1. In the case shown, $v_1 = \beta_1$ while $v_2 = \alpha_2$ and $v_3 = \alpha_3$. The solution at the point illustrated is thus

$$u(x,t) = \beta_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3. \tag{6.39}$$

Note that the solution is the same at any point in the wedge between the $x' = \lambda_1$ and $x' = \lambda_2$ characteristics. As we cross the pth characteristic, the value of $x - \lambda_p t$ passes through 0 and the corresponding v_p jumps from α_p to β_p. The other coefficients v_i $(i \neq p)$ remain constant.

The solution is constant in each of the wedges as shown in Figure 6.2. Across the pth characteristic the solution jumps with the jump given by

$$[u] = (\beta_p - \alpha_p) r_p. \tag{6.40}$$

Note that these jumps satisfy the Rankine-Hugoniot conditions (3.33), since $f(u) = Au$ leads to

$$\begin{aligned} [f] &= A[u] \\ &= (\beta_p - \alpha_p) A r_p \\ &= \lambda_p [u] \end{aligned}$$

and λ_p is precisely the speed of propagation of this jump. The solution $u(x,t)$ in (6.38) can alternatively be written in terms of these jumps as

$$u(x,t) = u_l + \sum_{\lambda_p < x/t} (\beta_p - \alpha_p) r_p \tag{6.41}$$

$$= u_r - \sum_{\lambda_p \geq x/t} (\beta_p - \alpha_p) r_p \tag{6.42}$$

It might happen that the initial jump $u_r - u_l$ is already an eigenvector of A, if $u_r - u_l = (\beta_i - \alpha_i) r_i$ for some i. In this case $\alpha_p = \beta_p$ for $p \neq i$. Then this discontinuity simply propagates with speed λ_i, and the other characteristics carry jumps of zero strength.

In general this is not the case, however, and the jump $u_r - u_l$ cannot propagate as a single discontinuity with any speed without violating the Rankine-Hugoniot condition. We can view "solving the Riemann problem" as finding a way to split up the jump $u_r - u_l$ into a sum of jumps

$$u_r - u_l = (\beta_1 - \alpha_1) r_1 + \cdots + (\beta_m - \alpha_m) r_m, \tag{6.43}$$

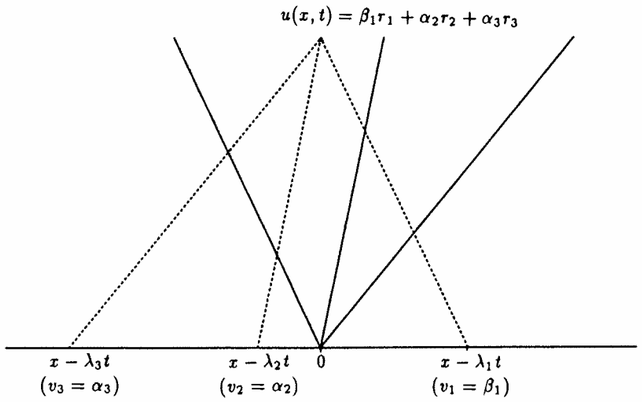


Figure 6.1. Construction of solution to Riemann problem at (x,t).

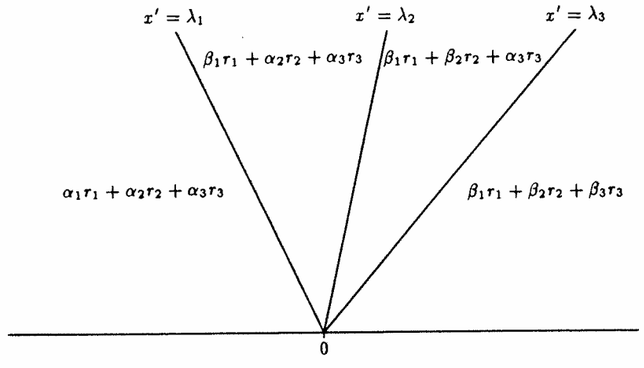


Figure 6.2. Values of solution u in each wedge of x–t plane.

each of which *can* propagate at an appropriate speed λ_i with the Rankine-Hugoniot condition satisfied.

For nonlinear systems we solve the Riemann problem in much the same way: The jump $u_r - u_l$ will usually not have the property that $[f]$ is a scalar multiple of $[u]$, but we can attempt to find a way to split this jump up as a sum of jumps, across each of which this property does hold. (Although life is complicated by the fact that we may need to introduce rarefaction waves as well as shocks.) In studying the solution of the Riemann problem, the jump in the pth family, traveling at constant speed λ_p, is often called the p-wave.

6.5.1 The phase plane

For systems of two equations, it is illuminating to view this splitting in the phase plane. This is simply the u_1–u_2 plane, where $u = (u_1, u_2)$. Each vector $u(x,t)$ is represented by a point in this plane. In particular, u_l and u_r are points in this plane and a discontinuity with left and right states u_l and u_r can propagate as a single discontinuity only if $u_r - u_l$ is an eigenvector of A, which means that the line segment from u_l to u_r must be parallel to the eigenvector r_1 or r_2. Figure 6.3 shows an example. For the state u_l illustrated there, the jump from u_l to u_r can propagate as a single discontinuity if and only if u_r lies on one of the two lines drawn through u_l in the direction r_1 and r_2. These lines give the locus of all points that can be connected to u_l by a 1-wave or a 2-wave. This set of states is called the **Hugoniot locus**. We will see that there is a direct generalization of this to nonlinear systems in the next chapter.

Similarly, there is a Hugoniot locus through any point u_r that gives the set of all points u_l that can be connected to u_r by an elementary p-wave. These curves are again in the directions r_1 and r_2.

For a general Riemann problem with arbitrary u_l and u_r, the solution consists of two discontinuities travelling with speeds λ_1 and λ_2, with a new constant state in between that we will call u_m. By the discussion above,

$$u_m = \beta_1 r_1 + \alpha_2 r_2 \tag{6.44}$$

so that $u_m - u_l = (\beta_1 - \alpha_1) r_1$ and $u_r - u_m = (\beta_2 - \alpha_2) r_2$. The location of u_m in the phase plane must be where the 1-wave locus through u_l intersects the 2-wave locus through u_r. This is illustrated in Figure 6.4a.

Note that if we interchange u_r and u_l in this picture, the location of u_m changes as illustrated in Figure 6.4b. In each case we travel from u_l to u_r by first going in the direction r_1 and then in the direction r_2. This is required by the fact that $\lambda_1 < \lambda_2$ since clearly the jump between u_l and u_m must travel slower than the jump between u_m and u_r if we are to obtain a single-valued solution.

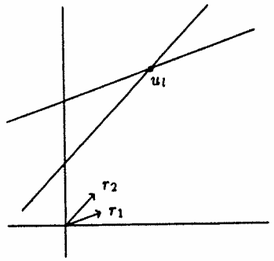
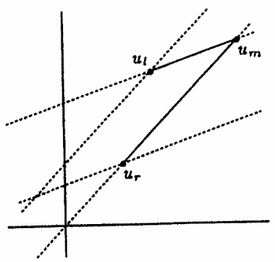


Figure 6.3. The Hugoniot locus of the state u_l consists of all states that differ from u_l by a scalar multiple of r_1 or r_2.



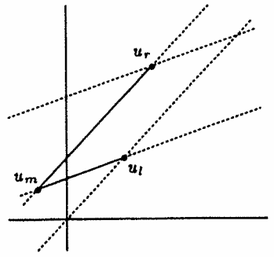


Figure 6.4. The new state u_m arising in the solution to the Riemann problem for two different choices of u_l and u_r.

For systems with more than two equations, the same interpretation is possible but becomes harder to draw since the phase space is now m dimensional. Since the m eigenvectors r_p are linearly independent, we can decompose any jump $u_r - u_l$ into the sum of jumps in these directions, obtaining a piecewise linear path from u_l to u_r in m-dimensional space.

7 Shocks and the Hugoniot Locus

We now return to the nonlinear system $u_t + f(u)_x = 0$, where $u(x,t) \in \mathbb{R}^m$. As before we assume strict hyperbolicity, so that $f'(u)$ has disctinct real eigenvalues $\lambda_1(u) < \cdots < \lambda_m(u)$ and hence linearly independent eigenvectors. We choose a particular basis for these eigenvectors, $\{r_p(u)\}_{p=1}^m$, usually chosen to be normalized in some manner, *e.g.* $\|r_p(u)\| \equiv 1$.

In the previous chapter we constructed the solution to the general Riemann problem for a linear hyperbolic system of conservation laws. Our goal in the next two chapters is to perform a similar construction for the nonlinear Riemann problem. In the linear case the solution consists of m waves, which are simply discontinuities traveling at the characteristic velocities of the linear system. In the nonlinear case our experience with the scalar equation leads us to expect more possibilities. In particular, the physically relevant vanishing viscosity solution may contain rarefaction waves as well as discontinuities. In this chapter we will first ignore the entropy condition and ask a simpler question: is it possible to construct a weak solution of the Riemann problem consisting only of m discontinuities propagating with constant speeds $s_1 < s_2 < \cdots < s_m$? As we will see, the answer is yes for $\|u_l - u_r\|$ sufficiently small.

7.1 The Hugoniot locus

Recall that if a discontinuity propagating with speed s has constant values $\hat{u}$ and $\tilde{u}$ on either side of the discontinuity, then the Rankine-Hugoniot jump condition must hold,

$$f(\tilde{u}) - f(\hat{u}) = s(\tilde{u} - \hat{u}). \tag{7.1}$$

Now suppose we fix the point $\hat{u} \in \mathbb{R}^m$ and attempt to determine the set of all points $\tilde{u}$ which can be connected to $\hat{u}$ by a discontinuity satisfying (7.1) for some s. This gives a system of m equations in $m+1$ unknowns: the m components of $\tilde{u}$, and s. This leads us to expect one parameter families of solutions.

We know that in the linear case there are indeed m such families for any $\hat{u}$. In the pth family the jump $\tilde{u} - \hat{u}$ is some scalar multiple of r_p, the pth eigenvector of A. We can

parameterize these families of solutions using this scalar multiple, say ξ, and we obtain the following solution curves:

$$\begin{aligned} \tilde{u}_p(\xi;\hat{u}) &= \hat{u} + \xi r_p \\ s_p(\xi;\hat{u}) &= \lambda_p \end{aligned}$$

for $p = 1, 2, \ldots, m$. Note that $\tilde{u}_p(0;\hat{u}) = \hat{u}$ for each p and so through the point $\hat{u}$ in phase space there are m curves (straight lines in fact) of possible solutions. This is illustrated in Figure 6.4 for the case $m = 2$. The two lines through each point are the states that can be connected by a discontinuity with jump proportional to r_1 or r_2.

In the nonlinear case we also obtain m curves through any point $\hat{u}$, one for each characteristic family. We again parameterize these curves by $\tilde{u}_p(\xi;\hat{u})$ with $\tilde{u}_p(0;\hat{u}) = \hat{u}$ and let $s_p(\xi;\hat{u})$ denote the corresponding speed. To simplify notation, we will frequently write these as simply $\tilde{u}_p(\xi)$, $s_p(\xi)$ when the point $\hat{u}$ is clearly understood.

The Rankine-Hugoniot condition gives

$$f(\tilde{u}_p(\xi)) - f(\hat{u}) = s_p(\xi)(\tilde{u}_p(\xi) - \hat{u}). \tag{7.2}$$

Differentiating this expression with respect to ξ and setting $\xi = 0$ gives

$$f'(\hat{u})\tilde{u}_p'(0) = s_p(0)\tilde{u}_p'(0) \tag{7.3}$$

so that $\tilde{u}_p'(0)$ must be a scalar multiple of the eigenvector $r_p(\hat{u})$ of $f'(\hat{u})$, while $s_p(0) = \lambda_p(\hat{u})$. The curve $\tilde{u}_p(\xi)$ is thus tangent to $r_p(\hat{u})$ at the point $\hat{u}$. We have already observed this, by a slightly different argument, in discussing weak shocks in Chapter 6. For a system of $m = 2$ equations, this is easily illustrated in the phase plane. An example for the isothermal equations of gas dynamics is discussed below, see Figure 7.1.

For smooth f, it can be shown using the implicit function theorem that these solution curves exist locally in a neighborhood of $\hat{u}$, and that the functions $\tilde{u}_p$ and s_p are smooth. See Lax[45] or Smoller[77] for details. These curves are called Hugoniot curves. The set of all points on these curves is often collectively called the **Hugoniot locus** for the point $\hat{u}$. If $\tilde{u}_p$ lies on the pth Hugoniot curve through $\hat{u}$, then we say that $\hat{u}$ and $\tilde{u}_p$ are connected by a p-shock.

EXAMPLE 7.1. The isothermal equations of gas dynamics (5.32) provide a relatively simple example of the nonlinear theory.

If we let m represent the momentum, $m = \rho v$, then the system becomes

$$\begin{aligned} \rho_t + m_x &= 0 \\ m_t + (m^2/\rho + a^2\rho)_x &= 0 \end{aligned} \tag{7.4}$$

or $u_t + f(u)_x = 0$ where $u = (\rho, m)$.

The Jacobian matrix is

$$f'(u) = \begin{bmatrix} 0 & 1 \\ a^2 - m^2/\rho^2 & 2m/\rho \end{bmatrix}. \tag{7.5}$$

The eigenvalues are

$$\lambda_1(u) = m/\rho - a, \qquad \lambda_2(u) = m/\rho + a \tag{7.6}$$

with eigenvectors

$$r_1(u) = \begin{bmatrix} 1 \\ m/\rho - a \end{bmatrix}, \qquad r_2(u) = \begin{bmatrix} 1 \\ m/\rho + a \end{bmatrix}. \tag{7.7}$$

These could be normalized but it is easiest to leave them in this simple form.

Now let's fix a state $\hat{u}$ and determine the set of states $\tilde{u}$ that can be connected by a discontinuity. The Rankine-Hugoniot condition (7.1) becomes, for this system,

$$\begin{aligned} \tilde{m} - \hat{m} &= s(\tilde{\rho} - \hat{\rho}) \\ (\tilde{m}^2/\tilde{\rho} + a^2\tilde{\rho}) - (\hat{m}^2/\hat{\rho} + a^2\hat{\rho}) &= s(\tilde{m} - \hat{m}). \end{aligned} \tag{7.8}$$

This gives two equations in the three unknowns $\tilde{\rho}$, $\tilde{m}$, and s. These equations can be easily solved for $\tilde{m}$ and s in terms of $\tilde{\rho}$, giving

$$\tilde{m} = \tilde{\rho}\hat{m}/\hat{\rho} \pm a\sqrt{\tilde{\rho}/\hat{\rho}}\,(\tilde{\rho} - \hat{\rho}) \tag{7.9}$$

and

$$s = \hat{m}/\hat{\rho} \pm a\sqrt{\tilde{\rho}/\hat{\rho}}. \tag{7.10}$$

The $\pm$ signs in these equations give two solutions, one for each family. Since $\tilde{m}$ and s can be expressed in terms of $\tilde{\rho}$, we can easily parameterize these curves by taking, for example,

$$\tilde{\rho}_p(\xi; \hat{u}) = \hat{\rho}(1 + \xi), \qquad p = 1,\ 2. \tag{7.11}$$

We then have

$$\tilde{u}_1(\xi; \hat{u}) = \hat{u} + \xi \begin{bmatrix} \hat{\rho} \\ \hat{m} - a\hat{\rho}\sqrt{1+\xi} \end{bmatrix}, \qquad s_1(\xi; \hat{u}) = \hat{m}/\hat{\rho} - a\sqrt{1+\xi}. \tag{7.12}$$

and

$$\tilde{u}_2(\xi; \hat{u}) = \hat{u} + \xi \begin{bmatrix} \hat{\rho} \\ \hat{m} + a\hat{\rho}\sqrt{1+\xi} \end{bmatrix}, \qquad s_2(\xi; \hat{u}) = \hat{m}/\hat{\rho} + a\sqrt{1+\xi}. \tag{7.13}$$

The choice of signs for each family is determined by the behavior as $\xi \to 0$, where the relation (7.3) must hold. It is easy to check that with the above choice we have

$$\begin{aligned} \frac{\partial}{\partial \xi}\tilde{u}_p(0; \hat{u}) &= \hat{\rho} r_p(\hat{u}) \propto r_p(\hat{u}), \\ s_p(0; \hat{u}) &= \lambda_p(\hat{u}), \end{aligned}$$

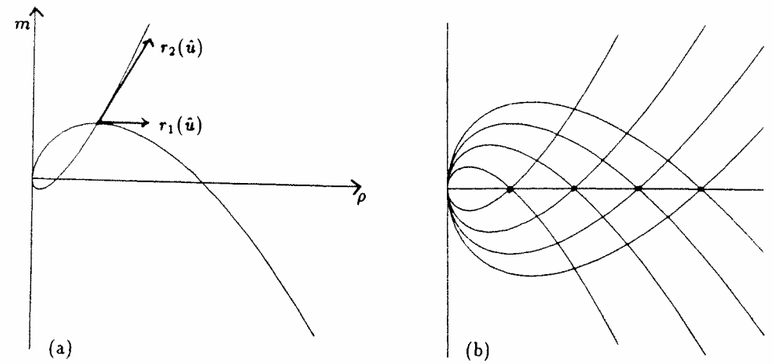


Figure 7.1. a) Hugoniot locus for the state $\hat{u} = (1, 1)$ in the isothermal gas dynamics equations with $a = 1$. b) Variation of these curves for $\hat{u} = (\hat{\rho}, 0)$, $\hat{\rho} = 1,\ 2,\ 3,\ 4$.

as expected.

Notice that real-valued solutions exist only for $\xi > -1$ and that $\tilde{u}_p(-1; \hat{u}) = (0, 0)$ for $p = 1, 2$ and any $\hat{u}$. Thus, each Hugoniot locus terminates at the origin (the *vacuum state*, since $\rho = 0$). There are no states with $\rho < 0$ that can be connected to $\hat{u}$ by a propagating discontinuity. The curves $\tilde{u}_p(\xi)$ are illustrated in Figure 7.1a for one particular choice of $\hat{u}$ and $a = 1$. Figure 7.1b shows how these curves vary with $\hat{u}$. In this case we see $\tilde{u}_p(\xi; \hat{u})$ for $\hat{u} = (\hat{\rho}, 0)$, $\hat{\rho} = 1,\ 2,\ 3,\ 4$.

EXERCISE 7.1. *Determine the Hugoniot locus for the shallow water equations (5.38).*

7.2 Solution of the Riemann problem

Now suppose that we wish to solve the Riemann problem with left and right states u_l and u_r (and recall that we are ignoring the entropy condition at this point). Just as in the linear case, we can accomplish this by finding an intermediate state u_m such that u_l and u_m are connected by a discontinuity satisfying the Rankine-Hugoniot condition, and so are u_m and u_r. Graphically we accomplish this by drawing the Hugoniot locus for each of the states u_l and u_r and looking for intersections. See Figure 7.2 for an example with the isothermal equations.

In this example there are two points of intersection, labelled u_m and u_m^*, but only u_m gives a single-valued solution to the Riemann problem since we need the jump from

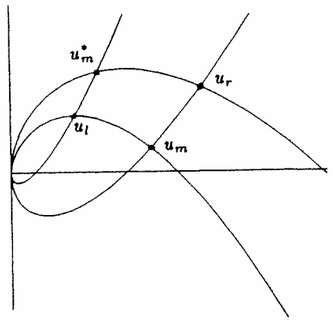


Figure 7.2. Construction of a weak solution to the Riemann problem with left and right states u_l and u_r.

u_l to u_m to travel more slowly than the jump from u_m to u_r. This requires that u_m be connected to u_l by a 1-shock while u_r is connected to u_m by a 2-shock, due to our convention that $\lambda_1(u) < \lambda_2(u)$. Of course our requirement really involves the shock speeds, not the characteristic speeds, but note that any 1-shock connected to u_m has speed

$$s_1(\xi; u_m) = m_m/\rho_m - a\sqrt{1 + \xi/\rho_m} < m_m/\rho_m \qquad \forall \xi > -\rho_m$$

while any 2-shock has speed

$$s_2(\xi; u_m) = m_m/\rho_m + a\sqrt{1 + \xi/\rho_m} > m_m/\rho_m \qquad \forall \xi > -\rho_m$$

and consequently $s_1(\xi_1; u_m) < s_2(\xi_2; u_m)$ for all $\xi_1,\ \xi_2 > -\rho_m$.

The state u_m can be found algebraically by using our explicit expressions for the Hugoniot locus. We wish to find a state (ρ_m, m_m) which is connected to u_l by a 1-shock and to u_r by a 2-shock. Consequently equation (7.9) with the minus sign should hold for $\tilde{u} = u_m,\ \hat{u} = u_l$, and the same equation with the plus sign should hold for $\tilde{u} = u_m, \hat{u} = u_r$. Equating the two resulting expressions for m_m gives

$$\rho_m m_l/\rho_l - a\sqrt{\rho_m/\rho_l}\,(\rho_m - \rho_l) = \rho_m m_r/\rho_r + a\sqrt{\rho_m/\rho_r}\,(\rho_m - \rho_r). \tag{7.14}$$

Setting $z = \sqrt{\rho_m}$, we obtain a quadratic equation for z,

$$\left(\frac{a}{\sqrt{\rho_r}} + \frac{a}{\sqrt{\rho_l}}\right) z^2 + \left(\frac{m_r}{\rho_r} - \frac{m_l}{\rho_l}\right) z - a\,(\sqrt{\rho_r} + \sqrt{\rho_l}) = 0. \tag{7.15}$$

This equation has a unique positive solution $\bar{z}$, and $\rho_m = \bar{z}^2$. We can then determine m_m by evaluating either side of (7.14).

More generally, for a system of m equations we can attempt to solve the Riemann problem by finding a sequence of states $u_1,\ u_2,\ \ldots,\ u_{m-1}$ such that u_l is connected to u_1 by a 1-shock, u_1 is connected to u_2 by a 2-shock, and so on, with u_{m-1} connected to u_r by an m-shock. If u_l and u_r are sufficiently close together then this can always be achieved. Lax proved a stronger version of this in his fundamental paper [44]. (By considering rarefaction waves also, the entropy satisfying solution can be constructed in a similar manner.) See also Lax[45] and Smoller[77] for detailed proofs. The idea is quite simple and will be summarized here following Lax[45]. From u_l we know that we can reach a one parameter family of states $u_1(\xi_1)$ through a 1-shock. From $u_1(\xi_1)$ we can reach another one parameter family of states $u_2(\xi_1, \xi_2)$ through a 2-shock. Continuing, we see that from u_l we can reach an m parameter family of states $u_m(\xi_1, \xi_2, \ldots, \xi_m)$. Moreover, we know that

$$\left.\frac{\partial u_m}{\partial \xi_p}\right|_{\xi_1 = \cdots = \xi_m = 0} \propto r_p(u_l), \qquad p = 1,\ 2,\ \ldots,\ m.$$

These vectors are linearly independent by our hyperbolicity assumption, and hence the Jacobian of the mapping $u_m : \mathbb{R}^m \to \mathbb{R}^m$ is nonsingular. It follows that the mapping u_m is bijective in a neighborhood of the origin. Hence for any u_r sufficiently close to u_l there is a unique set of parameters $\xi_1,\ \ldots,\ \xi_m$ such that $u_r = u_m(\xi_1, \ldots, \xi_m)$.

7.2.1 Riemann problems with no solution

For a general nonlinear system the local result quoted above is the best we can expect, and there are examples where the Riemann problem has no weak solution when $\|u_l - u_r\|$ is sufficiently large. In such examples the shock curves simply do not intersect in the required way. One such example is given by Smoller[76].

For specific systems of conservation laws one can often prove that solutions to the Riemann problem exist more globally. For example, in the isothermal case we can clearly always solve the Riemann problem provided $\rho_l,\ \rho_r > 0$. The case where ρ_l or ρ_r is negative is of little interest physically, but even then the Riemann problem can be solved by taking the vacuum state $u_m = (0, 0)$ as the intermediate state. We encounter problems only if $\rho = 0$ and $m \neq 0$ for one of the states, but since the equations involve m/ρ this is not surprising.

7.3 Genuine nonlinearity

In defining the Hugoniot locus above, we ignored the question of whether a given discontinuity is physically relevant. The state $\tilde{u}$ is in the Hugoniot locus of $\hat{u}$ if the jump

satisfies the Rankine-Hugoniot condition, regardless of whether this jump could exist in a vanishing viscosity solution. We would now like to define an entropy condition that can be applied directly to a discontinuous weak solution to determine whether the jumps should be allowed. In Chapter 3 we considered several forms of the entropy condition for scalar equations. In the convex case, the simplest condition is simply that characteristics should go *into* the shock as time advances, resulting in the requirement (3.45),

$$f'(u_l) > s > f'(u_r).$$

Lax[44] proposed a simple generalization of this entropy condition to systems of equations that are genuinely nonlinear, a natural generalization of the convex scalar equation. The pth characteristic field is said to be **genuinely nonlinear** if

$$\nabla\lambda_p(u) \cdot r_p(u) \neq 0 \qquad \text{for all } u, \tag{7.16}$$

where $\nabla\lambda_p(u) = (\partial\lambda_p/\partial u_1, \ \ldots, \ \partial\lambda_p/\partial u_m)$ is the gradient of $\lambda_p(u)$. Note that in the scalar case, $m = 1$ and $\lambda_1(u) = f'(u)$ while $r_1(u) = 1$ for all u. The condition (7.16) reduces to the convexity requirement $f''(u) \neq 0 \ \forall u$. This implies that the characteristic speed $f'(u)$ is monotonically increasing or decreasing as u varies, and leads to a relatively simple solution of the Riemann problem.

For a system of equations, (7.16) implies that $\lambda_p(u)$ is monotonically increasing or decreasing as u varies along an integral curve of the vector field $r_p(u)$. These integral curves will be discussed in detail in the next chapter, where we will see that through a rarefaction wave u varies along an integral curve. Since monotonicity of the propagation speed λ_p is clearly required through a rarefaction wave, genuine nonlinearity is a natural assumption.

7.4 The Lax entropy condition

For a genuinely nonlinear field, Lax's entropy condition says that a jump in the pth field (from u_l to u_r, say) is admissible only if

$$\lambda_p(u_l) > s > \lambda_p(u_r) \tag{7.17}$$

where s is again the shock speed. Characteristics in the pth family disappear into the shock as time advances, just as in the scalar case.

EXAMPLE 7.2. For isothermal gas dynamics we can easily verify that both fields are genuinely nonlinear. Since $\lambda_p = m/\rho \pm a$, we compute that in each case

$$\nabla\lambda_p(u) = \begin{bmatrix} -m/\rho^2 \\ 1/\rho \end{bmatrix}, \quad p = 1, \ 2. \tag{7.18}$$

Using (7.7), we compute that

$$\nabla\lambda_1(u) \cdot r_1(u) = -a/\rho \tag{7.19}$$

$$\nabla\lambda_2(u) \cdot r_2(u) = a/\rho. \tag{7.20}$$

These quantities are both nonzero for all u.

Now suppose u_l and u_r are connected by a 1-shock. Then u_l lies in the Hugoniot locus of u_r and also u_r lies in the Hugoniot locus of u_l. We can thus evaluate the shock speed s using (7.10) (with the minus sign, since the jump is a 1-shock) in two different ways, obtaining

$$s = \frac{m_l}{\rho_l} - a\sqrt{\rho_r/\rho_l} = \frac{m_r}{\rho_r} - a\sqrt{\rho_l/\rho_r}. \tag{7.21}$$

Since $\lambda_1(u) = m/\rho - a$, the entropy condition (7.17) becomes

$$\frac{m_l}{\rho_l} - a > \frac{m_l}{\rho_l} - a\sqrt{\rho_r/\rho_l} = \frac{m_r}{\rho_r} - a\sqrt{\rho_l/\rho_r} > \frac{m_r}{\rho_r} - a \tag{7.22}$$

and is clearly satisfied if and only if $\rho_r > \rho_l$.

Notice that since the fluid velocity is $v = m/\rho$, 1-shocks always travel slower than the fluid on either side, and so a given fluid particle passes through the shock from left to right (*i.e.* its state jumps from u_l to u_r). A consequence of the entropy condition is that the density of the gas must *increase* as it goes through the shock. This is also true more generally in the full Euler equations. The gas can only be compressed as the shock passes, not rarefied (rarefaction occurs, naturally enough, through a rarefaction wave rather than a shock).

For 2-shocks the entropy condition requires

$$\frac{m_l}{\rho_l} + a > \frac{m_l}{\rho_l} + a\sqrt{\rho_r/\rho_l} = \frac{m_r}{\rho_r} + a\sqrt{\rho_r/\rho_l} > \frac{m_r}{\rho_r} + a \tag{7.23}$$

which is now satisfied only if $\rho_r < \rho_l$. But note that 2-shocks travel faster than the fluid on either side, so that particles pass through the shock from right to left. So the entropy condition has the same physical interpretation as before: the density must jump from ρ_r to a higher value ρ_l as the gas goes through the shock.

We can now reconsider the Hugoniot locus of a point $\hat{u}$ and retain only the points $\tilde{u}$ that can be connected to $\hat{u}$ by an entropy-satisfying shock, discarding the entropy-violating shocks. In order to do this, we must first decide whether $\hat{u}$ is to lie to the left of the discontinuity or to the right. The entropy condition (7.17), unlike the Rankine-Hugoniot condition, is not symmetric in the two states.

Figure 7.3a shows the set of states that can be connected to the right of a given state $\hat{u}$ by an entropy-satisfying shock. Figure 7.3b shows the set of states that can be connected to the left of the same state $\hat{u}$. Note that the union of these curves gives the full Hugoniot locus. Each branch of the Hugoniot locus splits into two parts at $\hat{u}$; states on one side can only be connected to the left, states on the other side can only be connected to the right.

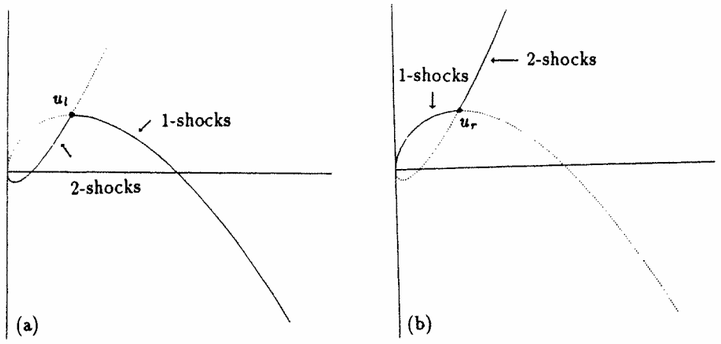


Figure 7.3. a) States u_r that can be connected to $u_l = \hat{u}$ by an entropy-satisfying shock. b) States u_l that can be connected to $u_r = \hat{u}$ by an entropy-satisfying shock. In each case the entropy-violating branches of the Hugoniot locus are shown as dashed lines.

7.5 Linear degeneracy

The assumption of genuine nonlinearity is obviously violated for a constant coefficient linear system, in which $\lambda_p(u)$ is constant and hence $\nabla\lambda_p \equiv 0$. More generally, for a nonlinear system it might happen that in one of the characteristic fields the eigenvalue $\lambda_p(u)$ is constant along integral curves of this field, and hence

$$\nabla\lambda_p(u) \cdot r_p(u) \equiv 0 \qquad \forall u. \tag{7.24}$$

(Of course the value of $\lambda_p(u)$ might vary from one integral curve to the next.) In this case we say that the pth field is **linearly degenerate**. This may seem rather unlikely, and not worth endowing with a special name, but in fact the Euler equations have this property. We will see later that for this system of three equations, two of the fields are genuinely nonlinear while the third is linearly degenerate.

A discontinuity in a linearly degenerate field is called a **contact discontinuity**. This name again comes from gas dynamics. In a shock tube problem the gas initially on one side of the diaphram never mixes with gas from the other side (in the inviscid Euler equations). As time evolves these two gases remain in contact along a ray in the x-t plane along which, in general, there is a jump in density. This is the contact discontinuity.

For general systems, if the pth field is linearly degenerate and u_l and u_r are connected by a discontinuity in this field, then it can be shown that u_l and u_r lie on the same integral curve of $r_p(u)$, so that $\lambda_p(u_l) = \lambda_p(u_r)$. Moreover, the propagation speed s is also equal to

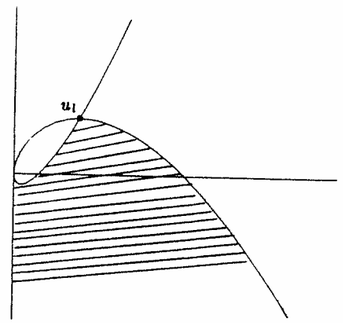


Figure 7.4. The shaded region shows the set of points u_r for which the Riemann problem with states u_l, u_r will have an entropy satisfying solution consisting of two shocks.

$\lambda_p(u_l)$. Consequently, the p-characteristics are parallel to the propagating discontinuity on each side, just as in the case of a linear system. Contact discontinuities can occur in vanishing viscosity solutions, and so for systems with both genuinely nonlinear and linearly degenerate fields, we should modify the entropy condition (7.17) to read

$$\lambda_p(u_l) \geq s \geq \lambda_p(u_r). \tag{7.25}$$

7.6 The Riemann problem

Returning to the Riemann problem, we see that for general data u_l and u_r, the weak solution previously constructed will not be the physically correct solution if any of the resulting shocks violate the entropy condition. In this case we need to also use rarefaction waves in our construction of the solution. This will be done in the next chapter.

EXAMPLE 7.3. In particular, the weak solution to the isothermal equations constructed in Figure 7.2 is nonphysical, since the 2-shock does not satisfy the entropy condition. For the state u_l illustrated there, a physically correct Riemann solution consisting of two shocks will exist only if the state u_r lies in the region shaded in Figure 7.4.

EXERCISE 7.2. *Consider the shallow water equations (5.38).*

1. *Show that a weak solution to the Riemann problem consisting only of shocks always exists if φ_l, $\varphi_r > 0$. Determine the intermediate state u_m for given states u_l and u_r.*

2. *Show that both fields are genuinely nonlinear.*

3. *Give a physical interpretation to the entropy condition for this system.*

EXERCISE 7.3. *Consider the system*

$$\begin{aligned} v_t + [vg(v,\phi)]_x &= 0 \\ \phi_t + [\phi g(v,\phi)]_x &= 0 \end{aligned} \tag{7.26}$$

where $g(v,\phi)$ is a given function. Systems of this form arise in two-phase flow. As a simple example, take $g(v,\phi) = \phi^2$ and assume $\phi > 0$.

1. *Determine the eigenvalues and eigenvectors for this system and show that the first field is linearly degenerate while the second field is genuinely nonlinear.*

2. *Show that the Hugoniot locus of any point $\hat{u}$ consists of a pair of straight lines, and that each line is also the integral curve of the corresponding eigenvector.*

3. *Obtain the general solution to the Riemann problem consisting of one shock and one contact discontinuity. Show that this solution satisfies the entropy condition if and only if $\phi_l \geq \phi_r$.*

EXERCISE 7.4 (A NONSTRICTLY HYPERBOLIC SYSTEM). *Consider the system*

$$\rho_t + (\rho v)_x = 0 \tag{7.27}$$

$$v_t + \left(\frac{1}{2}v^2\right)_x = 0 \tag{7.28}$$

1. *Show that the Jacobian matrix has eigenvalues $\lambda_1 = \lambda_2 = v$ and a one-dimensional space of eigenvectors proportional to $r = (1, v)$. Hence the Jacobian matrix is not diagonalizable, and the system is hyperbolic but not strictly hyperbolic.*

2. *Note that equation (7.28) decouples from (7.27) and is simply Burgers' equation for v. We can compute $v(x,t)$ from this equation, independent of ρ, and then (7.27) becomes*

$$\rho_t + v\rho_x = -\rho v_x$$

where the right hand side is now a source term and v is known. This is an evolution equation for ρ along characteristics. What happens when v becomes discontinuous?

3. *The system (7.27), (7.28) is the isothermal system (7.4) in the case $a = 0$. Use the theory of this chapter to investigate the limit $a \to 0$, and determine how the solution to a Riemann problem behaves in this limit. Relate this to the results of Part 2 above.*

8 Rarefaction Waves and Integral Curves

All of the Riemann solutions considered so far have the following property: the solution is constant along all rays of the form $x = \xi t$. Consequently, the solution is a function of x/t alone, and is said to be a "similarity solution" of the PDE. A rarefaction wave solution to a system of equations also has this property and takes the form

$$u(x,t) = \begin{cases} u_l & x \leq \xi_1 t \\ w(x/t) & \xi_1 t < x < \xi_2 t \\ u_r & x \geq \xi_2 t \end{cases} \tag{8.1}$$

where w is a smooth function with $w(\xi_1) = u_l$ and $w(\xi_2) = u_r$.

When does a system of equations have a solution of this form? As in the case of shocks, for arbitrary states u_l and u_r there might not be a solution of this form. But in general, starting at each point u_l there are m curves consisting of points u_r which can be connected to u_l by a rarefaction wave. These turn out to be subsets of the integral curves of the vector fields $r_p(u)$.

8.1 Integral curves

An integral curve for $r_p(u)$ is a curve which has the property that the tangent to the curve at any point u lies in the direction $r_p(u)$. The existence of smooth curves of this form follows from smoothness of f and strict hyperbolicity, since $r_p(u)$ is then a smooth function of u. If $u_p(\xi)$ is a parameterization (for $\xi \in \mathbb{R}$) of an integral curve in the pth family, then the tangent vector is proportional to $r_p(u_p(\xi))$ at each point, *i.e.*

$$u_p'(\xi) = \alpha(\xi) r_p(u_p(\xi)) \tag{8.2}$$

where $\alpha(\xi)$ is some scalar factor.

EXAMPLE 8.1. For the isothermal equations we can easily sketch the integral curves of r_p in the phase plane by drawing a selection of eigenvectors r_p and then sketching in

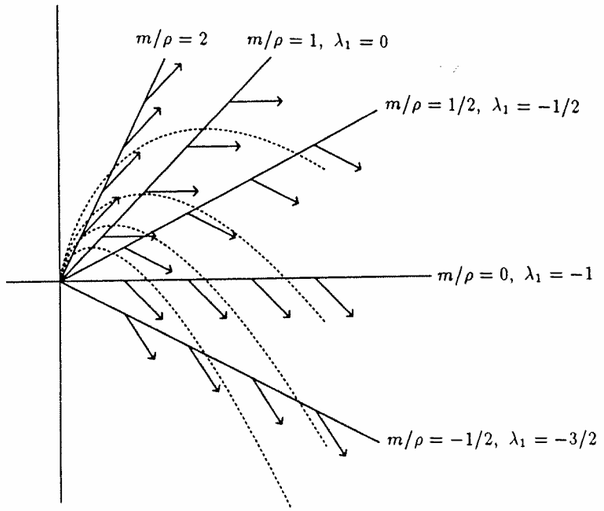


Figure 8.1. Integral curves of r_1 in the phase plane.

the smooth integral curves of this vector field. Figure 8.1 shows an example for the first characteristic field, in the case where the sound speed is taken as $a = 1$. Notice that along any ray $m/\rho = c$ in the phase plane the vector $r_1(u)$ is constant, since by (7.7), $r_1(u) = (1, c - a)$. This observaion has been used in Figure 8.1 to sketch in a number of eigenvectors.

Also notice that along $m/\rho = c$ we have $\lambda_1(u) = c - a$, so λ_1 is also constant along these rays. These are level lines of the function $\lambda_1(u)$. Since this characteristic field is genuinely nonlinear by (7.19), the function $\lambda_1(u)$ varies monotonically along any integral curve of r_1, as is clearly seen in Figure 8.1.

8.2 Rarefaction waves

To see that rarefaction curves lie along integral curves, and to explicitly determine the function $w(x/t)$ in (8.1), we differentiate $u(x,t) = w(x/t)$ to obtain

$$u_t(x,t) = -\frac{x}{t^2} w'(x/t) \tag{8.3}$$

$$u_x(x,t) = \frac{1}{t} w'(x/t) \tag{8.4}$$

so that $u_t + f'(u)u_x = 0$ yields

$$-\frac{x}{t^2} w'(x/t) + \frac{1}{t} f'(w(x/t))w'(x/t) = 0. \tag{8.5}$$

Multiplying by t and rearranging gives

$$f'(w(\xi))w'(\xi) = \xi w'(\xi), \tag{8.6}$$

where $\xi = x/t$. one possible solution of (8.6) is $w'(\xi) \equiv 0$, *i.e.*, w constant. Any constant function is a similarity solution of the conservation law, and indeed the rarefaction wave (8.1) takes this form for $\xi < \xi_1$ and $\xi > \xi_2$. In between, w is presumably smoothly varying and $w' \neq 0$. Then (8.6) says that $w'(\xi)$ must be proportional to some eigenvector $r_p(w(\xi))$ of $f'(w(\xi))$,

$$w'(\xi) = \alpha(\xi) r_p(w(\xi)) \tag{8.7}$$

and hence the values $w(\xi)$ all lie along some integral curve of r_p. In particular, the states $u_l = w(\xi_1)$ and $u_r = w(\xi_2)$ both lie on the same integral curve. This is a necessary condition for the existence of a rarefaction wave connecting u_l and u_r, but note that it is not sufficient. We need $\xi = x/t$ to be monotonically increasing as $w(\xi)$ moves from u_l to u_r along the integral curve; otherwise the function (8.1) is not single-valued. Note that our parameterization of the integral curve by ξ is not at all arbitrary at this point, since (8.6) requires that ξ be an eigenvalue of $f'(w(\xi))$,

$$\xi = \lambda_p(w(\xi)). \tag{8.8}$$

This particular parameterization is forced by our definition $\xi = x/t$. Note that (8.8) implies that w is constant along the ray $x = \lambda_p(w)t$, and hence each constant value of w propagates with speed $\lambda_p(w)$, just as in the scalar problem.

By (8.8), monotonicity of ξ is equivalent to monotonicity of $\lambda_p(w)$ as w moves from u_l to u_r. From a given state u_l we can move along the integral curve only in the direction in which λ_p is increasing. If λ_p has a local maximum at u_l in the direction r_p, then there are no rarefaction waves with left state u_l. In the generic nonlinear case, there is a one parameter family of states that can be connected to u_l by a p-rarefaction – all those states lying on the integral curve of r_p in the direction of increasing λ_p up to the next local maximum of λ_p.

If the pth field is genuinely nonlinear then λ_p is monotonically varying along the entire integral curve. We need not worry about local maxima and we see that u_l and u_r can always be connected by a rarefaction wave provided they lie on the same integral curve and

$$\lambda_p(u_l) < \lambda_p(u_r). \tag{8.9}$$

If the pth field is linearly degenerate, then λ_p is constant on each integral curve and there are no possible rarefaction waves in this family.

In order to explicitly determine the function $w(\xi)$, we first determine the scale factor $\alpha(\xi)$ in (8.7) by differentiating (8.8) with respect to ξ. This gives

$$\begin{aligned} 1 &= \nabla\lambda_p(w(\xi)) \cdot w'(\xi) \\ &= \alpha(\xi)\,\nabla\lambda_p(w(\xi)) \cdot r_p(w(\xi)) \end{aligned}$$

using (8.7), and hence

$$\alpha(\xi) = \frac{1}{\nabla\lambda_p(w(\xi)) \cdot r_p(w(\xi))}. \tag{8.10}$$

Using this in (8.7) gives a system of ordinary differential equations for $w(\xi)$:

$$w'(\xi) = \frac{r_p(w(\xi))}{\nabla\lambda_p(w(\xi)) \cdot r_p(w(\xi))}, \qquad \xi_1 \le \xi \le \xi_2 \tag{8.11}$$

with initial data

$$w(\xi_1) = u_l$$

where $\xi_1 = \lambda_p(u_l)$ and $\xi_2 = \lambda_p(u_r)$. Note that the denominator in (8.11) is finite for $\xi_1 \le \xi \le \xi_2$ only if λ_p is monotone between ξ_1 and ξ_2.

EXAMPLE 8.2. We will construct 1-rarefactions for the isothermal equations. Using (7.7) and (7.19), the system of ODEs (8.11) takes the form

$$\begin{aligned} \rho'(\xi) &= -\rho(\xi)/a, & \rho(\xi_1) &= \rho_l \\ m'(\xi) &= \rho(\xi) - m(\xi)/a, & m(\xi_1) &= m_l \end{aligned} \tag{8.12}$$

where $\xi_1 = \lambda_1(u_l) = m_l/\rho_l - a$. The first ODE is decoupled from the second and has solution

$$\rho(\xi) = \rho_l e^{-(\xi-\xi_1)/a}. \tag{8.13}$$

The second ODE is then

$$m'(\xi) = \rho_l e^{-(\xi-\xi_1)/a} - m(\xi)/a, \qquad m(\xi_1) = m_l \tag{8.14}$$

with solution

$$\begin{aligned} m(\xi) &= \left(\rho_l(\xi - \xi_1) + m_l\right) e^{-(\xi-\xi_1)/a} \\ &= \rho_l(\xi + a)e^{-(\xi-\xi_1)/a}. \end{aligned} \tag{8.15}$$

From the solutions $(\rho(\xi), m(\xi))$ it is also useful to eliminate ξ and solve for m as a function of ρ. This gives explicit expressions for the integral curves in the phase plane. If we solve for ξ in (8.13) and use this in (8.15) we obtain

$$m(\rho) = \rho m_l/\rho_l - a\rho\log(\rho/\rho_l). \tag{8.16}$$

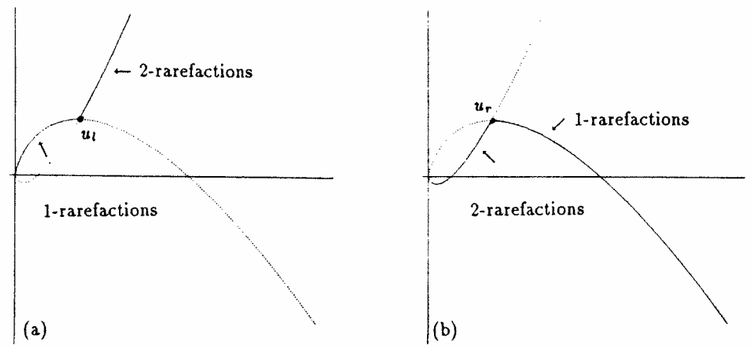


Figure 8.2. a) Set of states that can be connected to $u_l = \hat{u}$ by a rarefaction wave. b) Set of states that can be connected to $u_r = \hat{u}$ by a rarefaction wave. In each case the full integral curves are shown as dashed lines.

We can construct 2-rarefactions in exactly the same manner, obtaining

$$\rho(\xi) = \rho_l e^{(\xi-\xi_1)/a}, \tag{8.17}$$

$$m(\xi) = \rho_l(\xi - a)e^{(\xi-\xi_1)/a}. \tag{8.18}$$

and consequently

$$m(\rho) = \rho m_l/\rho_l + a\rho\log(\rho/\rho_l). \tag{8.19}$$

EXERCISE 8.1. *Derive the expressions for 2-rarefactions displayed above.*

EXERCISE 8.2. *From (8.16) and (8.19) verify that $m'(\rho_l) = \lambda_p(u_l)$ and explain why this should be so.*

For a given state $\hat{u} = u_l$ we can plot the set of all states u_r which can be connected to u_l by a rarefaction wave in either the first or second family. This is shown in Figure 8.2a for a particular choice of u_l. Note that if we consider this same state $\hat{u}$ to be u_r and now plot the set of all states u_l that can be connected to $\hat{u} = u_r$ by a rarefaction, we obtain a different picture as in Figure 8.2b. We must now have ξ *decreasing* as we move away from u_r and so it is the opposite side of each integral curve that is now relevant.

Note that these integral curves are very similar to the Hugoniot locus, *e.g.*, Figure 7.3. Locally, near the point $\hat{u}$, they must in fact be very similar. We know already that in the pth family each of these curves is tangent to $r_p(\hat{u})$ at $\hat{u}$. Moreover, it can be shown that the curvature of both curves is the same (See Lax[45]).

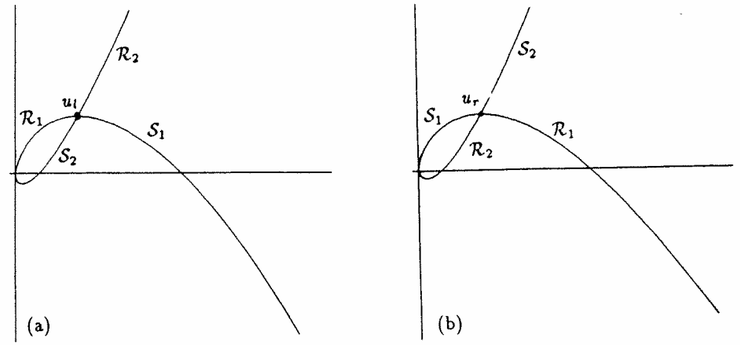


Figure 8.3. a) Set of states that can be connected to u_l by an entropy-satisfying 1-wave or 2-wave. b) Set of states that can be connected to u_r. In each case, $\mathcal{R}_p$ denotes p-rarefactions and $\mathcal{S}_p$ denotes p-shocks.

EXERCISE 8.3. *Verify this for the isothermal equations by computing $m''(\hat{\rho})$ for the Hugoniot locus from (7.9) and for the integral curves from (8.16) or (8.19) and seeing that they are the same. Also verify that the third derivatives are not equal.*

8.3 General solution of the Riemann problem

We can combine Figures 7.3 and 8.2 to obtain a plot showing all states that can be connected to a given $\hat{u}$ by entropy-satisfying waves, either shocks or rarefactions. Again, the nature of this plot depends on whether $\hat{u}$ is to be the left state or right state, so we obtain two plots as shown in Figure 8.3. Here $\mathcal{S}_p$ is used to denote the locus of states that can be connected by a p-shock and $\mathcal{R}_p$ is the locus of states that can be connected by a p-rarefaction. Notice that the shock and rarefaction curves match up smoothly (with the same slope and curvature) at the point $\hat{u}$.

To solve the general Riemann problem between two different states u_l and u_r, we simply superimpose the appropriate plots and look for the intersection u_m of a 1-wave curve from u_l and a 2-wave curve from u_r. An example for the isothermal equations is shown in Figure 8.4. This is the same example considered in Figure 7.2. We now see that the entropy-satisfying weak solution consists of a 1-shock from u_l to u_m followed by a 2-rarefaction from u_m to u_r.

To analytically determine the state u_m, we must first determine whether each wave is a shock or rarefaction, and then use the appropriate expressions relating m and ρ along

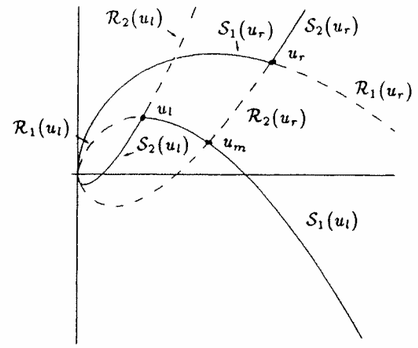


Figure 8.4. Construction of the entropy-satisfying weak solution to the Riemann problem with left and right states u_l and u_r.

each curve to solve for the intersection. We have already seen how to do this for the case of two shocks, by solving the equation (7.14). If the solution consists of two rarefactions then the intermediate state must satisfy

$$m_m = \rho_m m_l/\rho_l - a\rho_m \log(\rho_m/\rho_l) \tag{8.20}$$
$$m_m = \rho_m m_r/\rho_r + a\rho_m \log(\rho_m/\rho_r). \tag{8.21}$$

Equating the two right hand sides gives a single equation for ρ_m alone, with solution

$$\rho_m = \sqrt{\rho_l \rho_r}\, \exp\left(\frac{1}{2a}\left(\frac{m_l}{\rho_l} - \frac{m_r}{\rho_r}\right)\right). \tag{8.22}$$

We then obtain m_m from either (8.20) or (8.21).

If the solution consists of one shock and one rarefaction wave, as in Figure 8.4, then we must solve for ρ_m and m_m from the equations

$$m_m = \frac{\rho_m m_l}{\rho_l} - a\sqrt{\frac{\rho_m}{\rho_l}}\,(\rho_m - \rho_l) \tag{8.23}$$
$$m_m = \frac{\rho_m m_r}{\rho_r} + a\rho_m \log(\rho_m/\rho_l),$$

for example, in the case of a 1-shock followed by a 2-rarefaction. In this case it is not possible to obtain a closed form solution (ρ_m, m_m). Instead, it is necessary to solve these two equations by an iterative method such as Newton's method.

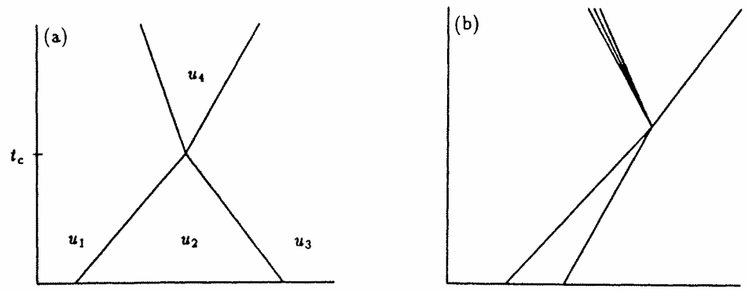


Figure 8.5. (a) Collision of a 2-shock and a 1-shock. (b) Collision of two 2-shocks.

8.4 Shock collisions

When two shocks collide the two discontinuities merge into one discontinuity that no longer satisfies the R-H condition. We can determine what happens beyond this point by solving a new Riemann problem. Consider Figure 8.5a where we have initial data consisting of three constant states u_1, u_2 and u_3 chosen such that u_1 and u_2 are connected by a 2-shock while u_2 and u_3 are connected by a 1-shock. At the collision time t_c, the function $u(x, t_c)$ has only a single discontinuity from u_1 to u_3. The behavior beyond this point is determined by solving the Riemann problem with $u_l = u_1$ and $u_r = u_3$.

EXERCISE 8.4. *Consider the isothermal equations with $a = 1$ and take data*

$$u(x,0) = \begin{cases} u_1 & x < 0.5 \\ u_2 & 0.5 < x < 1 \\ u_3 & x > 1 \end{cases}$$

where

$$u_1 = \begin{bmatrix} .28260 \\ .098185 \end{bmatrix} \qquad u_2 = \begin{bmatrix} .2 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} .35 \\ -0.19843 \end{bmatrix}.$$

Determine the solution $u(x, t)$.

EXERCISE 8.5. *For the isothermal equations, use the structure of the integral curves and Hugoniot loci to argue that:*

1. *When two shocks of different families collide, the result is again two shocks.*
2. *When two shocks in the same family collide, the result is a shock in that family and a rarefaction wave in the other family. (see Figure 8.5b).*

9 The Riemann problem for the Euler equations

I will not attempt to present all of the details for the case of the Euler equations. In principle we proceed as in the examples already presented, but the details are messier. Instead, I will concentrate on discussing one new feature seen here, contact discontinuities, and see how we can take advantage of the linear degeneracy of one field to simplify the solution process for a general Riemann problem. Full details are available in many sources, for example [11], [77], [97].

If we compute the Jacobian matrix $f'(u)$ from (2.14), with the polytropic equation of state (5.22), we obtain

$$f'(u) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)v^2 & (3-\gamma)v & (\gamma-1) \\ \frac{1}{2}(\gamma-1)v^3 - v(E+p)/\rho & (E+p)/\rho - (\gamma-1)v^2 & \gamma v \end{bmatrix}. \tag{9.1}$$

The eigenvalues are

$$\lambda_1(u) = v - c, \quad \lambda_2(u) = v, \quad \lambda_3(u) = v + c \tag{9.2}$$

where c is the sound speed,

$$c = \sqrt{\frac{\gamma p}{\rho}}. \tag{9.3}$$

9.1 Contact discontinuities

Of particular note in these equations is the fact that the second characteristic field is linearly degenerate. It is easy to check from (9.1) that

$$r_2(u) = \begin{bmatrix} 1 \\ v \\ \frac{1}{2}v^2 \end{bmatrix} \tag{9.4}$$

is an eigenvector of $f'(u)$ with eigenvalue $\lambda_2(u) = v = (\rho v)/\rho$. Since

$$\nabla\lambda_2(u) = \begin{bmatrix} -v/\rho \\ 1/\rho \\ 0 \end{bmatrix} \tag{9.5}$$

we find that $\nabla\lambda_2 \cdot r_2 \equiv 0$.

The first and third fields are genuinely nonlinear (somewhat more tedious to check).

Since the second field is linearly degenerate, we can have neither rarefaction waves nor shocks in this field. Instead we have contact discontinuities, which are linear discontinuities that propagate with speed equal to the characteristic speed λ_2 on each side.

Note that because $\lambda_2 = v$ is constant on the integral curves of r_2, and since r_2 depends only on v, the vector r_2 is itself constant on these curves, and hence the integral curves are straight lines in phase space. Moreover, these integral curves also form the Hugoniot locus for contact discontinuities. To see this, consider the Rankine-Hugoniot condition between states u and $\hat{u}$:

$$\begin{aligned} \rho v - \hat{\rho}\hat{v} &= s(\rho - \hat{\rho}) \\ \left(\frac{1}{2}\rho v^2 + p\right) - \left(\frac{1}{2}\hat{\rho}\hat{v}^2 + \hat{p}\right) &= s(\rho v - \hat{\rho}\hat{v}) \\ v(E+p) - \hat{v}(\hat{E} + \hat{p}) &= s(E - \hat{E}). \end{aligned} \tag{9.6}$$

These equations are clearly satisfied if we take $s = v = \hat{v}$ and $p = \hat{p}$. But then

$$u - \hat{u} = \begin{bmatrix} \rho - \hat{\rho} \\ \rho v - \hat{\rho}\hat{v} \\ \left(\frac{1}{2}\rho v^2 + p/(\gamma-1)\right) - \left(\frac{1}{2}\hat{\rho}\hat{v}^2 + p/(\gamma-1)\right) \end{bmatrix} = (\rho - \hat{\rho})r_2(u). \tag{9.7}$$

The jump is in the direction $r_2(u)$ and so the Hugoniot locus coincides with the integral curve of r_2.

We see from this that across a contact discontinuity the velocity v and pressure p are constant, only the density jumps (with resulting jumps in the other conserved quantities momentum and energy). Note that this is a special case of (6.32). Also notice that the speed of propagation is the same as the fluid velocity v. Thus particle paths agree with the characteristics, and move parallel to the contact discontinuity.

Since particle paths do not cross the discontinuity, two gases which are initially in contact with one another and have the same velocity and pressure will stay distinct, and continue to be in contact along this discontinuity.

It may seem strange that this discontinuity can sustain a jump in density — it seems that the denser gas should try to expand into the thinner gas. But that's because our intuition tends to equate higher density with higher pressure. It is only a *pressure* difference that can provide the force for expansion, and here the pressures are equal.

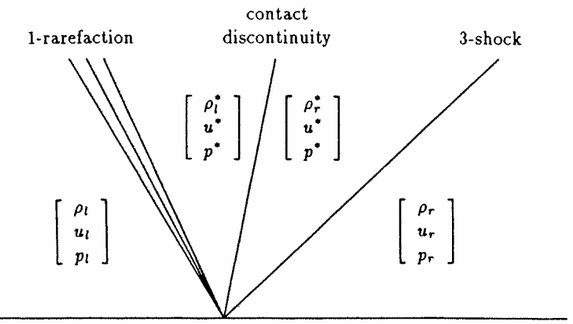


Figure 9.1. Typical solution to the Riemann problem for the Euler equations.

We can achieve two different densities at the same pressure by taking gases at two different temperatures. In fact, from (5.21) it is clear that there must be a jump in temperature if $\rho \neq \hat{\rho}$ while $p = \hat{p}$. There must also be a jump in entropy by (5.23). This explains why contact discontinuities do not appear in solutions to the isothermal or isentropic equations considered previously. In the reduction of the Euler equations to one of these systems of only two equations, it is this linearly degenerate characteristic field that disappears.

9.2 Solution to the Riemann problem

The first and third characteristic fields are genuinely nonlinear and have behavior similar to the two characteristic fields in the isothermal equations. The solution to a Riemann problem typically has a contact discontinuity and two nonlinear waves, each of which might be either a shock or a rarefaction wave depending on u_l and u_r. A typical solution is shown in Figure 9.1.

Because v and p are constant across the contact discontinuity, it is often easier to work in the variables (ρ, v, p) rather than $(\rho, \rho v, E)$, although of course the jump conditions must be determined using the conserved variables. The resulting Hugoniot locus and integral curves can be transformed into (ρ, v, p) space.

If the Riemann data is (ρ_l, v_l, p_l) and (ρ_r, v_r, p_r), then the two new constant states that appear in the Riemann solution will be denoted by $u_l^* = (\rho_l^*, v^*, p^*)$ and $u_r^* = (\rho_r^*, v^*, p^*)$. (See Figure 9.1.) Note that across the 2-wave we know there is a jump only in density.

Solution of the Riemann problem proceeds in principle just as in the previous chapter. Given the states u_l and u_r in the phase space, we need to determine the two intermediate states in such a way that u_l and u_l^* are connected by a 1-wave, u_l^* and u_r^* are connected

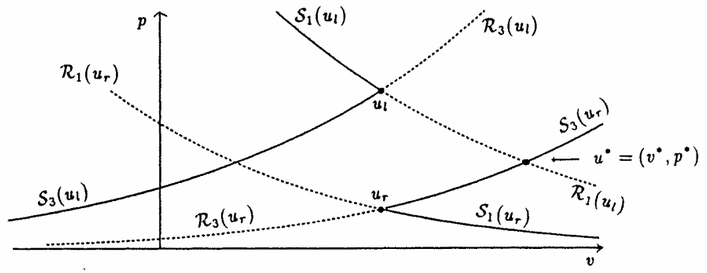


Figure 9.2. Projection of shock and rarefaction curves onto the two-dimensional v–p plane, and determination of u^*.

by a 2-wave, and finally u_r^* and u_r are connected by a 3-wave.

This seems difficult, but we can take advantage of the fact that we know the 2-wave will be a contact discontinuity across which v and p are constant to make the problem much simpler. Instead of considering the full three dimensional (ρ, v, p) phase space, consider the v–p plane and project the integral curves and Hugoniot loci for the 1-waves and 3-waves onto this plane. In particular, project the locus of all states that can be connected to u_l by a 1-wave (entropy satisfying shocks or rarefactions) onto this plane and also the locus of all states that can be connected to u_r by a 3-wave. This gives Figure 9.2.

We see in this example that we can go from u_l (or actually, the projection of u_l) to u^* by a 1-rarefaction and from u^* to u_r by a 3-shock. The problem with this construction, of course, is that these curves are really curves in 3-space, and just because their projections intersect does not mean the original curves intersect. However, the curve $\mathcal{R}_1(u_l)$ must go through some state $u_l^* = (\rho_l^*, v^*, p^*)$ for some ρ_l^* (so that it's projection onto the v–p plane is (v^*, p^*)). Similarly, the curve $\mathcal{S}_3(u_r)$ must pass through some state $u_r^* = (\rho_r^*, v^*, p^*)$. But these two states differ only in ρ, and hence can be connected by a 2-wave (contact discontinuity). We have thus achieved our objective. Note that this technique depends on the fact that *any* jump in ρ is allowed across the contact discontinuity.

In practice the calculation of u^* can be reduced to a single nonlinear equation for p^*, which is solved by an iterative method. Once p^* is known, u^*, ρ_l^* and ρ_r^* are easily determined. Godunov first proposed a numerical method based on the solution of Riemann problems and presented one such iterative method in his paper[24] (also described

in §12.15 of [63]). Chorin[6] describes an improvement of this method. More details on the solution of the Riemann problem can also be found in §81 of [11].