17.1 Vector Fields

Definition A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x, y) (or (x, y, z)) a two (or three dimensional) vector given by $\vec{F}(x, y)$ (or $\vec{F}(x, y, z)$).

Example

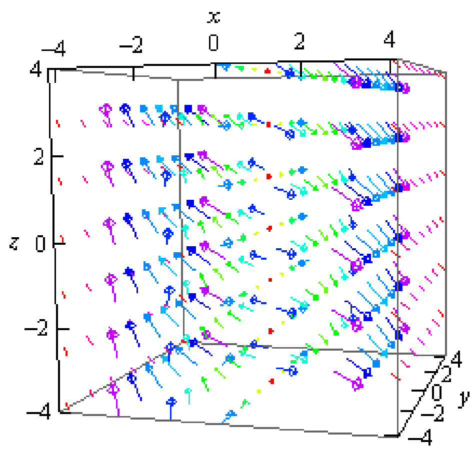

Figure 1: 3 dimensional velocity vector field of a flow, the arrow at each point (x, y, z) represents the velocity vector at that point

Notation for the vector function $\vec{F}$:

$$\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$$

$$\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

The **component functions** P, Q, R (if it is present) are sometimes called **scalar functions**.

Example 1 Sketch each of the following vector fields.
(a) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$
(b) $\vec{F}(x, y, z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$
Soln:

To graph the vector field we need to get some values of the function, which means plugging in some points into the functions.
(a) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$
Here are a couple of evaluations.

$$\vec{F}(1/2, 1/2) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

$$\vec{F}(1/2, -1/2) = -(-\frac{1}{2})\vec{i} + \frac{1}{2}\vec{j} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

The first evaluation tells us that at the point $(1/2, 1/2)$ we will plot the vector $-\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$.

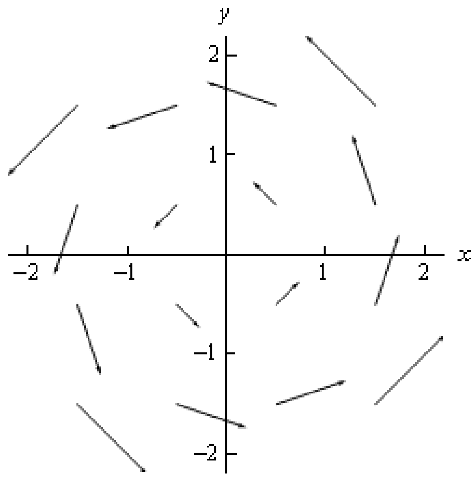

Figure 2: Vector field of **(a)**

(b) $\vec{F}(x, y, z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$
Do the same type of evaluations as for **(a)**

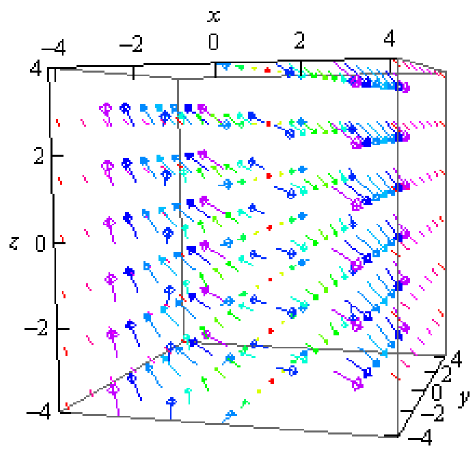

Figure 3: Vector field of **(b)**

Gradient Fields

Recall that given a function $f(x, y, z)$, the gradient vector is defined by $\nabla f = \langle f_x, f_y, f_z \rangle$. ∇f is a vector field and is often called a **gradient vector field.**

Example Sketch the gradient vector field for $f(x, y) = x^2 + y^2$ as well as several contours for this function.

Soln:

The gradient vector field for this function is

$$\nabla f = 2x\vec{i} + 2y\vec{j}$$

Recall that the contours for a function are nothing more than curves defined by $f(x, y) = k$ for various values of k. So the contours of function $f(x, y)$ are defined by the equation

$$x^2 + y^2 = k$$

. Thus they are circles centered at the origin with radius $\sqrt{k}$.

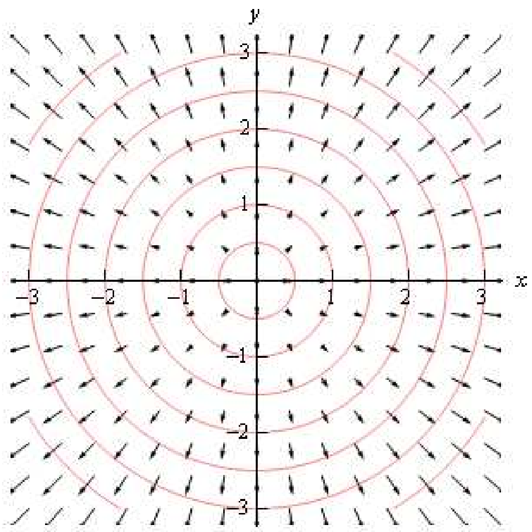

Figure 4: The gradient vector field of $x^2 + y^2$ and contours

Note: 1. The vectors of the gradient vector field of a function are all perpendicular (or orthogonal) to the contours of the function.

2. The direction of fastest change for a function is given by the gradient vector at that point.

conservative vector fields

A vector field $\vec{F}$ is called a **conservative vector fields** if there exists a function f such that $\vec{F} = \nabla f$. If $\vec{F}$ is a conservative vector fields, then the function f, is called a **potential function** for $\vec{F}$.

Example the vector field $\vec{F}(x, y) = y\vec{i} + x\vec{j}$ is a conservative vector field with a potential function of $f(x, y) = xy$ because $\nabla f =< y, x >$.

17.2 Line Integrals

Recall How to parameterize equations?

Some of the more basic curves that we need to know how to write down parametric equations as well as limits on the parameter if they are required.

(a) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parametric equation: $x = a\cos(t), \quad y = b\sin(t), \quad 0 \le t \le 2\pi$ (Counter-Clockwise direction)

or $x = a\cos(t), \quad y = -b\sin(t), \quad 0 \le t \le 2\pi$ (Clockwise direction)

(b) Circle $x^2 + y^2 = r^2$ Parametric equation: $x = r\cos(t), \quad y = r\sin(t), \quad 0 \le t \le 2\pi$ (Counter-Clockwise direction)

(c) y = f(x)

Parametric equation: $x = t, \quad y = f(y)$

(d) x = g(y)

Parametric equation: $y = t, \quad x = g(t)$

(e) Line segment from (x_0, y_0, z_0) to (x_1, y_1, z_1)

Parametric equation: $\vec{r}(t) = (1-t) < x_0, y_0, z_0 > +t < x_1, y_1, z_1 > \quad 0 \le t \le 1$

or

$$x = (1-t)x_0 + tx_1,$$
$$y = (1-t)y_0 + ty_1, \qquad 0 \le t \le 1$$
$$z = (1-t)z_0 + tz_1$$

or

$$\vec{r}(t) =< x_0, y_0, z_0 > +t < x_1 - x_0, y_1 - y_0, z_1 - z_0 > \quad 0 \le t \le 1$$

Note: 1. A parametrization of a curve gives the **orientation** (or the direction) of the curve. The positive direction corresponds to increasing values of the parameter t.

2. We will eventually see that the direction that the curve is traced out can, on occasion, change the answer of the line integrals.

Line Integrals (along the two-dimensional curve (in xy-plane))

With line integrals we will start with integrating the function $f(x, y)$ and the values of x and y that we are going to use will be the points ,(x, y), that lie on a curve C.

Assume that the curve C is smooth (defined shortly) and is given by the parametric equations, $x = h(t)$, $y = g(t)$ $a \le t \le b$ (or $\vec{r}(t) = h(t)\vec{i} + g(t)\vec{j}, \quad a \le t \le b$ if we write the parameterization of the curve as a vector function.)

Definition The curve is called **smooth** if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \ne \vec{0}$ for all t.

Definition The **line integral** of $f(x, y)$ along C is denoted by

$$\int_C f(x, y)ds$$

. Sometimes it is called the **line integral of f with respect to arc length**.

Note: 1. ds indicates that we are moving along the curve, C, instead of the x-axis (denoted by dx) or the y-axis (denoted by dy).

2.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

. The ds is the same for both the arc length integral $L = \int_a^b ds$ and the notation for the line integral.

The line integral is then

$$\int_C f(x,y)ds = \int_a^b f(h(t),g(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Note: 1. Do not forget to plug the parametric equations into the function.

2. As long as the parameterization of the curve C is traced out exactly once as t increases from a to b, the value of the line integral will be independent of the parameterization of the curve.

Vector form notations:

1.

$$\int_C f(x,y)ds = \int_a^b f(h(t),g(t))|\vec{r}'(t)|dt$$

because

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |\vec{r}'(t)|$$

where $|\vec{r}'(t)|$ is the magnitude of $\vec{r}'(t)$.

2.

$$\int_C f(x,y)ds = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt$$

Example 1. Let C be the line segment from $(1,1,1)$ to $(3,-2,-1)$. Find $\int_C x^2 - yds$

Soln:

$$\vec{r}(t) =< 1,1,1 > +t < 2,-3,-2 >=< 1+2t, 1-3t, 1-2t > \quad 0 \le t \le 1$$
$$\Rightarrow \vec{r}'(t) =< 2,-3,-2 >\Rightarrow |\vec{r}'(t)| = \sqrt{4+9+4} = \sqrt{17}$$

So

$$\int_C x^2 - yds = \int_0^1 [(1+2t)^2 - (1-3t)]\sqrt{17}dt = \frac{29\sqrt{17}}{6}$$

Line integrals over ***piecewise smooth curves***

A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves, C_1,, C_n where the end point of C_i is the starting point of C_{i+1}.

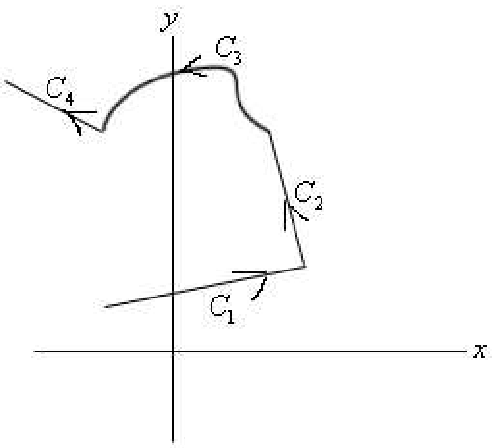

Figure 5: A piecewise smooth curve which is a union of C_1, C_2, C_3, C_4

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \int_{C_3} f(x,y)ds + \int_{C_4} f(x,y)ds$$

Example 2 Let C be composed of the two curves $C_1 : x = 1 - y^2$ and $C_2 : x = y^2 - 1$ shown below. Evaluate $\int_C \sqrt{1+4y^2}ds$

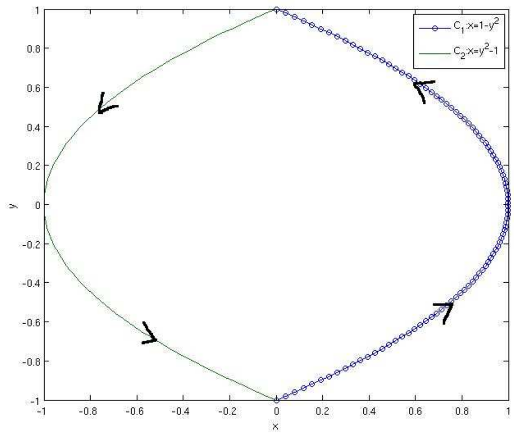

Figure 6: C_1 and C_2 curves

Soln:
Parameterization of $C_1 : x = 1 - y^2$:

$$\vec{r}_1(t) =< 1 - t^2, t >, \quad -1 \leq t \leq 1$$

The direction of C_1 is from $(0, -1)$ to $(0, 1)$

Parameterization of $C_2 : x = y^2 - 1$:

$$\vec{r}_2(t) =< t^2 - 1, t >, \quad -1 \le t \le 1$$

The direction of C_2 is from $(0, -1)$ to $(0, 1)$.

The orientations of C_1 and C_2 are not consistent by these parameterizations!!!

We change the orientation of C_2.

Replace t in parameterization of C_2 with $-t$:

$$\vec{r}_2(t) =< (-t)^2 - 1, -t >=< t^2, -t >, \quad -1 \le t \le 1$$

Now the orientations of C_1 and C_2 are consistent.

$$|\vec{r}_1'(t)| = \sqrt{1+4t^2}$$
$$|\vec{r}_2'(t)| = \sqrt{1+4t^2}$$

$$\begin{aligned}\int_C \sqrt{1+4y^2}ds &= \int_{C_1} \sqrt{1+4y^2}ds + \int_{C_2} \sqrt{1+4y^2}ds \\ &= \int_{-1}^{1} 1+4t^2dt + \int_{-1}^{1} 1+4t^2dt = 28/3\end{aligned}$$

Example 3. Let C be the line segment from $(3, -2, -1)$ to $(1, 1, 1)$. Find $\int_C x^2 - yds$

Soln:

$$\vec{r}(t) =< 3, -2, -1 > +t < -2, 3, 2 >=< 3-2t, -2+3t, -1+2t > \quad 0 \le t \le 1$$
$$\Rightarrow \vec{r}'(t) =< -2, 3, 2 >\Rightarrow |\vec{r}'(t)| = \sqrt{4+9+4} = \sqrt{17}$$

So

$$\int_C x^2 - yds = \int_0^1 [(3-2t)^2 - (-2+3t)]\sqrt{17}dt = \frac{29\sqrt{17}}{6}$$

Note: Compare **Example 3** with **Example 1**. When we switch the direction of the curve, these kinds of line integrals will not change. (There are other kinds of line integrals in which this will not be the case.)

Fact:

$$\int_C f(x,y)ds = \int_{-C} f(x,y)ds$$

Here the curve $-C$ is the same curve as C except the direction has been reversed.

One application of line integrals

If $\rho(x, y)$ represents the density of the wire shaped like a curve C, the **mass** m of the wire:

$$m = \int_C \rho(x,y)ds,$$

the **center of mass** $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds$$
$$\bar{y} = \frac{1}{m}\int_C y\rho(x,y)ds$$

Line Integrals along the three-dimensional curve

Suppose that the three-dimensional curve C is given by the parameterization,

$$x = x(t)$$
$$y = y(t)$$
$$z = z(t), \quad a \leq t \leq b$$

then the line integral is given by

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

Vector notation:
if we write the parametrization as a vector function $\vec{r}(t) =< x(t), y(t), z(t) >$,

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))|\vec{r}'(t)|dt = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt$$

Example 4 Let C be a wire in the shape of $\vec{r}(t) =< \cos(t), \sin(t), t/\pi > 0 \leq t \leq 4\pi$, when constant density $\rho(x,y,z) = \rho$. Compute the mass of C.

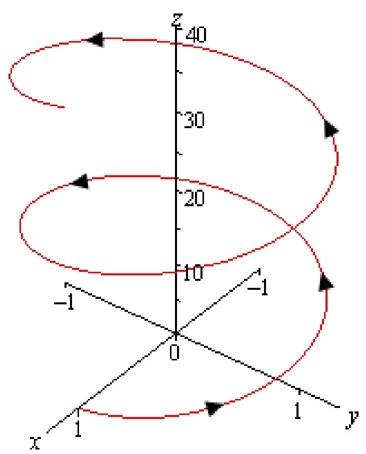

Figure 7: Sketch of the helix

Soln:

$$m = \int_C \rho ds = \int_0^{4\pi} \rho\sqrt{\sin^2(t) + \cos^2(t) + 1/\pi^2}dt$$
$$= 4\pi\rho\sqrt{1 + 1/\pi^2} = 4\rho\sqrt{\pi^2 + 1}$$

Line integrals with respect to x and/or y
Suppose that a two-dimensional curve C is given by the parameterization,

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

The **line integral of f with respect to x** is,

$$\int_C f(x, y)dx = \int_a^b f(x(t), y(t))x'(t)dt$$

The **line integral of f with respect to y** is,

$$\int_C f(x, y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

Note: When evaluating line integrals be careful to first note which differential you have got so you do not work the wrong kind of line integral.

$$\int_C Pdx + Qdy = \int_C Pdx + \int_C Qdy$$

Example 6 Evaluate $\int_C \sin(\pi y)dy + yx^2dx$ where
(a) C is the line segment from $(0, 2)$ to $(1, 4)$;
(a) C is the line segment from $(1, 4)$ to $(0, 2)$.
Soln:
(a)
The parameterization of the curve is

$$\vec{r}(t) = (1 - t) < 0, 2 > +t < 1, 4 >=< t, 2 + 2t > \qquad 0 \leq t \leq 1$$

The line integral is

$$\int_C \sin(\pi y)dy + yx^2dx = \int_C \sin(\pi y)dy + \int_C yx^2dx =$$
$$\int_0^1 \sin(\pi(2 + 2t))2dt + \int_0^1 (2 + 2t)t^2dt = \frac{7}{6}$$

(b) The parameterization of the curve is

$$\vec{r}(t) = (1 - t) < 1, 4 > +t < 0, 2 >=< 1 - t, 4 - 2t > \qquad 0 \leq t \leq 1$$

Verify that

$$\int_C \sin(\pi y)dy + yx^2dx = -\frac{7}{6}$$

Note: Switching the direction of the curve got us the opposite sign of the values.

Fact: If C is any curve, then

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx$$

$$\int_{-C} f(x,y)dy = -\int_C f(x,y)dy$$

and

$$\int_{-C} Pdx + Qdy = -\int_C Pdx + Qdy$$