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Line Integrals w. r. t. x, y, and/or z

If a three-dimensional curve C is parameterized by $x = x(t), \quad y = y(t), \quad z = z(t), \quad a \leq t \leq b$,

The **line integral of** $f(x, y, z)$ **w. r. t.** x is

$$\int_C f(x, y, z)dx = \int_a^b f(x(t), y(t), z(t))x'(t)dt$$

The **line integral of** $f(x, y, z)$ **w. r. t.** y is

$$\int_C f(x, y, z)dy = \int_a^b f(x(t), y(t), z(t))y'(t)dt$$

The **line integral of** $f(x, y, z)$ **w. r. t.** z is

$$\int_C f(x, y, z)dz = \int_a^b f(x(t), y(t), z(t))z'(t)dt$$

If the above integrals occur together, we denote

$$\int_C Pdx + Qdy + Rdz = \int_C Pdx + \int_C Qdy + \int_C Rdz \tag{1}$$

Line Integrals of Vector Fields

Applications of Line Integrals of Vector Fields: Add up the tangential component of a vector field along a curves. Examples: work ($\vec{F}$ = force), flow ($\vec{F}$ = velocity field)

Let $\vec{F}$ be a vector field and let a curve C be parameterized by $\vec{r}(t)$ so the unit tangent vector is $\vec{T} = \vec{r'}(t)/|\vec{r'}(t)|$.

$$\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k} \ ;$$

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \qquad a \leq t \leq b.$$

Then

$$\int_C \text{comp}_{\vec{T}}\vec{F}ds = \int_C \vec{F} \cdot \vec{T}ds = \int_a^b \vec{F} \cdot \frac{\vec{r'}(t)}{|\vec{r'}(t)|}|\vec{r'}|dt = \int_a^b \vec{F} \cdot \vec{r'}dt$$

Definition The **line integral of** $\vec{F}$ **along** C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r'}dt$$

Note: $\vec{F}(\vec{r}(t))$ is a shorthand for $\vec{F}(x(t), y(t), z(t))$.

Alternate notation:

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b (P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt})dt = \int_C Pdx + Qdy + Rdz$$

Note: Compare this alternate notation with Eq. (1). This shows that line integrals of vector fields can be defined in terms of line integrals with respect to x, y, and z. This give us another approach for evaluating line integrals of vector fields.

Example 1 Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = 8x^2yz\vec{i} + 5z\vec{j} - 4xy\vec{k}$ and C is the curve given by $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $0 \le t \le 1$

Soln:

We first evaluate the vector field along the curve.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\vec{i} + 5t^3\vec{j} - 4t(t^2)\vec{k} = 8t^7\vec{i} + 5t^3\vec{j} - 4t^3\vec{k}$$

We evaluate the derivative of the parametrization

$$\vec{r'}(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

We take the dot product

$$\vec{F}(\vec{r}(t))\cdot\vec{r'}(t) = 8t^7 + 10t^4 - 12t^5$$

The line integral is

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 8t^7 + 10t^4 - 12t^5 dt = (t^8 + 2t^5 - 2t^6)|_0^1 = 1$$

Example 2: Let C be $\vec{r}(t) =< \cos(t), \sin(t), t >$, $0 \le t \le 2\pi$. Evaluate $\int_C yzdx + xzdy + xydz$.

Soln:

We have

$$x(t) = \cos(t), \quad y(t) = \sin(t), \quad z(t) = t \quad .$$

$$\int_C yzdx + xzdy + xydz = \int_0^{2\pi} \sin(t)t(-\sin(t))dt + \cos(t)t(\cos(t))dt + \cos(t)\sin(t)(1)dt =$$

$$\int_0^{2\pi} t(\cos^2(t) - 1) + t\cos^2(t) + \cos(t)\sin(t)dt = \int_0^{2\pi} 2t\cos^2(t) - t + \cos(t)\sin(t)dt =$$

$$\int_0^{2\pi} 2t\cos^2(t) - tdt + \frac{1}{2}\sin^2(t)|_0^{2\pi} = \int_0^{2\pi} t(1+\cos(2t)) - tdt = \int_0^{2\pi} t\cos(2t)dt =$$

$$\text{(use integration by parts)} = \frac{t}{2}(\sin(2t) + \frac{1}{2}\cos(2t))|_0^{2\pi} = 0$$

Example 3 Let C be the part of the unit circle in the first quadrant joined with the two unit line segments along the axes, oriented counter-clock wise. Compute $\int_C x\sqrt{y}dx + 2y\sqrt{x}dy$.

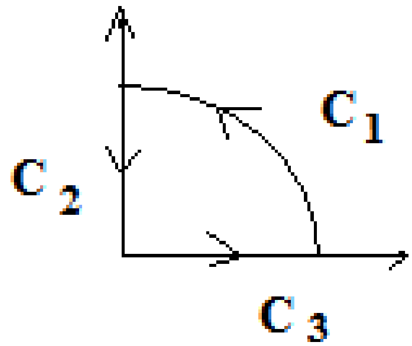

Figure 1: C **of Example 3**

Soln:
Parametrization of C_1, C_2, C_3:
C_1:

$$x = \cos(t), \quad y = \sin(t), \quad 0 \le t \le \pi/2$$

C_2:

$$x = 0, \quad y = 1 - t, \quad 0 \le t \le 1$$

C_3:

$$x = t, \quad y = 0, \quad 0 \le t \le 1$$

$$\begin{aligned}\int_C x\sqrt{y}dx + 2y\sqrt{x}dy = \int_{C_1} x\sqrt{y}dx + 2y\sqrt{x}dy + \int_{C_2} x\sqrt{y}dx + 2y\sqrt{x}dy + \int_{C_3} x\sqrt{y}dx + 2y\sqrt{x}dy \\ = \int_0^{\pi/2} \cos(t)\sqrt{\sin(t)}(-\sin(t)) + 2\sin(t)\sqrt{\cos(t)}(\cos(t))dt + \int_0^1 0dt + \int_0^1 0dt = \\ = \int_0^{\pi/2} -\cos(t)\sin^{3/2}(t) + 2\sin(t)\cos^{3/2}(t)dt \text{ (use substitution)} = \\ -\frac{2}{5}\sin^{5/2}(t) - \frac{4}{5}\cos^{5/2}(t)|_0^{\pi/2} = 2/5\end{aligned}$$

Fact:

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$$

Note: Do not forget that this is also true for line integrals with respect to x, y, and/or z.