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A Graph-Theoretic Characterization of Structural Controllability for Multi-Agent System with Switching Topology

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Abstract— **This paper considers the controllability problem for multi-agent systems. In particular, the structural controllability of multi-agent systems with a single leader under switching topologies is investigated. The structural controllability of multi-agent systems is a generalization of the traditional controllability concept for dynamical systems, and purely based on the communication topologies among agents. The main contribution of the paper is a graph-theoretic characterization of the structural controllability for multi-agent systems. It turns out that the multi-agent system with switching topology is structurally controllable if and only if the union graph $\mathcal{G}$ of the underlaying communication topologies is connected. Finally, the paper concludes with several illustrative examples and discussions of the results and future work.**

I. INTRODUCTION

Due to the latest advances in communication and computation, the distributed control and coordination of the networked dynamic agents has rapidly emerged as a hot multidisciplinary research area [1]-[8], which lies at the intersection of systems control theory, communication and mathematics. In addition, the advances of the research in multi-agent systems are strongly supported by their promising civilian and military applications, such as cooperative control of unmanned air vehicles(UAVs), autonomous underwater vehicles(AUVs), space exploration, congestion control in communication networks, air traffic control and so on. Much work has been done on the formation stabilization and consensus seeking. Approaches like graph Laplacians for the associated neighborhood graphs, artificial potential functions, and navigation functions for distributed formation stabilization with collision avoidance constraints have been developed. Furthermore, inspired by the cooperative behavior of natural swarms, such as bee flocking, ant colonies and fish schooling, people try to obtain experiences from how the group units make their whole group motions under control just through limited and local interactions among them.

The control of such large scale complex systems poses several new challenges that fall beyond the traditional methods. Part of the difficulty comes from the fact that the global behavior of the whole group combined by multiple agents is not a simple summation of the individual agent's behavior. Actually, the group behavior can be greatly impacted by the communication protocols or interconnection topology between the agents, which makes the global behavior display high complexities. Hence, the cooperative control of multi-agent systems is still in its infancy and attracts more and more researchers' attention. One basic question in multi-agent systems that attracts control engineers' interest is what is the necessary information exchanging among agents to make the whole group well-behaved, e.g., controllable. This can be formulated as a controllability problem for multi-agent systems under the leader-follower framework. Roughly speaking, a multi-agent system is controllable if and only if we can drive the whole group of agents to any desirable configurations only based on local interactions between agents and possibly some limited commands to a few agents that serve as leaders. The basic issue is the interplay between control and communication. In particular, we would like to investigate what is the necessary and/or sufficient condition on the graph of communication topologies among agents for the controllability of multi-agent systems.

This problem was first proposed by Tanner in [1], which formulated it as the controllability of a linear system and proposed a necessary and sufficient algebraic condition based on the eigenvectors of the graph Laplacian. Reference [1] focused on fixed topology situation with a particular member which acted as the single leader. The problem was then developed in [2][3][4][5][8], and got some interesting results. For example, in [3], it was shown that a necessary and sufficient condition for controllability is not sharing any common eigenvalues between the Laplacian matrix of the follower set and the Laplacian matrix of the whole topology. However, it remains elusive on what exactly the graphical meaning of these algebraic conditions related to the Laplacian matrix. This motivates several research activities on illuminating the controllability of multi-agent systems from a graph theoretical point of view. For example, a notion of anchored systems was introduced in [8] and it was shown that symmetry with respect to the anchored vertices makes the system uncontrollable. However, a satisfactory graphical interpretation of these algebraic controllability conditions turns out to be very challenging. Recently, we proposed a new notation for the controllability of multi-agent systems, called structural controllability in [4], and investigate the problem directly through the graph-theoretic approach for control systems. In contrast to the existing literature, [4] considered a weighted graph and focused on the case of a single leader under a fixed topology. The system is called structurally controllable if one may find a set of weights such

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that the corresponding multi-agent system is controllable in a classical sense. It turns out that this controllability notation only depends on the topology of the communication scheme, and the multi-agent system is structurally controllable if and only if the graph is connected.

Notice that the results in [1], [2], [3], [4], [5], [8] are all focused on multi-agent systems under fixed communication topologies which may restrict their impacts on real applications. In practice, the communication linkages between agents are unavoidably influenced by many factors that are out of control, such as distance, noise disturbance and signal strength. For certain applications, it may become impossible to keep the communication topology fixed for the whole period. Therefore, it is of practical importance to consider time varying communication topologies. A natural framework to study the time variance of communication topology is through switched systems, see e.g., [9][10][11]. In this paper, we will focus on multi-agent systems under switching topologies in the framework of switched systems. Some early efforts have been observed in the literature. Necessary and sufficient algebraic conditions for the controllability of multi-agent systems under switching topology were derived in [6][7] based on the developments of controllability study in switched systems. However, these algebraic results lacks graphically intuitive interpretations, which are important since they can provide us significant guidelines for the communication protocol design for multi-agent systems. Therefore, this paper aims to fill this gap and propose a graphic interpretation of these algebraic conditions for the controllability of multi-agent systems under switching topology.

In particular, we follow the setup in [4] and investigate the structural controllability of multi-agent systems with single leader and time varying communication topologies. It is assumed that the leader acts as the external or control signal and will not be affected by any other group members. Based on this structural controllability, we propose a necessary and sufficient graph theoretic condition for the structural controllability of multi-agent system with switching topologies. It turns out that the structural controllability is completely determined by the interconnection topology, and the multi-agent system is structurally controllable if and only if the union graph $\mathcal{G}$ is connected. Some examples are given to underscore our theoretical analysis.

The outline of this paper is as follows: In Section II, we introduce some basic preliminaries, followed by structural controllability study in Section III, where a graphic necessary and sufficient condition for the structural controllability is given. In Section IV, some examples are presented to give the readers deeper understanding of our theoretical results. Finally, some concluding remarks are drawn in the paper.

II. Preliminaries and Problem Formulation

A. Graph Theory Preliminaries

A weighted graph is an appropriate representation for the communication or sensing links among agents because it can represent both existence and strength of these links among agents. The weighted graph $\mathcal{G}$ with N vertices consists of a vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{I} = \{e_1, e_2, \ldots, e_{N'}\}$, which is the interconnection links among the vertices. Each edge in the weighted graph represents a bidirectional communication or sensing media. Two vertices are known to be *neighbors* if $(i,j) \in \mathcal{I}$, and the number of neighbors for each vertex is its *valency*. An alternating sequence of distinct vertices and edges in the weighted graph is called a *path*. The weighted graph is said to be *connected* if there exists at least one path between any distinct vertices, and *complete* if all vertices are neighbors to each other.

The *adjacency matrix*, $\mathcal{A}$ is defined as

$$\mathcal{A}_{(i,j)} = \begin{cases} \mathcal{W}_{ij} & (i,j) \in \mathcal{I}, \\ 0 & otherwise. \end{cases} \tag{1}$$

where $\mathcal{W}_{ij} \neq 0$ stands for the weight of edge (i,j). Here, the adjacency matrix $\mathcal{A}$ is $\mid \mathcal{V} \mid \times \mid \mathcal{V} \mid$ and $\mid . \mid$ is the cardinality of a set.

The *Laplacian matrix* of a graph $\mathcal{G}$, denoted as $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ or $\mathcal{L}$ for simplicity, is defined as

$$\mathcal{L}_{(i,j)} = \begin{cases} \Sigma_{i\neq j}\ \mathcal{W}_{ij} & i = j, \\ -\mathcal{W}_{ij} & (i,j) \in \mathcal{I}, \\ 0 & otherwise. \end{cases} \tag{2}$$

B. Multi-agent Structural Controllability with Switching Topology

Our objective in this paper is to control N agents based on the leader-follower framework. Specifically, we will consider the case of a single leader and switching topology. Without loss of generality, assume that the N-th agent serves as the leader and take commands and controls from outside operators directly, while the rest $N-1$ agents are followers and take controls as the nearest neighbor law.

Mathematically, each agent's dynamics can be seen as a point mass and follows

$$\dot{x}_i = u_i. \tag{3}$$

The control strategy for driving all follower agents is

$$u_i = -\sum\nolimits_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j), \tag{4}$$

where $\mathcal{N}_i$ is the neighbor set of the agent i, and w_{ij} is weight of the edge from agent i to agent j. On the other hand, the leader's control signal is not influenced by the followers and need to be designed, which can be represented as

$$\dot{x}_N = u_N. \tag{5}$$

In other words, the leader affects its nearby agents, but it does not get directly affected by the followers since it only accepts the control input from an outside operator. For simplicity, we will use z to stand for x_N in the sequel.

It is known that the whole system equipped with m subsystems can be written in a compact form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{aq_i} & B_{aq_i} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u_N \end{bmatrix} \quad . \tag{6}$$

Or, equivalently,

$$\left\{\begin{array}{lcl}\dot{x} & = & A_{aq_i}x + B_{aq_i}z, \\ \dot{z} & = & u_N.\end{array}\right. \quad (7)$$

where $i \in \{1,\ldots,m\}$. $A_{aq_i} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $B_{aq_i} \in \mathbb{R}^{(N-1)\times 1}$ are both sub-matrices of the corresponding graph Laplacian matrix $\mathcal{L}$. The matrix A_{aq_i} reflects the interconnection among followers, and the column vector B_{aq_i} represents the relation between followers and the leader under corresponding subsystems. Since the communication topologies among agents are time-varying, so the matrices A_{aq_i} and B_{aq_i} are also varying as a function of time. Therefore, the dynamical system described in (6) can be naturally modeled as a switched system [9], [10], [11]. Here, we will study the controllability problem for multi-agent systems under the framework of switched systems.

According to [1], it turned out that the complete topology graph makes the system uncontrollable, which shows that too much information exchange may damage the controllability of our system. In contrast, if we set weights of unnecessary connections to be zero and impose appropriate weights to other connections so as to use the communication information in a selective way, then it is possible to make the system controllable [4]. Accordingly, we follow the notation in [4] and introduce the following definition of structural controllability for switched systems:

Definition 1. The linear system (6), whose matrix elements are zeros or undetermined parameters, is said to be structurally controllable if and only if there exist a set of weights w_{ij} that can make the system (6) controllable in a classical sense.

Our main task here is to find out under what kinds of communication topologies, it is possible to make the group motions under control and steer the agents to the specific geometric positions or formation as a whole group. Now this controllability problem reduces to whether we can find a set of weights w_{ij} such that the multi-agent system (6) is controllable. Then the controllability problem of multi-agent system can now be formulated as the structural controllability problem of switched linear system (6). Before the discussion of our main results in this paper, we recall a known result in the literature for the controllability of switched systems.

C. Switched Linear System and Controllability Matrix

In general, a switched linear system is composed of a family of subsystems and a rule that governs the switching among them, and is mathematically described by

$$\dot{x}(t) = \quad A_\sigma x(t) + B_\sigma u(t), \quad (8)$$

σ is the control signal and a piecewise constant signal taking value from an index set $M \triangleq \{1,\ldots,m\}$. Suppose that there are m subsystems (A_i, B_i), $i \in \{1,\ldots,m\}$.

For the controllability of switched systems, an algebraic criterion is given in [11] as follows :

Lemma 1. If the matrix:

$$\begin{array}{l}[B_1, B_2, B_3, \ldots, B_m, \\ A_1B_1, A_2B_1, A_3B_1, \ldots, A_mB_m, \\ \ldots, \\ A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m]\end{array} \quad (9)$$

has full row rank, the switched linear system (8) is controllable, and vice versa.

This matrix is called the controllability matrix of the corresponding switched linear system (8).

III. STRUCTURAL CONTROLLABILITY

A. Union Graph

We have formulated the controllability problem of multi-agent system with a single leader under switching topology to the structural controllability of switched linear system. Rewrite the multi-agent system (6) here:

$$\left[\begin{array}{c}\dot{x} \\ \dot{z}\end{array}\right] = \left[\begin{array}{cc}A_{aq_i} & B_{aq_i} \\ 0 & 0\end{array}\right]\left[\begin{array}{c}x \\ z\end{array}\right] + \left[\begin{array}{c}0 \\ u_N\end{array}\right]. \quad (10)$$

$i \in \{1,\ldots,m\}$. The multi-agent system contains m interconnection topologies means that the corresponding switched system consists of m subsystems.

To derive our main result for the structural controllability problem, a new notation is proposed here: union graph. Before this definition, we want to introduce another necessary notation for the subsystems.

Notation 1. The matrix pair (A_{aq_i}, B_{aq_i}) can be represented by a digraph, defined as a flow structure, G_i, with vertex set $\mathcal{V'}_i = \{v'_1, v'_2, \ldots, v'_N\}$. There exists an edge from v'_k to v'_j in the flow structure if and only if $A_{aq_i}(j,k) \neq 0$, and an edge from v'_N to v'_k if and only if $B_{aq_i}(k) \neq 0$.

For each subsystem, we have got a graph G_i with vertex set $\mathcal{V'}_i$ and edge set $\mathcal{I'}_i$ to represent the underlaying communication topologies. As to the whole switched system, the corresponding graph, which is called union graph, is defined as follows:

Notation 2. The switched linear system (A_{aq_i}, B_{aq_i}), $i \in \{1,\ldots,m\}$ can be represented by a union digraph, defined as a flow structure $\mathcal{G}$. Mathematically, $\mathcal{G}$ is defined as

$$G_1 \bigcup G_2 \bigcup G_3 \bigcup \ldots \bigcup G_m = \{\mathcal{V'}_1 \bigcup \mathcal{V'}_2 \bigcup \mathcal{V'}_3 \ldots \bigcup \mathcal{V'}_m ; \mathcal{I'}_1 \bigcup \mathcal{I'}_2 \bigcup \mathcal{I'}_3 \ldots \bigcup \mathcal{I'}_m\}$$

The union graph $\mathcal{G}$ is built in the way that we overlay all the graph G_i together without changing the relative relations between the leader and the followers. Actually, it turns out that the union graph $\mathcal{G}$ is the representation of the system: $(A_1 + A_2 + A_3 + \ldots + A_m, B_1 + B_2 + B_3 + \ldots + B_m)$.

In the following discussion, the notation of union graph is employed to propose the necessary and sufficient condition for the structural controllability.

B. Main Results on Structural Controllability

Before considering the structural controllability, we first discuss the controllability of multi-agent system (10) when all the weights are fixed. Apparently, the leader z is always controllable, so all attentions should be paid to the controllability of the rest *N-1* follower agents.

Directly applying *Lemma 1* to the switched system (10), the following lemma can be easily proved.

Lemma 2. For switched linear system (10) with fixed weighting, the following statements are equivalent:

i) The subsystem (A_{aq_i}, B_{aq_i}) of system (10) is controllable.
ii) The controllability matrix

$$\left[\begin{array}{ccccc} B_{aq_i} & A_{aq_i}B_{aq_i} & A_{aq_i}^2 B_{aq_i} & \dots & A_{aq_i}^{N-2}B_{aq_i}\end{array}\right],$$

has full row rank.

It follows that the controllability of the system (10) coincides with the controllability of the following system:

$$\dot{x} \;=\; A_{aq_i}x + B_{aq_i}z \qquad i \in \{1,\dots,m\} \;. \tag{11}$$

Therefore, the structural controllability of system (10) coincides with the structural controllability of switched linear system (11). For simplicity, we use $(A_i, B_i)\;\; i \in \{1,\dots,m\}$ to represent the switched linear system (11) in the sequel. Consequently, the multi-agent structural controllability problem is formulated to the structural controllability problem of system (11).

For the structural controllability of multi-agent system, we need the following definition from [12]:

Definition 2. The pair (A, B) is said to be reducible or have form I if they can be written in the following form:

$$A = \left[\begin{array}{cc} A_{11} & 0 \\ A_{21} & A_{22}\end{array}\right], B = \left[\begin{array}{c} 0 \\ B_{22}\end{array}\right], \tag{12}$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $A_{21} \in \mathbb{R}^{(N-p)\times p}$, $A_{22} \in \mathbb{R}^{(N-p)\times(N-p)}$ and $B_{22} \in \mathbb{R}^{(N-p)\times m}$.

Whenever the matrix pair (A, B) is in form I, the system is structurally uncontrollable and meanwhile, the controllability matrix

$$Q = \left[B, AB, \dots, A^{n-1}B\right],$$

will now have at least one row which is identically zero for all parameter values.

Before the discussion of our main results, we recall a known result in the literature for the structural controllability of multi-agent system with fixed topology [4]:

Lemma 3. The multi-agent system with fixed topology under the communication topology $\mathcal{G}$ is structurally controllable if and only if graph $\mathcal{G}$ is connected.

This lemma proposed an interesting graphic condition for structural controllability in fixed topology situation and revealed that the controllability is totally determined by the communication topology. However, how about in the switching topology situation? According to *Lemma 1*, once we impose proper scalars for the parameters of the system matrix (A_i, B_i) to satisfy the full rank condition, the multi-agent system (10) is structurally controllable. However, this only proposed an algebraic condition. Do we still have very good graphic interpretation for the relationship between the structural controllability and switching interconnection topologies? The following theorem answers this question and gives a graphic necessary and sufficient condition for structural controllability under switching topologies.

Theorem 1. The multi-agent system (10) with the communication topologies $\mathcal{G}_i\;\; i \in \{1,\dots,m\}$ is structurally controllable if and only if the union graph $\mathcal{G}$ is connected.

Proof: Necessity: Assume that the multi-agent switched system is structurally controllable, we want to prove that the union graph $\mathcal{G}$ is connected, which is equivalent with that the system has no isolated agents in the union graph $\mathcal{G}$ [4].

Here, we suppose that the union graph $\mathcal{G}$ is disconnected. For simplicity, we will prove by contradiction for the case that there exits only one disconnected agent. The proof can be straightforwardly extended to more general cases with more than one disconnected agent. If there is one isolated agent in the union graph, there are two possible situations: the isolated agent is the leader or one of the followers. On one hand, if the isolated agent is the leader, it follows that $B_1 + B_2 + B_3 + \ldots + B_m$ is identically a null vector. So every B_i is a null vector. Easily we can conclude that the controllability matrix for the switched system is never of full row rank $N-1$, which means that the multi-agent system is not structurally controllable. On the other hand, if the isolated agent is one follower, we get that the matrix pair $(A_1+A_2+A_3+\ldots+A_m, B_1+B_2+B_3+\ldots+B_m)$ is reducible. By *Definition 2*, the controllability matrix

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&(A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),\\
&(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m),\\
&,\dots,\\
&(A_1 + A_2 + \ldots + A_m)^{N-2}(B_1 + B_2 + \ldots + B_m)],
\end{aligned}$$

always has at least one row that is identically zero. Expanding the matrix yields

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2\\
&+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m \ldots + A_mB_m\\
&,\dots,\\
&A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_m^{N-2}B_m].
\end{aligned}$$

The zero row is identically zero for every parameter. Consequently we can know that every component of the matrix, such as $B_i, A_iB_j \; and \; A_i^pA_j^qB_r$ has the same row always to be zero. As a result, the controllability matrix

$$\begin{aligned}
&[B_1, B_2, B_3, \dots, B_m,\\
&A_1B_1, A_2B_1, A_3B_1, \dots, A_mB_m,\\
&,\dots,\\
&A_1^{N-2}B_1, A_2A_1^{N-3}B_1, \dots, A_1A_m^{N-3}B_1, \dots, A_m^{N-2}B_m],
\end{aligned}$$

always has one zero row. Therefore, the multi-agent system (10) not controllable and not structurally controllable. Until now, we have got the necessity proved.

Sufficiency: If the union graph $\mathcal{G}$ is connected, we want to prove that the switched system (11) is structurally controllable, which coincides with that the multi-agent system (10) is structurally controllable.

Here the union graph $\mathcal{G}$: $G(A_1, B_1) \bigcup G(A_2, B_2) \bigcup G(A_3, B_3) \bigcup \ldots \bigcup G(A_m, B_m)$ is connected. According to *Lemma 3*, the corresponding system $(A_1 + A_2 + A_3 + \ldots + A_m, B_1 + B_2 + B_3 + \ldots + B_m)$ is structurally controllable. Then there exist some scalars for the parameters in the A_i and B_i matrices that make the controllability matrix

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&(A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),\\
&(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m),\\
&,\ldots,\\
&(A_1 + A_2 + \ldots + A_m)^{N-2}(B_1 + B_2 + \ldots + B_m)],
\end{aligned}$$

has full row rank $N-1$. Expanding the matrix, it follows that the matrix

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2\\
&+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m \ldots + A_mB_m\\
&,\ldots,\\
&A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_m^{N-2}B_m],
\end{aligned}$$

has full rank $N-1$. Next, we add some column vectors to the above matrix and get

$$\begin{aligned}
&[B_1 + B_2 + B_3 + \ldots + B_m, B_2, B_3, \ldots, B_m,\\
&A_1B_1 + A_2B_1 + A_3B_1 + \ldots + A_mB_1\\
&+ A_1B_2 + A_2B_2 + A_3B_2 + \ldots + A_mB_2\\
&+ A_1B_3 + A_2B_3 + A_3B_3 + \ldots + A_mB_3\\
&+ \ldots + A_1B_m + A_2B_m + A_3B_m + \ldots + A_mB_m,\\
&A_2B_1, A_3B_1, \ldots, A_mB_m\\
&,\ldots,\\
&A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_1A_m^{N-3}B_1 + \ldots\\
&+ A_m^{N-2}B_m, A_2A_1^{N-3}B_1, \ldots, A_1A_m^{N-3}B_1, \ldots, A_m^{N-2}B_m].
\end{aligned}$$

This matrix still have $N-1$ linear independent column vectors, so it has full row rank. Next, subtract $B_2, B_3, \ldots, B_m$ from $B_1 + B_2 + B_3 + \ldots + B_m$; subtract $A_2B_1, A_3B_1, \ldots, A_mB_m$ from $A_1B_1 + A_2B_1 + A_3B_1 + \ldots + A_mB_1 + \ldots + A_1B_m + \ldots + A_mB_m$ and subtract $A_2A_1^{N-3}B_1, \ldots, A_1A_m^{N-3}B_1, \ldots, A_m^{N-2}B_m$ from $A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_1A_m^{N-3}B_1 + \ldots + A_m^{N-2}B_m$. Because this column fundamental transformation will not change the matrix rank, the matrix still has full row rank. Now the matrix becomes

$$\begin{aligned}
&[B_1, B_2, B_3, \ldots, B_m,\\
&A_1B_1, A_2B_1, A_3B_1, \ldots, A_mB_m,\\
&\ldots,\\
&A_1^{N-2}B_1, A_2A_1^{N-3}B_1, \ldots, A_1A_m^{N-3}B_1, \ldots, A_m^{N-2}B_m],
\end{aligned}$$

which is the controllability matrix for switched linear systems in *Lemma 1* and has full row rank $N-1$. Therefore, the switched system is structurally controllable. And finally, we get that if the union graph $\mathcal{G}$ is connected, then the multi-agent switched system (10) is structurally controllable. ■

To make sure the controllability of a group of agents under switching topology is simply to keep their union graph's connectivity. Just having to keep the connectivity, this offers us high freedom to consider other factors, such as communication and control cost, so as to realize an optimal control effect to steer the agents to the desired positions.

IV. NUMERICAL EXAMPLES

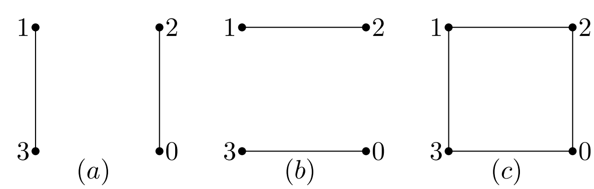

Fig. 1. Switched network with two subsystems

To illustrate the main result, we consider here a four-agent network with agent 0 as the leader and with switching topology described by the graphs in Fig. 1(a)-(b). Overlay the subgraphs together to get the union graph $\mathcal{G}$ of this example shown in Fig. 1(c). It turns out that the union graph of the switched system is connected. By our main result *Theorem 1*, we get that the multi-agent system is structurally controllable.

Next, we will check the rank condition of this switched system to see whether it is structurally controllable.

From Fig. 1, we can compute the system matrices of subgraphs from Laplacian matrix to be:

$$A_{aq_1} = \begin{bmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_2 & 0 \\ \lambda_4 & 0 & \lambda_3 \end{bmatrix}, \quad B_{aq_1} = \begin{bmatrix} 0 \\ \lambda_5 \\ 0 \end{bmatrix}.$$

$$A_{aq_2} = \begin{bmatrix} \lambda_6 & \lambda_9 & 0 \\ \lambda_9 & \lambda_7 & 0 \\ 0 & 0 & \lambda_8 \end{bmatrix}, \quad B_{aq_2} = \begin{bmatrix} 0 \\ 0 \\ \lambda_{10} \end{bmatrix}.$$

According to *Lemma 1*, we have the controllability matrix for this switched linear system here:

$$\begin{aligned}
&[B_1, B_2, A_1B_1, A_2B_1, A_1B_2, A_2B_2,\\
&A_1^2B_1, A_2A_1B_1, A_1A_2B_1, A_2^2B_1,\\
&A_1^2B_2, A_2A_1B_2, A_1A_2B_2, A_2^2B_2].
\end{aligned} \tag{13}$$

Apply (13) to this example, we can easily find three column vectors here:

$$\begin{bmatrix} 0 \\ \lambda_5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ \lambda_{10} \end{bmatrix}, \quad \begin{bmatrix} \lambda_4\lambda_{10} \\ 0 \\ \lambda_3\lambda_{10} \end{bmatrix}.$$

We impose all the parameters scalar 1. Then it turns out that the three column vectors are linear independent. As a result, the matrix has full row rank and by *Lemma* 1, the multi-agent system is controllable and therefore structurally controllable and finally can be steered to the desired positions as a whole group.

Next is another example, we still consider a four-agent network with agent 0 as the leader and with switching topology described by the graphs in Fig. 2(a)-(b). Overlay the subgraphs together to get the union graph $\mathcal{G}$ of this example shown in Fig. 2(c). It turns out that the union graph of the switched system is disconnected, because agent 2 is isolated. By our main result *Theorem 1*, we get that the multi-agent system is not structurally controllable.

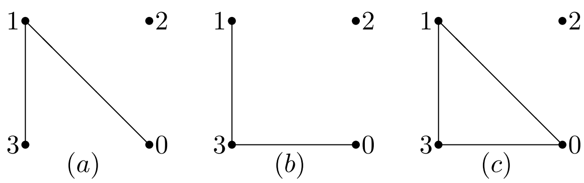

Fig. 2. Another switched network with two subsystems

Next, we will check the rank condition of this switched linear system to see whether it is structurally controllable or not.

From Fig. 2, we can compute the system matrices of subgraphs from Laplacian matrix to be:

$$A_{aq_1} = \begin{bmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_2 & 0 \\ \lambda_4 & 0 & \lambda_3 \end{bmatrix}, \quad B_{aq_1} = \begin{bmatrix} \lambda_5 \\ 0 \\ 0 \end{bmatrix}.$$

$$A_{aq_2} = \begin{bmatrix} \lambda_6 & 0 & \lambda_9 \\ 0 & \lambda_7 & 0 \\ \lambda_9 & 0 & \lambda_8 \end{bmatrix}, \quad B_{aq_2} = \begin{bmatrix} 0 \\ 0 \\ \lambda_{10} \end{bmatrix}.$$

Compute the controllability matrix for this example:

$$\begin{bmatrix} \lambda_5 & 0 & \lambda_1\lambda_5 & \dots & \lambda_6\lambda_9\lambda_{10} + \lambda_8\lambda_9\lambda_{10} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_{10} & \lambda_4\lambda_5 & \dots & \lambda_8^2\lambda_{10} + \lambda_9^2\lambda_{10} \end{bmatrix}.$$

This matrix has the second row always to be zero for all the parameter scalars, which makes the matrix can not have full row rank. Therefore, this switched system is not structurally controllable and not controllable. So the multi-agent system is not structurally controllable and not controllable.

From the above two examples, we illustrate our main results and present an intuitive interpretation that the multi-agent system with a single leader under switching topology is structurally controllable if and only if the union graph $\mathcal{G}$ is connected.

V. Conclusions and Future Work

In this paper, the structural controllability problem of the multi-agent systems interconnected via a switching weighted topology has been considered. Based on known results in the literature of switched systems and graph theory, a graphic necessary and sufficient condition for the structural controllability of multi-agent systems under switching communication topologies was derived. It was shown that the multi-agent system is structurally controllable if and only if the union graph $\mathcal{G}$ is connected. The graphic characterization shows a clear relationship between the controllability and interconnection topologies and gives us a foundation to design the optimal control effect for the switched multi-agent system.

Some interesting remarks can be made on this result. First, it gives us a clear understanding on what are the necessary information exchanges among agents to make the group of agents behavior in a desirable way. Second, it provides us a guideline to design communication protocols among dynamical agents. It is required that the resulted communication topology among agents should somehow remain connected as time goes on, which is quite intuitive and reasonable. Third, it is possible to reduce communication load by disable certain linkages or make them on and off as long as the union graph is connected. Several interesting research questions arise from this scenario. For example, what is the optimal switching sequence of topologies in the sense of minimum communication cost? How to co-design the switching topology path and control signals to achieve desirable configuration in an optimal way? We will investigate these questions in our future research.

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