Spines of manifolds and the freeness of fundamental groups of noncompact surfaces

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Abstract

We prove a theorem of Whitehead that says that a smooth noncompact n-manifold deformation retracts onto an $(n-1)$-dimensional spine. As a consequence, we deduce a theorem of Johansson that says that the fundamental group of a noncompact surface is free.

If M is a manifold, then a *spine* of M is a CW-complex X contained in M such that M deformation retracts to X. The goal of this note is to prove the following theorem of Whitehead.

Theorem 0.1 (Whitehead, [W, Lemma 2.1])**.** *Let M^n be a connected noncompact smooth n-manifold. Then M^n has an $(n-1)$-dimensional spine.*

Remark 0.2. This strengthens the familiar fact that $H^n(M^n;\mathbb{Z}) = 0$. □

Theorem 0.1 has the following corollary.

Corollary 0.3 (Johansson, [J])**.** *The fundamental group of a noncompact surface is free.*

Remark 0.4. Of course, this is elementary for the surfaces of finite type obtained by puncturing closed surfaces in finitely many places, but noncompact surfaces can be far wilder than this (e.g. a sphere minus a Cantor set). □

Proof of Corollary 0.3. If S is a noncompact connected surface, then Theorem 0.1 implies that S is homotopy equivalent to a one-dimensional CW-complex. In particular, the fundamental group of S is free. □

Remark 0.5. Here is a tempting but wrong way to prove Corollary 0.3 (but it does lead to the original proof of this Corollary; see [S, §4.2.2] for the additional ideas needed). Observe that if S is a connected noncompact surface, then we can write $S = \cup_{k=1}^{\infty} S_k$, where the S_k are an increasing sequence

$$S_1 \subset S_2 \subset S_3 \subset \cdots$$

of connected compact subsurfaces of S with nonempty boundary. The surfaces S_k have free fundamental groups, and it is not hard to also ensure that the induced maps $\pi_1(S_k) \to \pi_1(S_{k+1})$ are all injective. From this, we see that $\pi_1(S)$ is an increasing union of free groups. One might think that this implies that $\pi_1(S)$ is free, but alas this is wrong. For instance, we can write

$$\mathbb{Q} = \bigcup_{k=1}^{\infty} G_k,$$

where G_k is the subgroup consisting of all rational numbers that can be written as fractions whose denominators are divisible by the k^{th} power of the product of the first k primes. Though $\mathbb{Q}$ is not free, we have $G_k \cong \mathbb{Z}$, the free group on one generator. □

We now give what is essentially Whitehead's original proof of Theorem 0.1. For an alternate more analytic proof, see [NR, Theorem 2.2].

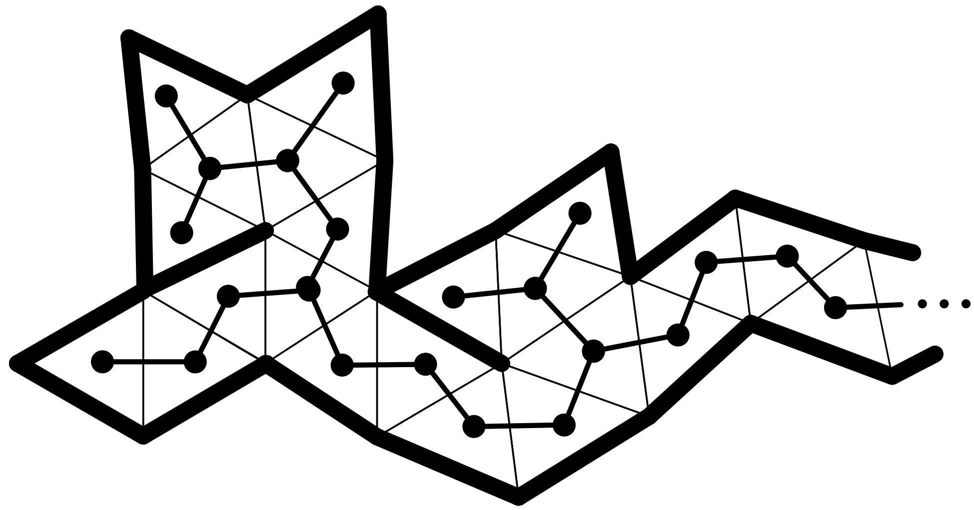

Figure 1: *A component of $Y = M^n \setminus \mathrm{Int}(N(X))$ containing a component of the forest F. The subspace $N(X)$ is in bold.*

Proof of Theorem 0.1. Fix a smooth triangulation of M^n. Let D be the dual graph to this triangulation, so D is the graph whose vertices are the n-simplices of M^n and whose edges connect the vertices associated to n-simplices that share an $(n-1)$-dimensional face. The graph D embeds in M^n in the obvious way. Since M^n is noncompact and connected, the graph D is infinite and connected.

The proof will now have two steps. For the first, a *special tree* is a tree T that can be written as the union of a ray R and a collection of disjoint finite trees each of which intersects R at a single vertex.

Step 1. *There exists a forest F in D containing every vertex of D such that each component of F is a special tree.*

Let $\mathcal{F}$ be the set of forests in D each of whose components is an infinite ray. Since D is connected and infinite, it contains a ray, so $\mathcal{F}$ is nonempty. Partially order $\mathcal{F}$ by saying that $F_1, F_2 \in \mathcal{F}$ satisfy $F_1 \preceq F_2$ when each component of F_1 is a component of F_2. Applying Zorn's lemma, we can choose a maximal element F' of $\mathcal{F}$. Let $\{Y_n\}_{n \in I}$ be the components of the complement of F' in D. All the Y_n must be finite graphs; indeed, if one of them was infinite, then it would contain an infinite ray that could be added to F', contradicting the maximality of F'. For $n \in I$, let T_n be a maximal tree in Y_n. We can find a component C_n of F' such that some vertex of T_n is adjacent to C_n along an edge e_n. Let F be the result of enlarging the components of F' by adding each T_n and e_n to C_n. The subgraph F of D clearly contains every vertex of D. Moreover, for each component C of F' and each vertex v of C, at most finitely many finite trees are attached to v when we form F. It follows that each component of F is a special tree.

Step 2. *Let X be the $(n-1)$-dimensional subcomplex of M^n obtained by starting with the $(n-2)$-skeleton of M^n and then adding each $(n-1)$-simplex that is not crossed by an edge of F. Then X is a spine of M^n.*

Let $N(X)$ be a small closed regular neighborhood of X. The subspace $N(X)$ deformation retracts to X, so it is enough to prove that M^n deformation retracts to $N(X)$. Define $Y = M^n \setminus \mathrm{Int}(N(X))$. The components of Y consist of closed regular neighborhoods of the components of F (see Figure 1). Since F is a special tree, these closed regular neighborhoods are homeomorphic to the result $\mathbb{D}^n \setminus \{p\}$ of removing a single point p from the boundary of the closed unit disc $\mathbb{D}^n$ in $\mathbb{R}^n$. Since $\mathbb{D}^n \setminus \{p\}$ deformation retracts to $S^n \setminus \{p\}$, we can deformation retract each component of Y to its boundary $Y \cap N(X)$. This implies that we can deformation retract M^n to $N(X)$, as desired. □

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References

[J] I. Johansson, Topologische Untersuchungen über unverzweigte Überlagerungsflächen, Skriften norske Vidensk.-Akad. Oslo Math.-natur. Kl. 1 (1931), 1–69.

[NR] T. Napier and M. Ramachandran, Elementary construction of exhausting subsolutions of elliptic operators, Enseign. Math. (2) 50 (2004), no. 3-4, 367–390.

[MO] Andy Putman (https://mathoverflow.net/users/317/andy-putman), Fundamental groups of noncompact surfaces, URL (version: 2015-10-27): https://mathoverflow.net/q/18454

[S] J. Stillwell, *Classical topology and combinatorial group theory*, second edition, Graduate Texts in Mathematics, 72, Springer-Verlag, New York, 1993.

[W] J. H. C. Whitehead, The immersion of an open 3-manifold in euclidean 3-space, Proc. London Math. Soc. (3) 11 (1961), 81–90.

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