# Conditional Probability and Tree Diagrams

Sometimes our computation of the probability of an event is changed by the knowledge that a related event has occurred (or is guaranteed to occur) or by some additional conditions imposed on the experiment. We see some examples below:

# Conditional Probability and Tree Diagrams

**Example** In a previous example, we estimated that the probability that LeBron James will make his next attempted field goal in a major league game is 0.567. We used the proportion of field goals made out of field goals attempted (FG%) in the 2013/2014 season to estimate this probability. If we look at the split statistics below, we see that the FG% changes when calculated under specified condition. For example the FG% for games played after 3 days or more rest is 0.614 which is much higher than the overall FG%.

# Conditional Probability and Tree Diagrams

Season: 2013-2014

**2013-2014 PER GAME SPLITS**

| SPLIT | GP | MIN | FGM-FGA | FG% |
|---|---|---|---|---|
| **Total** | **77** | **37.7** | **10.0-17.6** | **.567** |
| Home | 39 | 37.1 | 10.0-16.9 | .592 |
| Road | 38 | 38.3 | 9.9-18.2 | .543 |
| vs. Division | 15 | 37.1 | 10.1-17.9 | .561 |
| vs. Conference | 48 | 37.5 | 9.8-17.3 | .564 |
| 0 Days Rest | 15 | 38.0 | 10.4-18.4 | .565 |
| 1 Days Rest | 43 | 37.3 | 9.8-17.2 | .570 |
| 2 Days Rest | 14 | 38.1 | 9.9-18.1 | .545 |
| 3+ Days Rest | 5 | 39.0 | 10.2-16.6 | .614 |

If we know that LeBron's next field goal attempt will be made in a game after 3 days or more rest, it would be natural to use the statistic

$$0.614 = \frac{\text{Field goals made after 3+ days rest}}{\text{Field goals attempted after 3+ days rest}}$$

to estimate the probability that he will be successful.

# Conditional Probability and Tree Diagrams

Here we are estimating the probability that LeBron will make the field goal **given** the extra information that the attempt will be made in a game after 3 days + rest. This is referred to as a **conditional probability**, because we have some **prior information** about conditions under which the experiment will be performed.

Additional information may change the **sample space** and the **successful event subset.**

# Conditional Probability and Tree Diagrams

**Example** Let us consider the following experiment: A card is drawn at random from a standard deck of cards. Recall that there are 13 hearts, 13 diamonds, 13 spades and 13 clubs in a standard deck of cards.

- Let H be the event that a heart is drawn,
- let R be the event that a red card is drawn and
- let F be the event that a face card is drawn, where the face cards are the kings, queens and jacks.

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(a) If I draw a card at random from the deck of 52, what is $\mathbf{P}(H)$? $\frac{13}{52} = 25\%$.

# Conditional Probability and Tree Diagrams

(b) If I draw a card at random, and without showing you the card, I tell you that the card is red, then what are the chances that it is a heart?

# Conditional Probability and Tree Diagrams

(b) If I draw a card at random, and without showing you the card, I tell you that the card is red, then what are the chances that it is a heart? $\dfrac{13}{26} = 50\%$. If I had told you the card was black, then the sample space is all black cards, and there are 26 of those, but the successful outcomes consist of all black hearts, of which there are 0, so the probability then is $\frac{0}{26}$.

# Conditional Probability and Tree Diagrams

(b) If I draw a card at random, and without showing you the card, I tell you that the card is red, then what are the chances that it is a heart? $\dfrac{13}{26} = 50\%$. If I had told you the card was black, then the sample space is all black cards, and there are 26 of those, but the successful outcomes consist of all black hearts, of which there are 0, so the probability then is $\frac{0}{26}$.

Here we are calculating the probability that the card is a heart given that the card is red. This is denoted by $\mathbf{P}(H|R)$, where the vertical line is read as "given". Notice how the probability changes with the prior information. Note also that we can think of the prior information as restricting the sample space for the experiment in this case. We can think of all red cards or the set $R$ as a **reduced sample space**.

# Conditional Probability and Tree Diagrams

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(d) If I draw a card at random, and without showing you the card, I tell you that the card is red, then what are the chances that it is a face card (i.e. what is $\mathbf{P}\left(F \middle| R\right)$)?

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There are 6 red face cards and 26 red cards so
$\mathbf{P}\left(F \middle| R\right) = \frac{6}{26} = \frac{3}{13}$. *Notice that in this case, the probability does not change even though both the sample space and the event space do change.*

# Conditional Probability and Tree Diagrams

The calculations above were reasonably easy and intuitive. The probability that the card is a heart given (the prior information) that the card is red is denoted by

$$\mathbf{P}\left(H \middle| R\right)$$

Note that

$$\mathbf{P}\left(H \middle| R\right) = \frac{n(H \cap R)}{n(R)} = \frac{\mathbf{P}(H \cap R)}{\mathbf{P}(R)}.$$

This probability is called the **conditional probability of H given R.**

# Conditional Probability and Tree Diagrams

**Definition** If $A$ and $B$ are events in a sample space $S$, with $\mathbf{P}(B) \neq 0$, the **conditional probability** that an event A will occur, given that the event $B$ has occurred is given by

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**Note** From our example above, we saw that sometimes $\mathbf{P}\left(A \middle| B\right) = \mathbf{P}(A)$ and sometimes $\mathbf{P}\left(A \middle| B\right) \neq \mathbf{P}(A)$. When $\mathbf{P}(A|B) = \mathbf{P}(A)$, we say that the events $A$ and $B$ are *independent*. We will discuss this in more detail in the next section.

# Calculating Conditional Probabilities

**Example** Consider the data, in the following table, recorded over a month with 30 days:

Weather

| Mood | | S | NS |
|---|---|---|---|
| | G | 9 | 6 |
| | NG | 1 | 14 |

On each day I recorded, whether it was sunny, (S), or not, (NS), and whether my mood was good, G, or not (NG).

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On each day I recorded, whether it was sunny, (S), or not, (NS), and whether my mood was good, G, or not (NG).

(a)If I pick a day at random from the 30 days on record, what is the probability that I was in a good mood on that day, $\mathbf{P}(G)$? The sample space is the 30 days under discussion. I was in a good mood on $9 + 6 = 15$ of them so $\mathbf{P}(G) = \dfrac{15}{30} = 50\%$.

# Calculating Conditional Probabilities

(b)What is the probability that the day chosen was a Sunny day, $\mathbf{P}(S)$?

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The sample space is still the 30 days under discussion. It was sunny on $9 + 1 = 10$ of them so $\mathbf{P}(S) = \dfrac{10}{30} \approx 33\%$.

# Calculating Conditional Probabilities

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The sample space is still the 30 days under discussion. It was sunny on $9+1=10$ of them so $\mathbf{P}(S)=\dfrac{10}{30}\approx 33\%$.

(c) What is $\mathbf{P}\left(G \middle| S\right)$?

# Calculating Conditional Probabilities

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The sample space is still the 30 days under discussion. It was sunny on $9 + 1 = 10$ of them so $\mathbf{P}(S) = \dfrac{10}{30} \approx 33\%$.

(c) What is $\mathbf{P}\left(G \middle| S\right)$? $\mathbf{P}\left(G \middle| S\right) = \dfrac{\mathbf{P}(G \cap S)}{\mathbf{P}(S)}$. Hence we need to calculate $\mathbf{P}(G \cap S)$. Here the sample space is still the 30 days: $G \cap S$ consists of sunny days in which I am in a good mood and there were 9 of them. Hence $\mathbf{P}(G \cap S) = \dfrac{9}{30}$. Therefore $\mathbf{P}\left(G \middle| S\right) = \dfrac{\frac{9}{30}}{\frac{10}{30}} = \dfrac{9}{10} = 90\%$.

# Calculating Conditional Probabilities

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**Note** $\quad \mathbf{P}\left(G \middle| S\right) \neq \mathbf{P}\left(S \middle| G\right)$.

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**Note** $\quad \mathbf{P}\left(G \middle| S\right) \neq \mathbf{P}\left(S \middle| G\right)$.

**Note** $\quad$ If $\mathbf{P}\left(E \middle| F\right) \neq \mathbf{P}(E)$, then $\mathbf{P}\left(F \middle| E\right)$ will not equal $\mathbf{P}(F)$. If $\mathbf{P}\left(E \middle| F\right) = \mathbf{P}(E)$, then $\mathbf{P}\left(F \middle| E\right) = \mathbf{P}(F)$.

# Calculating Conditional Probabilities

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**Note** $\quad \mathbf{P}\left(G \middle| S\right) \neq \mathbf{P}\left(S \middle| G\right)$.

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**Note** $\quad \mathbf{P}\left(A \middle| B\right) \neq \mathbf{P}(A)$, it does not necessarily imply a cause and effect relationship. In the example above, the weather might have an effect on my mood, however it is unlikely that my mood would have any effect on the weather.

# Calculating Conditional Probabilities

**Example** Of the students at a certain college, 50% regularly attend the football games, 30% are first-year students and 40% are upper-class students who do not regularly attend football games.

# Calculating Conditional Probabilities

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# Calculating Conditional Probabilities

(a) What is the probability that a student selected at random is both is a first-year student and regularly attends football games? Let $R$ be the set of students who regularly attend football games; let $U$ be the set of upper-class students $F$ be the set of first-year students.
Given: $\mathbf{P}(R) = 50\%$; $\mathbf{P}(F) = 30\%$; $\mathbf{P}(U \cap R') = 40\%$.

# Calculating Conditional Probabilities

(a) What is the probability that a student selected at random is both is a first-year student and regularly attends football games? Let $R$ be the set of students who regularly attend football games; let $U$ be the set of upper-class students $F$ be the set of first-year students.
Given: $\mathbf{P}(R) = 50\%$; $\mathbf{P}(F) = 30\%$; $\mathbf{P}(U \cap R') = 40\%$.
Since $F \cup U$ is everybody and $F \cap U$ is empty, $\mathbf{P}(U) = 70\%$.

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Since $F \cup U$ is everybody and $F \cap U$ is empty, $\mathbf{P}(U) = 70\%$.
Since $\mathbf{P}(R' \cap U) = 40\%$ and $\mathbf{P}(U) = 70\%$, $\mathbf{P}(R \cap U) = 30\%$.

# Calculating Conditional Probabilities

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Since $F \cup U$ is everybody and $F \cap U$ is empty, $\mathbf{P}(U) = 70\%$.
Since $\mathbf{P}(R' \cap U) = 40\%$ and $\mathbf{P}(U) = 70\%$, $\mathbf{P}(R \cap U) = 30\%$.
Since $\mathbf{P}(R) = \mathbf{P}(R \cap F) + \mathbf{P}(R \cap U)$, $\mathbf{P}(R \cap F) = 20\%$.

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Since $\mathbf{P}(R) = \mathbf{P}(R \cap F) + \mathbf{P}(R \cap U)$, $\mathbf{P}(R \cap F) = 20\%$.
Note in this example that $U$ does not stand for the UNIVERSAL SET. If you want a relevant universal set it is $F \cup U$!

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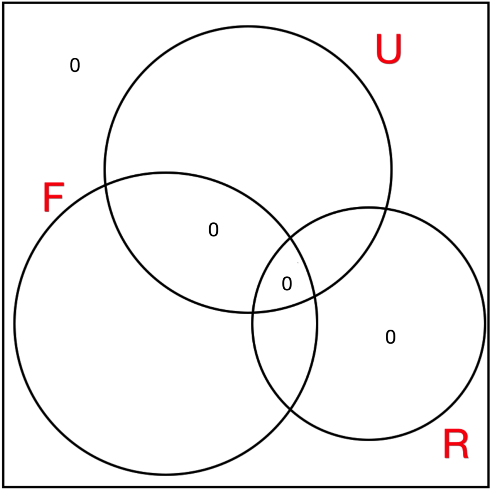
Since $\mathbf{P}(R) = \mathbf{P}(R \cap F) + \mathbf{P}(R \cap U)$, $\mathbf{P}(R \cap F) = 20\%$.
Note in this example that $U$ does not stand for the UNIVERSAL SET. If you want a relevant universal set it is $F \cup U$!
If you find these manipulations difficult to produce on your own (as opposed to verifying that the calculations are correct), you might want to try the Venn Diagram approach.

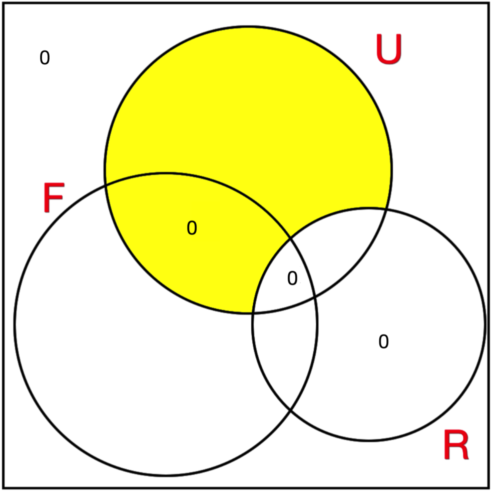
# Calculating Conditional Probabilities




We know $U \cup F$ is everybody, $U \cap F = \emptyset$ and $R \subset U \cup F$. Therefore we can fill in four 0's as indicated.

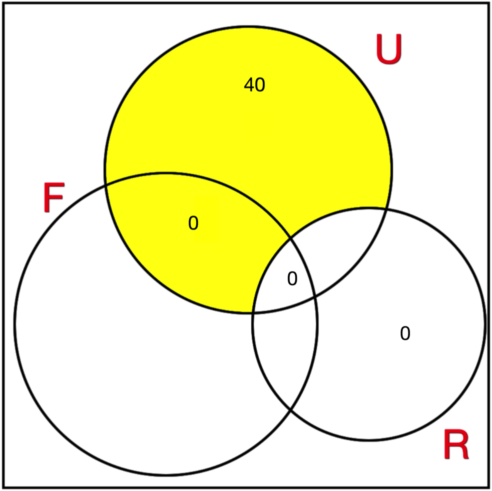
# Calculating Conditional Probabilities

Next identify what you are given.



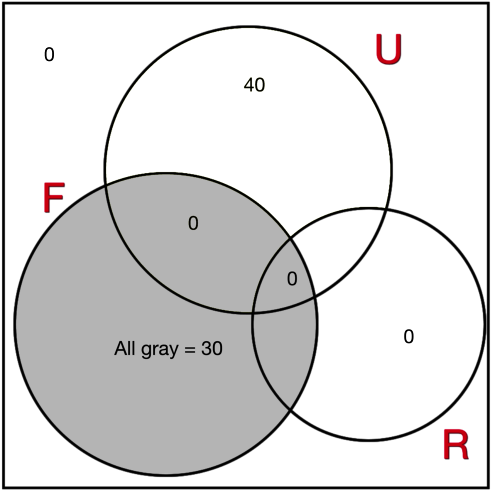

The yellow region is $U \cap R'$ and we know $\mathbf{P}(U \cap R') = 40\%$.

# Calculating Conditional Probabilities



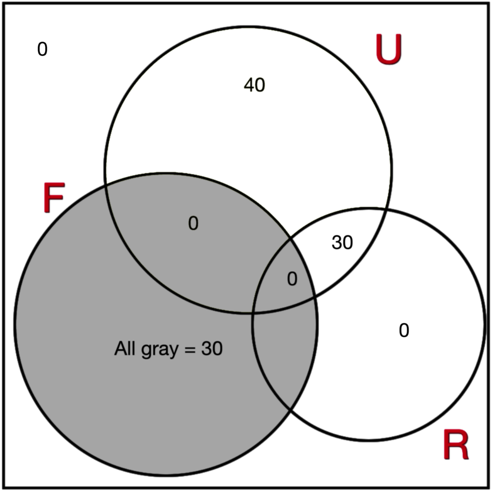

We also know the values for the disk $R$ and the disk $F$.

# Calculating Conditional Probabilities



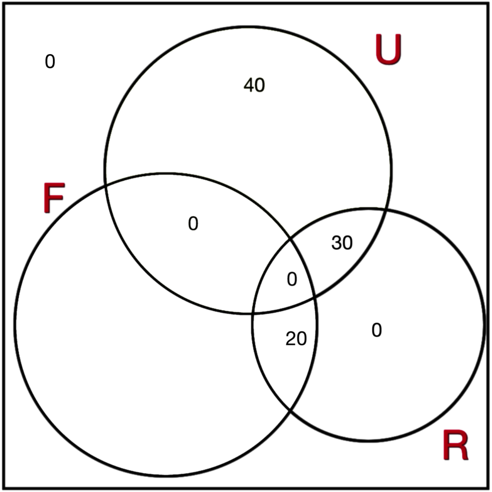

Since $F \cup U$ is everybody, $\mathbf{P}(F \cup U) = 100$. From the Inclusion-Exclusion Principle we see we can work out the unknown bit of $U$.

# Calculating Conditional Probabilities



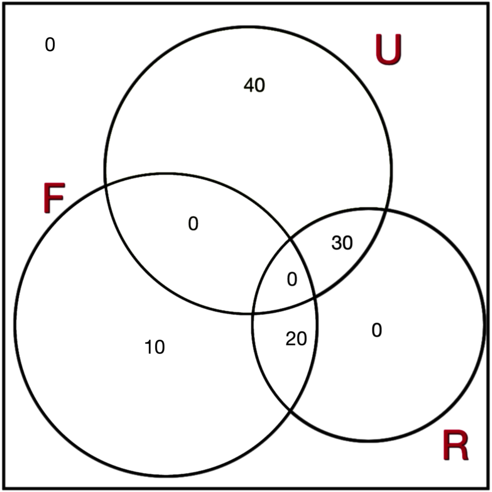

Since n(F) + n(U) = 100

# Calculating Conditional Probabilities




Since $\mathbf{P}(R) = 50$ we can fill in the last bit of $R$.

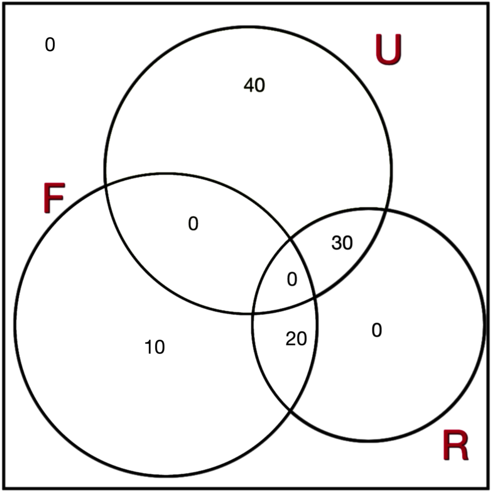
# Calculating Conditional Probabilities




Since $\mathbf{P}(F) = 30$ we can fill in the last bit of $F$.

# Calculating Conditional Probabilities

(b) What is the conditional probability that the person chosen attends football games, given that he/she is a first year student?



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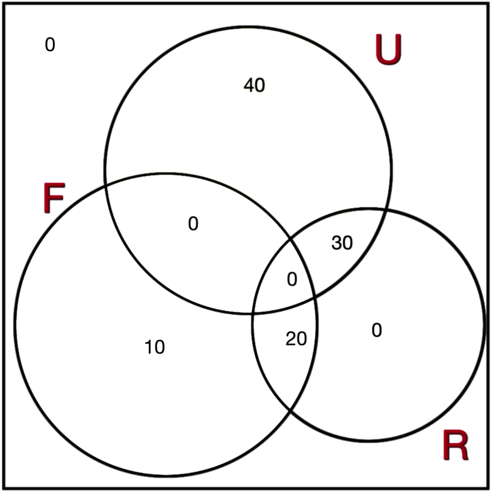
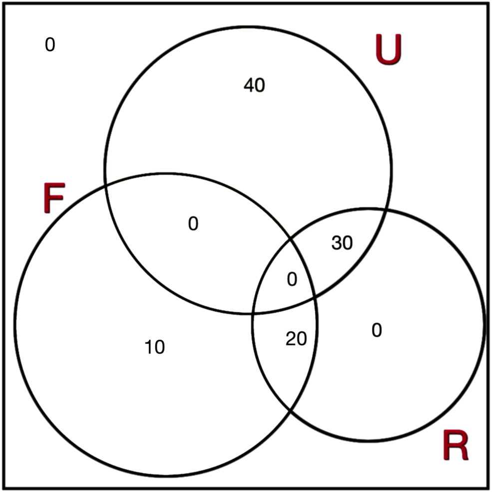



$$\mathbf{P}\left(R\middle|F\right) = \frac{\mathbf{P}(R \cap F)}{\mathbf{P}(F)} = \frac{0.2}{0.3} \approx 67\%.$$

# Calculating Conditional Probabilities

(c) What is the conditional probability that the person is a first year student given that he/she regularly attends football games?



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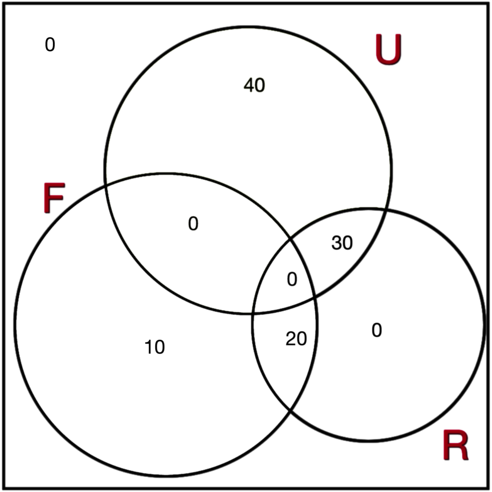


$$\mathbf{P}\left(F \middle| R\right) = \frac{\mathbf{P}(R \cap F)}{\mathbf{P}(R)} = \frac{0.2}{0.5} = 40\%.$$

# Calculating Conditional Probabilities

**Example** If $S$ is a sample space, and $E$ and $F$ are events with

$$\mathbf{P}(E) = .5, \mathbf{P}(F) = .4 \quad \text{and } \mathbf{P}(E \cap F) = .3,$$

(a) What is $\mathbf{P}\left(E \middle| F\right)$?

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$$\mathbf{P}\left(E \middle| F\right) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{0.3}{0.4} = 75\%.$$

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(b) What is $\mathbf{P}\left(F \middle| E\right)$?

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$$\mathbf{P}\left(E \middle| F\right) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{0.3}{0.4} = 75\%.$$

(b) What is $\mathbf{P}\left(F \middle| E\right)$?

$$\mathbf{P}\left(F \middle| E\right) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(E)} = \frac{0.3}{0.5} = 60\%.$$

# A formula for $\mathbf{P}(E \cap F)$.

We can rearrange the equation

$$\mathbf{P}\left(E \middle| F\right) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)}$$

to get

$$\mathbf{P}(F)\mathbf{P}\left(E \middle| F\right) = \mathbf{P}(E \cap F).$$

Also we have

$$\mathbf{P}\left(F \middle| E\right) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(E)}.$$

or

$$\boxed{\mathbf{P}(E)\mathbf{P}\left(F \middle| E\right) = \mathbf{P}(E \cap F).}$$

This formula gives us a **multiplicative formula** for $\mathbf{P}(E \cap F)$. In addition to giving a formula for calculating the probability of two events occurring simultaneously, it is very useful in calculating probabilities for **sequential events**.

$$\mathbf{P}(E \cap F) = \mathbf{P}(E)\mathbf{P}\left(F \middle| E\right).$$

If we wish to calculate the probability of $E$ and then $F$, it is equal to $\mathbf{P}(E) \cdot \mathbf{P}\left(F \middle| E\right)$, where $\mathbf{P}\left(F \middle| E\right)$ gives the probability that $F$ will happen <u>given</u> that $E$ has already occurred.

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**Example** If $\mathbf{P}\left(E \middle| F\right) = .2$ and $\mathbf{P}(F) = .3$, find $\mathbf{P}(E \cap F)$.

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$\mathbf{P}(E) = \mathbf{P}\left(E \middle| F\right) \cdot \mathbf{P}(F) = 0.2 \cdot 0.3 = 0.006 = 6\%.$

$$\mathbf{P}(E \cap F) = \mathbf{P}(E)\mathbf{P}\left(F \middle| E\right).$$

**Example**The probability that the maximum temperature tomorrow will be 30°F or below is 0.5. When the temperature is 30°F or below, the probability that my car will not start is 0.7. What is the probability that the maximum temperature tomorrow will be 30°F or below and my car will not start.

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Let $N$ be the event "my car will not start" and let $T$ be the event the "maximum temperature tomorrow will be 30°F or below". Then
$\mathbf{P}(N \cap T) = \mathbf{P}\left(N \middle| T\right) \cdot \mathbf{P}(T) = 0.5 \cdot 0.7 = 0.35 = 35\%.$

# Tree Diagrams.

Sometimes, if there are sequential steps in an experiment, or repeated trials of the same experiment, or if there are a number of stages of classification for objects sampled, it is very useful to represent the probability/information on a tree diagram.

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**Example** Given an Urn containing 6 red marbles and 4 blue marbles, I draw a marble at random from the urn and then, without replacing the first marble, I draw a second marble from the urn. What is the probability that both marbles are red?

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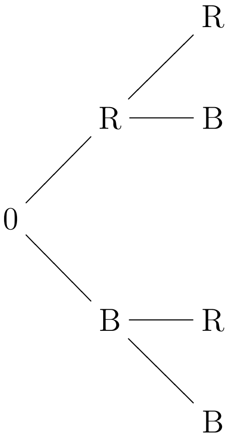
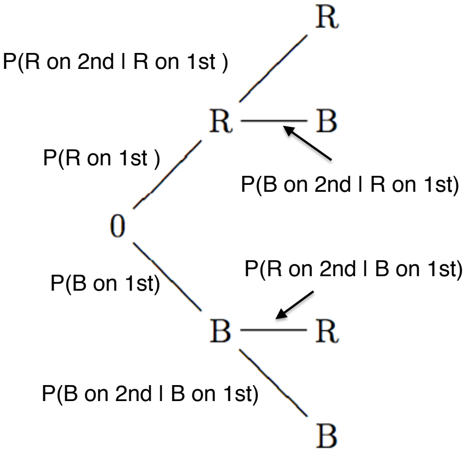
**Example** Given an Urn containing 6 red marbles and 4 blue marbles, I draw a marble at random from the urn and then, without replacing the first marble, I draw a second marble from the urn. What is the probability that both marbles are red?

We can draw a tree diagram to represent the possible outcomes of the above experiment and label it with the appropriate conditional probabilities as shown (where 1st denotes the first draw and 2nd denotes the second draw):

# Tree Diagrams.



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(a) Fill in the appropriate probabilities on the tree diagram on the left above (note: the "chemistry" in the urn changes when you do not replace the first ball drawn).

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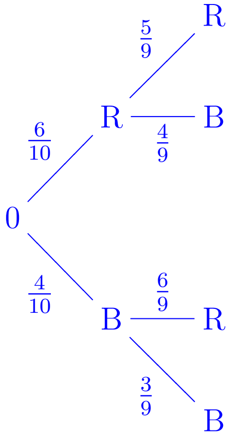


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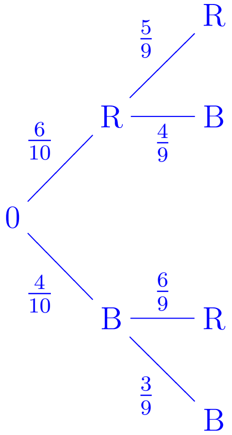


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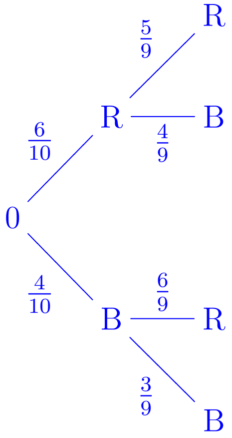


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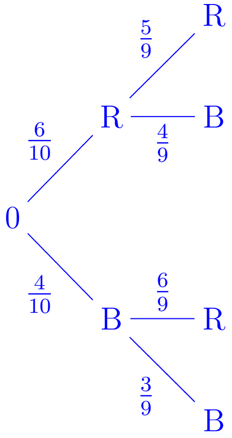


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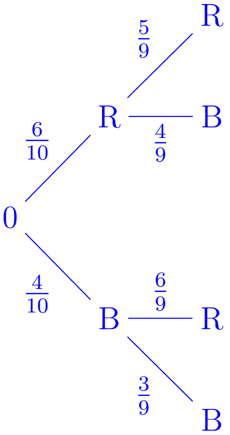


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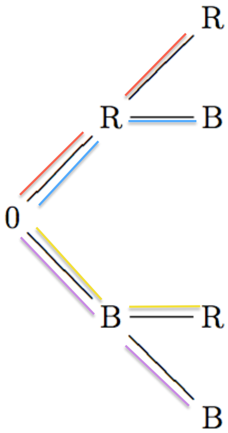
Note that each path on the tree diagram represents one outcome in the sample space.




| Outcome | Probability |
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| RR(red path) | |
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| BR(yellow path) | |
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| Outcome | Probability |
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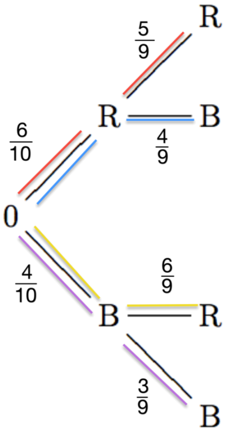
| RB(blue path) | $\frac{6}{10} \cdot \frac{4}{9} = \frac{24}{90}$ |
| BR(yellow path) | $\frac{4}{10} \cdot \frac{6}{9} = \frac{24}{90}$ |
| BB(purple path) | $\frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90}$ |



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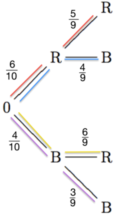
To find the probability of an event, we identify the outcomes (paths) in that event and add their probabilities.

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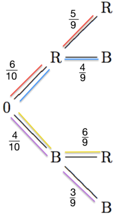


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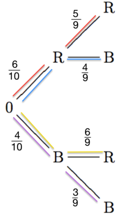


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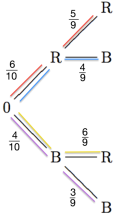


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$$\frac{6}{10} \cdot \frac{4}{9} + \frac{4}{10} \cdot \frac{3}{9} = \frac{24}{90} + \frac{12}{90} = \frac{36}{90} = 40\%.$$

# Tree Diagrams.

**Summary of “rules” for drawing tree diagrams**

1. The branches emanating from each point (that is branches on the immediate right) must represent all possible outcomes in the next stage of classification or in the next experiment.
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**Example 1** A box of 20 apples is ready for shipment, four of the apples are defective. An inspector will select at most four apples from the box. He selects each apple randomly, one at a time, inspects it and if it is not defective, sets it aside. The first time he selects a defective apple, he stops the process and the box will not be shipped. If the first four apples selected are good, he replaces the 4 apples and ships the box.

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**Example 1** A box of 20 apples is ready for shipment, four of the apples are defective. An inspector will select at most four apples from the box. He selects each apple randomly, one at a time, inspects it and if it is not defective, sets it aside. The first time he selects a defective apple, he stops the process and the box will not be shipped. If the first four apples selected are good, he replaces the 4 apples and ships the box.

(a) Draw a tree diagram representing the outcomes and assign probabilities appropriately.

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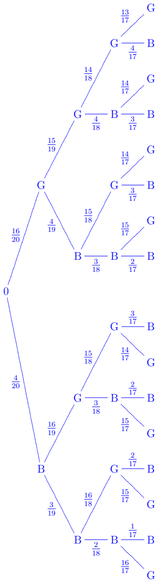

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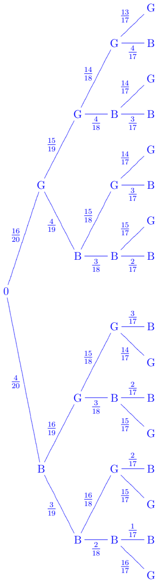


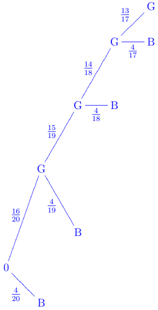

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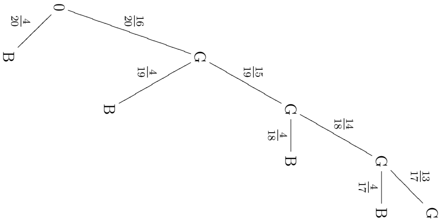


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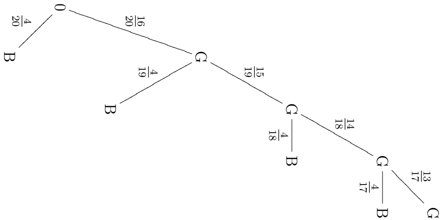


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$$\frac{16}{20} \cdot \frac{15}{19} \cdot \frac{14}{18} \cdot \frac{13}{17} = \frac{\mathbf{P}(16,4)}{\mathbf{P}(20,4)} \approx 0.3756449948$$

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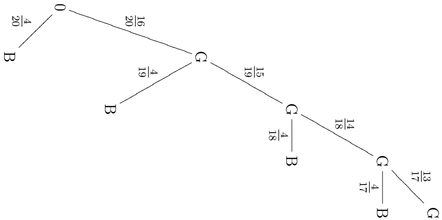


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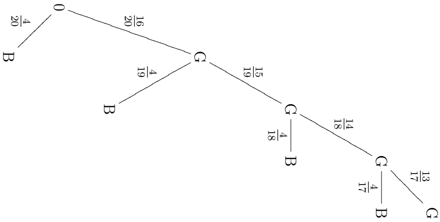

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$$\frac{4}{20} + \frac{16}{20} \cdot \frac{4}{19} \approx 0.3684210526.$$

# Tree Diagrams.

**Example** In a certain library, twenty percent of the fiction books are worn and need replacement. Ten percent of the non-fiction books are worn and need replacement. Forty percent of the library's books are fiction and sixty percent are non-fiction. What is the probability that a book chosen at random needs repair? Draw a tree diagram representing the data.

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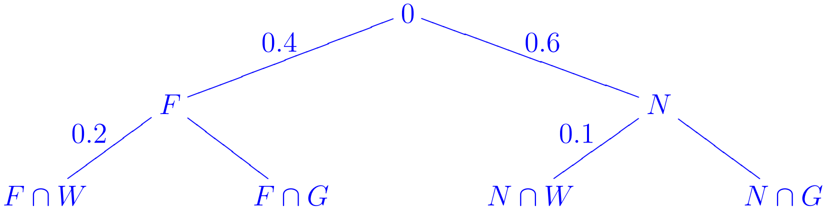
Let $F$ be the subset of fiction books and let $N$ be the subset of non-fiction books. Let $W$ be the subset of worn books and let $G$ be the subset of non-worn books.

# Tree Diagrams.

## What we are given.



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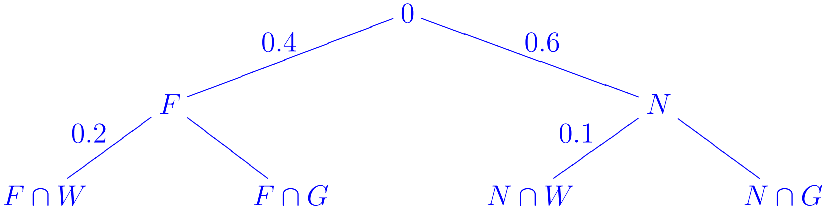


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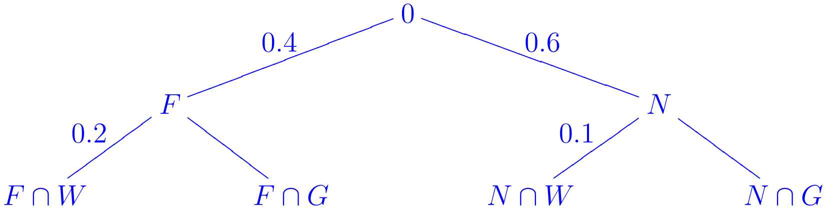


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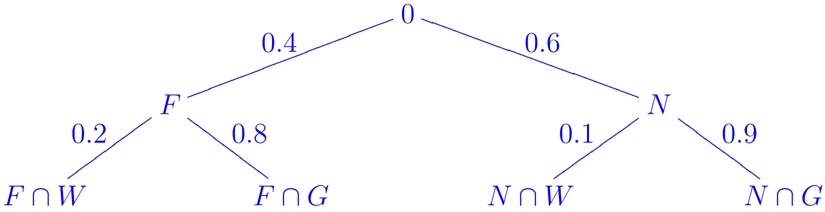


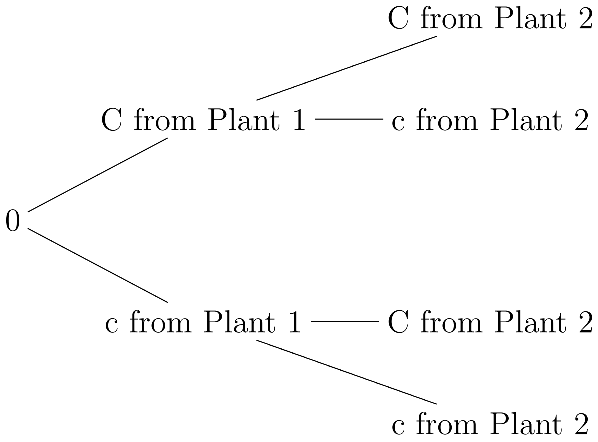
# Tree Diagrams.

**Example (Genetics)** Traits passed from generation to generation are carried by genes. For a certain type of pea plant, the color of the flower produced by the plant (either red or white) is determined by a pair of genes. Each gene is of one of the types $C$(dominant gene) or $c$(recessive gene). Plants for which both genes are of type $c$ (said to have genotype $cc$ ) produce white flowers. All other plants - that is, plants of genotypes $CC$ and $Cc$ - produce red flowers. When two plants are crossed, the offspring receives one gene from each parent. If the parent is of type Cc, both genes are equally likely to be passed on.

# Tree Diagrams.

I Suppose you cross two pea plants of genotype $Cc$,

I(a) fill in the probabilities on the tree diagram below.



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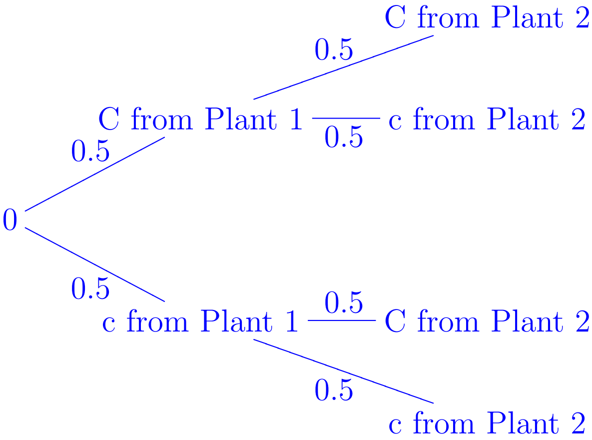

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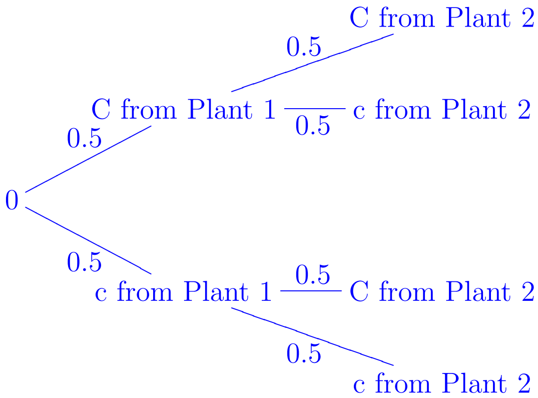


You only get white flowers from path “c from Plant 1” to “c from Plant 2” so the answer is $0.5 \cdot 0.5 = 0.25 = 25\%$.

# Tree Diagrams.

I(c) what is the probability that the offspring produces red flowers?

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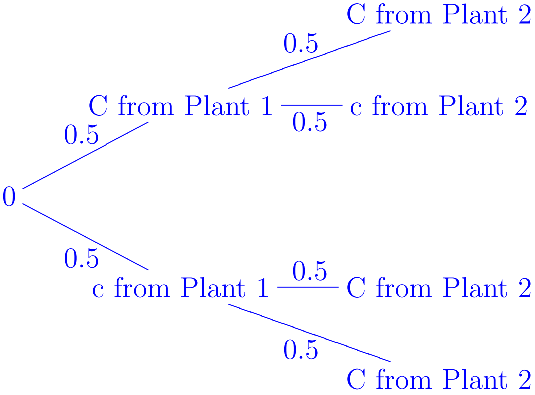



The answer is either $1 - \mathbf{P}(cc) = 75\%$ or $\mathbf{P}(cC) + \mathbf{P}(Cc) + \mathbf{P}(CC) = 75\%$.

# Tree Diagrams.

II Suppose you have a batch of red flowering pea plants, of which 60% have genotype $Cc$ and 40% have genotype $CC$. You select one of these plants at random and cross it with a white flowering pea plant.

II(a) what is the probability that the offspring will produce red flowers (use the tree diagram below to determine the probability).

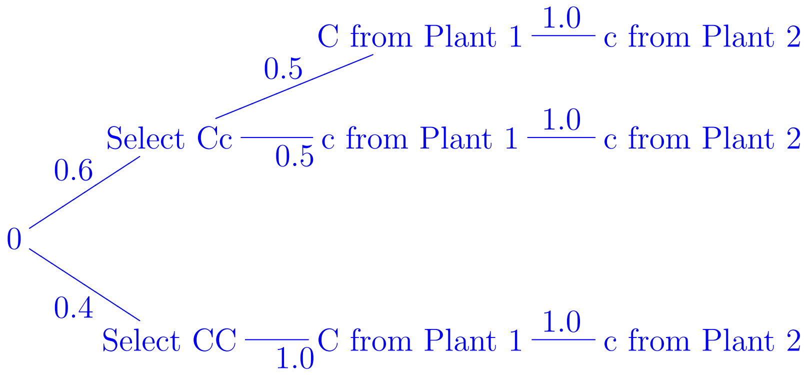
C from Plant 1 — c from Plant 2

Select Cc — c from Plant 1 — c from Plant 2

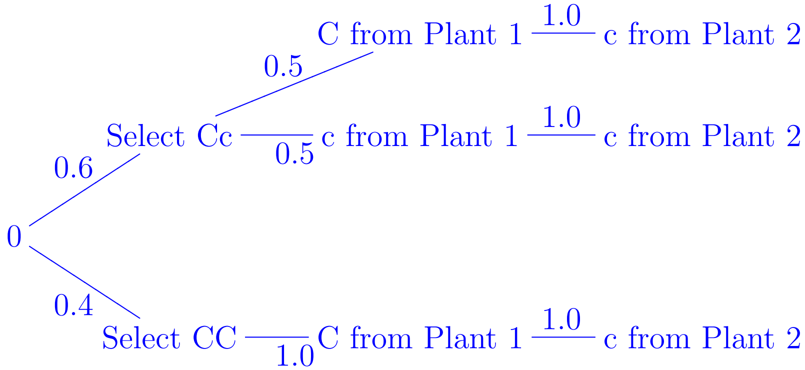
0

Select CC — C from Plant 1 — c from Plant 2

# Tree Diagrams.



# Tree Diagrams.




You get red flowers unless you get cc: $0.6 \cdot 0.5 \cdot 1.0 = 0.3$, so the requested probability is $1 - 0.3 = 0.7$. You can also get it as

$$0.6 \cdot 0.5 \cdot 1.0 + 0.4 \cdot 1.0 \cdot 1.0 = 0.3 + 0.4 = 0.7$$