Bayes' Theorem

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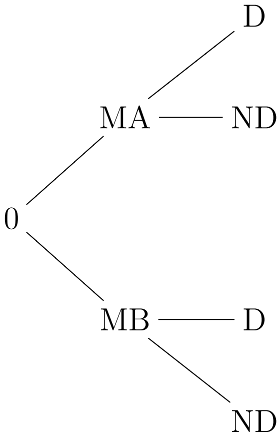
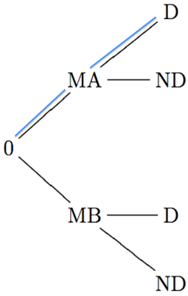
Example Suppose that a factory has two machines, Machine A and Machine B, both producing jPhone touch screens. Forty percent of their touch screens come from Machine A and 60% of their touch screens come from Machine B. Ten percent of the touch screens produced by Machine A are defective and five percent of the touch screens from Machine B are defective. If I randomly choose a touch screen from those produced by both machines and find that it is defective, what is the probability that it came from machine A?

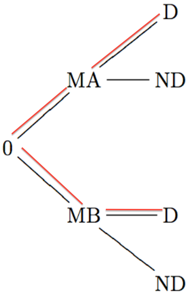
Bayes' Theorem

We can draw a tree diagram representing the information we are given. If we choose a touch screen at random from those produced in the factory, we let MA be the event that it came from Machine A and let MB be the event that it came from Machine B. We let D denote the event that the touch screen is defective and let ND denote the event that it is not defective. Fill in the appropriate probabilities on the tree diagram on the left on the next page.

Bayes' Theorem

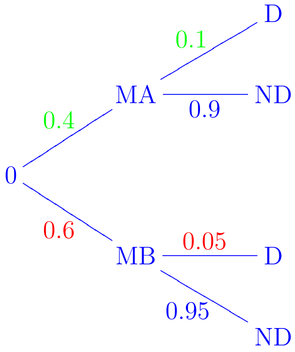




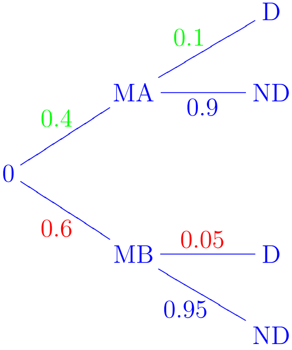


Bayes' Theorem

We can now calculate $\mathbf{P}\left(MA \middle| D\right) = \dfrac{\mathbf{P}(MA \cap D)}{\mathbf{P}(D)} = \dfrac{\mathbf{P}(MA \cap D)}{\mathbf{P}\left(D \middle| MA\right) \cdot \mathbf{P}(MA) + \mathbf{P}\left(D \middle| MB\right) \cdot \mathbf{P}(MB)}$. Note the event D is shown in red above and the event $MA \cap D$ is shown in blue.



Bayes' Theorem



$$\mathbf{P}\left(MA \middle| D\right) = \frac{\mathbf{P}(MA \cap D)}{\mathbf{P}(D)} =$$

$$\frac{\mathbf{P}(MA \cap D)}{\mathbf{P}\left(D \middle| MA\right) \cdot \mathbf{P}(MA) + \mathbf{P}\left(D \middle| MB\right) \cdot \mathbf{P}(MB)} =$$

$$\frac{0.4 \cdot 0.1}{0.4 \cdot 0.1 + 0.6 \cdot 0.05} = \frac{0.04}{0.04 + 0.03} = \frac{0.04}{0.07} = \frac{4}{7}$$

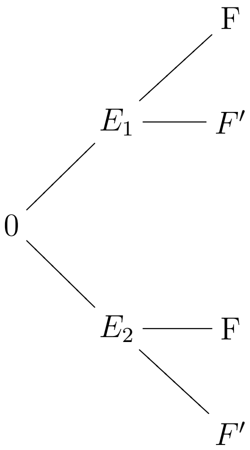
Bayes' Theorem

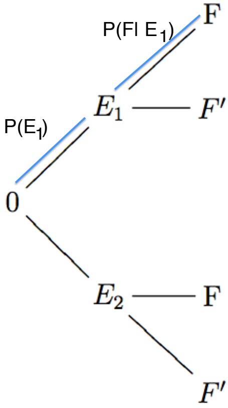
Let E_1 and E_2 be mutually exclusive events $(E_1 \cap E_2 = \emptyset)$ whose union is the sample space, i.e. $E_1 \cup E_2 = S$. Let F be an event in S for which $\mathbf{P}(F) \neq 0$. Then

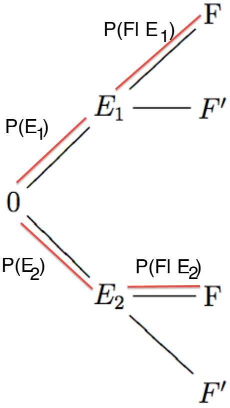
$$\boxed{\mathbf{P}\left(E_1 \middle| F\right) = \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(F)} = \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(E_1 \cap F) + \mathbf{P}(E_2 \cap F)} = \frac{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right)}{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right) + \mathbf{P}(E_2)\mathbf{P}\left(F \middle| E_2\right)}.}$$

Note that if we cross-classify outcomes in the sample space according to whether they belong to E_1 or E_2 and whether they belong to F or F', we get a tree diagram as above from which we can calculate the probabilities.

Bayes' Theorem







Predictive value of diagnostic test

The above analysis allows us to gain insight a commonly misunderstood point about the accuracy of tests for diseases and drugs. The predictive value of a diagnostic test does not depend entirely on the sensitivity of the test. It also depend on the prevalence of the disease. Many people when asked the following question "If a swimmer fails a drug test that is known to be 95 percent accurate(whether they have drugs in their system or not), how likely is it that he/she is really guilty?" will answer 95 percent, but of course you know that you need more information in order to answer the question. Check out the following article on the subject:

Doctors flunk quiz on screening-test math

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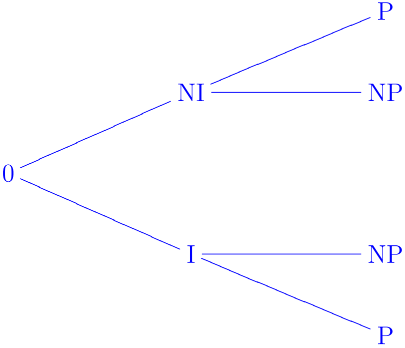
Let's see what happens in a few examples.

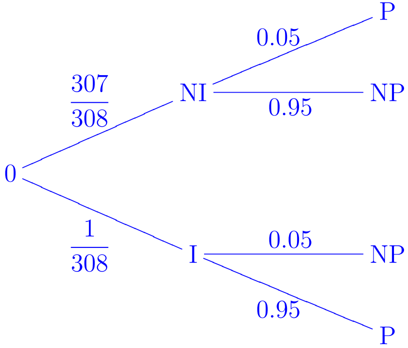
Predictive value of diagnostic test

Example Suppose, for example a test for the HIV virus is 95% accurate.The test gives a positive result for 95% of those taking the test who are HIV positive. Also the test gives a negative result for 95% of those taking the test who are not HIV positive.

(a) According to a recent estimate, approximately one million people in the U.S. are HIV positive. The population of the U.S. is approximately 308 million. Suppose a random U.S. resident takes the aids test and tests positive, what is the probability that the person is infected given that they have tested positive, That is what is $\mathbf{P}\left(I \middle| P\right)$?
(We let P denote the event that a person chosen at random from the population tests positive, we let I denote the event that a person chosen at random is infected.)

Predictive value of diagnostic test





$$\mathbf{P}\left(I \middle| P\right) \approx \frac{\frac{1}{308}\cdot 0.95}{\frac{307}{308}\cdot 0.05 + \frac{1}{308}\cdot 0.95} =$$

$$\frac{0.00308441558442}{0.04983766233766 + 0.00308441558442} =$$

$$\frac{0.00308441558442}{0.05292207792208} \approx 0.0583 = 5.83\%$$

Predictive value of diagnostic test

(b) In country X, forty percent of the residents are HIV positive. Suppose a random resident of Country X takes the aids test and tests positive, what is the probability that the person is infected given that they have tested positive, That is what is $\mathbf{P}\left(I \middle| P\right)$?

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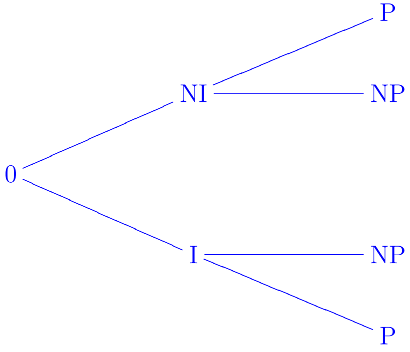

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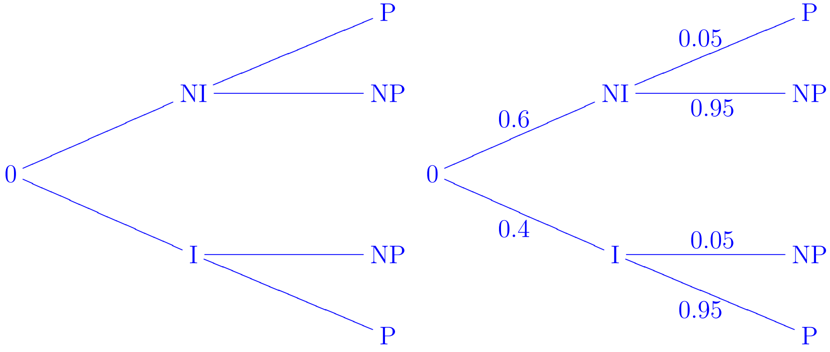

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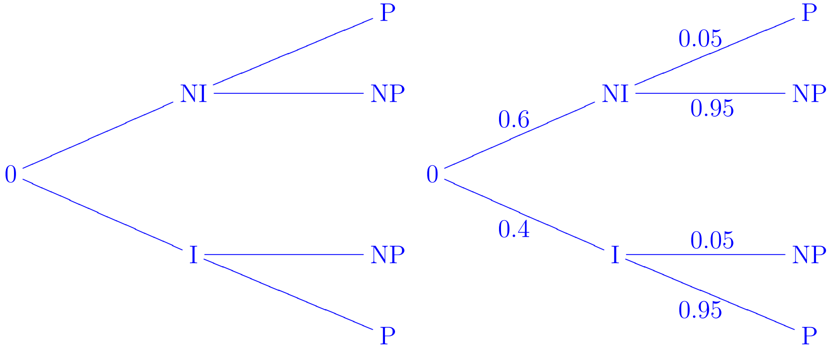


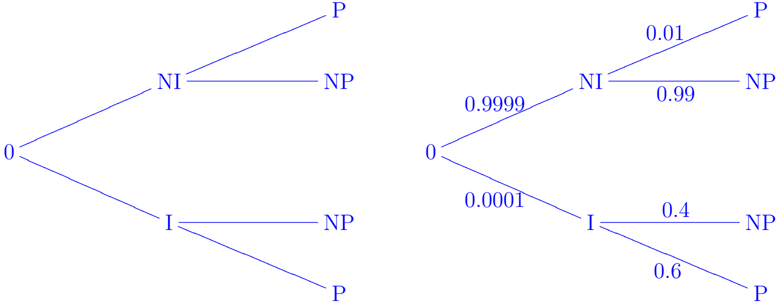
$$\mathbf{P}\left(I \middle| P\right) = \frac{0.4 \cdot 0.95}{0.6 \cdot 0.05 + 0.4 \cdot 0.95} = \frac{0.38}{0.03 + 0.38} = \frac{0.38}{0.41} \approx 0.9268 \approx 93\%$$

Predictive value of diagnostic test

Example A test for Lyme disease is 60% accurate when a person has the disease and 99% accurate when a person does not have the disease. In Country Y, 0.01% of the population has Lyme disease. What is the probability that a person chosen randomly from the population who test positive for the disease actually has the disease?

Predictive value of diagnostic test

Example A test for Lyme disease is 60% accurate when a person has the disease and 99% accurate when a person does not have the disease. In Country Y, 0.01% of the population has Lyme disease. What is the probability that a person chosen randomly from the population who test positive for the disease actually has the disease?



$\mathbf{P}\left(I \middle| P\right) = \frac{0.0001 \cdot 0.06}{0.9999 \cdot 0.01 + 0.0001 \cdot 0.06} = \frac{0.00006}{0.00006 + 0.009999} = \frac{0.00006}{0.010059} \approx$ 0.006.

A legal example

A crime has been committed and the only evidence is a blood spatter that could only have come from the perpetrator. The chance of a random individual having the same blood type as that of the spatter is 10%. Joe has been arrested and charged. The trial goes as follows.
Prosecutor: Since there is only a 10% chance that Joe's blood would match, there is a 90% chance that Joe did it. That's good enough for me.
Defence Lawyer: There are two hundred people in the neighborhood who could have done the crime. Twenty of them will have the same blood type as the sample. Hence the chances that Joe did it are $\frac{1}{20} = 5\%$ so there is a 95% chance that Joe is innocent. That's good enough for me.

The ghost of the Reverend Thomas Bayes: You're all nuts!

The ghost of the Reverend Thomas Bayes: You're all nuts!

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If $\mathbf{P}(I) = x$ and so $\mathbf{P}(G) = 1 - x$ then

$$\mathbf{P}\left(I \middle| M\right) = \frac{0.1 \cdot x}{0.1 \cdot x + 1 \cdot (1 - x)} = \frac{0.1x}{1 - 0.9x}$$

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If you think that Joe is guilty, then $x = 0$ and after seeing the evidence you still think Joe is guilty, $\mathbf{P}\left(I \middle| M\right) = 0$.

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If you think that Joe there is a 40% change that Joe is guilty, then $x = 0.4$ and after seeing the evidence $\mathbf{P}\left(I \middle| M\right) = 0.0625$.

If $\mathbf{P}(I) = x$ and so $\mathbf{P}(G) = 1 - x$ then

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If you think that Joe there is a 40% change that Joe is guilty, then $x = 0.4$ and after seeing the evidence $\mathbf{P}\left(I \middle| M\right) = 0.0625$.

If you think that the police just searched a blood type database until they came up with a name in the neighborhood, then you should probably start with the defense lawyer's idea that $x = P(I) = \frac{19}{20} = 95\%$. Now after seeing the evidence $\mathbf{P}\left(I \middle| M\right) = \frac{0.095}{1 - 0.855} = \frac{0.095}{0.145} = 0.66$.

General Bayes' Theorem

Let $E_1, E_2, \ldots, E_n$ be (pairwise) mutually exclusive events such that $E_1 \cup E_2 \cup \cdots \cup E_n = S$, where S denotes the sample space. Let F be an event such that $\mathbf{P}(F) \neq 0$, Then

$$\mathbf{P}\left(E_1 \middle| F\right) = \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(F)} =$$

$$\frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(E_1 \cap F) + \mathbf{P}(E_2 \cap F) + \cdots + \mathbf{P}(E_n \cap F)} =$$

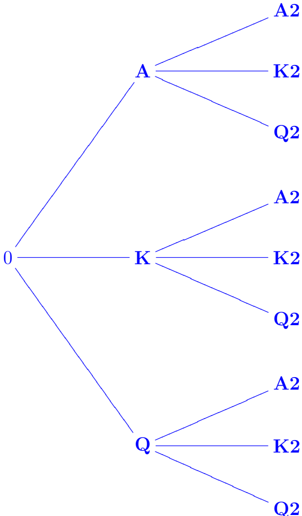
$$\frac{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right)}{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right) + \mathbf{P}(E_2)\mathbf{P}\left(F \middle| E_2\right) + \cdots + \mathbf{P}(E_n)\mathbf{P}\left(F \middle| E_n\right)}$$

General Bayes' Theorem

Example A pile of 8 playing cards has 4 aces, 2 kings and 2 queens. A second pile of 8 playing cards has 1 ace, 4 kings and 3 queens. You conduct an experiment in which you randomly choose a card from the first pile and place it on the second pile. The second pile is then shuffled and you randomly choose a card from the second pile. If the card drawn from the second deck was an ace, what is the probability that the first card was also an ace?

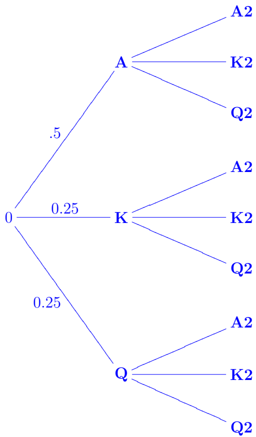
General Bayes' Theorem

Let **A** be the event that you draw an ace, **K** the event that you draw a king and **Q** be the event that you draw a queen.



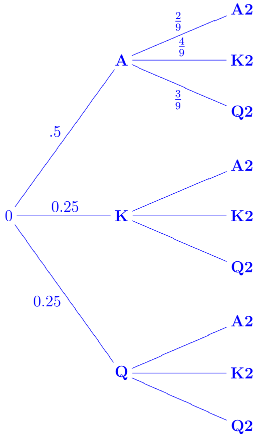
General Bayes' Theorem

In the first round there are $4 + 2 + 2 = 8$ cards so the probabilities in the first round are



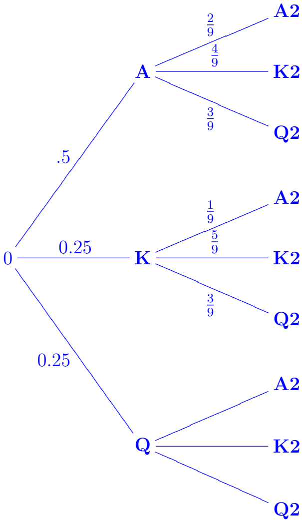
General Bayes' Theorem

In the second round there are $1 + 4 + 3 + 1 = 9$ cards and the probabilities are different at the various nodes. If you draw an ace in round 1 the cards are 2 aces, 4 kings and 3 queens so we get



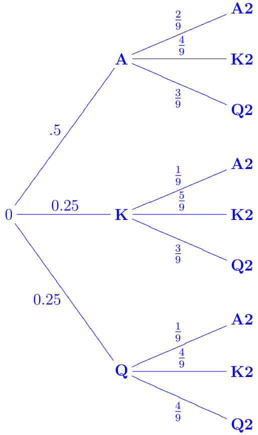
General Bayes' Theorem

If you draw a king in round 1 the cards are 1 ace, 5 kings and 3 queens so we get

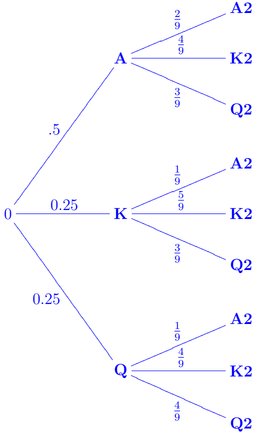


General Bayes' Theorem

If you draw a queen in round 1 the cards are 1 ace, 4 kings and 4 queens so we get



General Bayes' Theorem



The question asks for

$$\mathbf{P}\left(A \middle| A2\right) = \frac{\mathbf{P}(A \cap A2)}{\mathbf{P}(A_2)} = \frac{\frac{1}{2} \cdot \frac{2}{9}}{\frac{1}{2} \cdot \frac{2}{9} + \frac{1}{4} \cdot \frac{1}{9} + \frac{1}{4} \cdot \frac{1}{9}} = \frac{\frac{4}{36}}{\frac{4}{36} + \frac{1}{36} + \frac{1}{36}} = \frac{4}{6} = \frac{2}{3}.$$