Section 8.4: Random Variables and probability distributions of discrete random variables

In the previous sections we saw that when we have numerical data, we can calculate descriptive statistics such as the mean, the median, the range and the standard deviation. When we perform an experiment, we are often interested in recording more than one piece of numerical data for each trial. For example, when a patient visits the doctor's office, their height, weight, temperature and blood pressure are recorded. These observations vary from patient to patient, hence they are called variables.

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In fact they are called random variables, because we cannot predict what their value will be for the next trial of the experiment (for the next patient).

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A Random Variable is a rule that assigns a number to each outcome of an experiment. There may be more than one random variable associated with an experiment.

Example 1 An experiment consists of rolling a pair of dice, one red and one green, and observing the pair of numbers on the uppermost faces (red first). We let X denote the sums of the numbers on the uppermost faces. Below, we show the outcomes on the left and the values of X associated to some of the outcomes on the right:

$$\begin{matrix} \{(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\} \end{matrix}$$

Outcome	X
(1, 1)	2
(2, 1)	3
(3, 1)	4
(4, 1)	5
⋮	⋮

(a) What are the possible values of X?

$$\begin{array}{cccccc}
\{(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\}
\end{array}$$

Outcome	X
(1, 1)	2
(2, 1)	3
(3, 1)	4
(4, 1)	5
$\vdots$	$\vdots$

(a) What are the possible values of X?

$1 + 1 = 2,\ 1 + 2 = 3,\ \ldots,\ 6 + 6 = 12.$

$$\begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Outcome	X
(1, 1)	2
(2, 1)	3
(3, 1)	4
(4, 1)	5
$\vdots$	$\vdots$

(a) What are the possible values of X?

$1+1=2$, $1+2=3$, $\ldots$, $6+6=12$.

(b) We see that there are 2 outcomes of this experiment for which X has a value of 3, namely $(2,1)$ and $(1,2)$. How many outcomes of the above experiment are associated with the values of X shown in the table below:

Value of X	**Number of possible (associated) outcomes**
2	
3	2
4	
5	
6	
$\vdots$	$\vdots$

Value of X	**Number of possible (associated) outcomes**
2	
3	2
4	
5	
6	
$\vdots$	$\vdots$

Value of X	**Number of possible (associated) outcomes**	
2	1	(1,1)
3	2	(1,2), (2,1)
4	3	(1,3), (2,2), (3,1)
5	4	(1,4), (2,3), (3,2), (4,1)
6	5	(1,5), (2,4), (3,3), (4,2), (5,1)
7	6	(1,6), (2,4), (3,4), (4,3), (5,2), (6,1)
8	5	(2,6), (3,5), (4,4), (5,3), (6,2)
9	4	(3,6), (4,5), (5,4), (6,3)
10	3	(4,6), (5,5), (4,6)
11	2	(5,6), (6,5)
12	1	(6,6)

(c) We could also define other variables associated to this experiment. Let Y be the product of the numbers on the uppermost faces, fill in the values of Y associated to the outcomes given below:

Outcome	Y
(1, 1)	
(2, 1)	
(3, 1)	
(4, 1)	
$\vdots$	$\vdots$

Outcome	Y
(1, 1)	1
(2, 1), (1, 2)	2
(3, 1), (1, 3)	3
(4, 1), (1, 4)	4
(5, 1), (1, 5)	5
(6, 1), (1, 6)	6

Outcome	Y
(3, 3)	9
(4, 3), (3, 4)	12
(5, 3), (3, 5)	15
(6, 3), (3, 6)	18

Outcome	Y
(2, 2)	4
(3, 2), (2, 3)	6
(4, 2), (2, 4)	8
(5, 2), (2, 5)	10
(6, 2), (2, 6)	12

Outcome	Y
(4, 4)	16
(4, 5), (5, 4)	20
(4, 6), (6, 4)	24
(5, 5)	25
(5, 6), (6, 5)	30
(6, 6)	36

(d) What are the possible values of Y?

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$1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36$

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$1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36$

You weren't asked but

Value	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24	25	30	36
Frequency	1	2	2	3	2	4	1	1	2	4	2	1	2	2	2	1	2	1

Example An experiment consists of flipping a coin 4 times and observing the sequence of heads and tails. The random variable X is the number of heads in the observed sequence. Fill in the possible values of X and the number of outcomes associated to each value in the table below:

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Value of X	**Number of possible (associated) outcomes**

Example An experiment consists of flipping a coin 4 times and observing the sequence of heads and tails. The random variable X is the number of heads in the observed sequence. Fill in the possible values of X and the number of outcomes associated to each value in the table below:

Value of X	**Number of possible (associated) outcomes**

Value of X	**Number of possible (associated) outcomes**
4	HHHH
3	HHHT, HHTH, HTHH, THHH
2	HHTT, HTHT, HTTH, THHT, THTH, TTHH
1	HTTT, THTT, TTHT, TTTH
0	TTTT

For some random variables, the possible values of the variable can be separated and listed in either a finite list or and infinite list. These variables are called **discrete random variables**.Some examples are shown below:

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Experiment	**Random Variable**, X
Roll a pair of six-sided dice	Sum of the numbers
Roll a pair of six-sided dice	Product of the numbers
Toss a coin 10 times	Number of tails
Choose a small pack of M&M's at random	The number of blue M&M's in the pack
Choose a year at random	The number of people who ran the Boston Marathon in that year

On the other hand, **a continuous random variable** can assume any value in some interval. Some examples are:

Experiment	**Random Variable**, X
Choose a patient at random	Patient's Height
Choose an apple at random at your local grocery store	Weight of the apple
Choose a customer at random at Subway	The length of time the customer waits to be served

Probability Distributions For Random Variables

For a discrete random variable with finitely many possible values, we can calculate the probability that a particular value of the random variable will be observed by adding the probabilities of the outcomes of our experiment associated to that value of the random variable (assuming that we know those probabilities). This assignent of probabilities to each possible value of X is called the probability distribution of X.

Example If I roll a pair of fair six sided dice and observe the pair of numbers on the uppermost face, all outcomes are equally likely, each with a probability of $\frac{1}{36}$. Let X denote the sum of the pair of numbers observed. We saw that a value of 3 for X is associated to two outcomes in our sample space: $(2, 1)$ and $(1, 2)$. Therefore the probability that X takes the value 3 or $\mathbf{P}(X = 3)$ is the sum of the probabilities of the two outcomes $(2, 1)$ and $(1, 2)$ which is $\frac{2}{36}$. That is

$$\mathbf{P}(X = 3) = \frac{2}{36}.$$

If X is a discrete random variable with finitely many possible values, we can display the probability distribution of X in a table where the possible values of X are listed alongside their probabilities.

Example

I roll a pair of fair six sided dice and observe the pair of numbers on the uppermost face. Let X denote the sum of the pair of numbers observed. Complete the table showing the probability distribution of X below:

$$
\begin{array}{llllll}
\{(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\}
\end{array}
$$

X	**P**(X)
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

X	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{P}(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

X	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{P}(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

This table is an example of a probability distribution associated to a random variable.

Probability Distribution:

If a discrete random variable has possible values $x_1, x_2, x_3, \ldots, x_k$, then a **probability distribution** $\mathbf{P}(X)$ is a rule that assigns a probability $\mathbf{P}(x_i)$ to each value x_i. More specifically,

- $0 \leq \mathbf{P}(x_i) \leq 1$ for each x_i.
- $\mathbf{P}(x_1) + \mathbf{P}(x_2) + \cdots + \mathbf{P}(x_k) = 1$.

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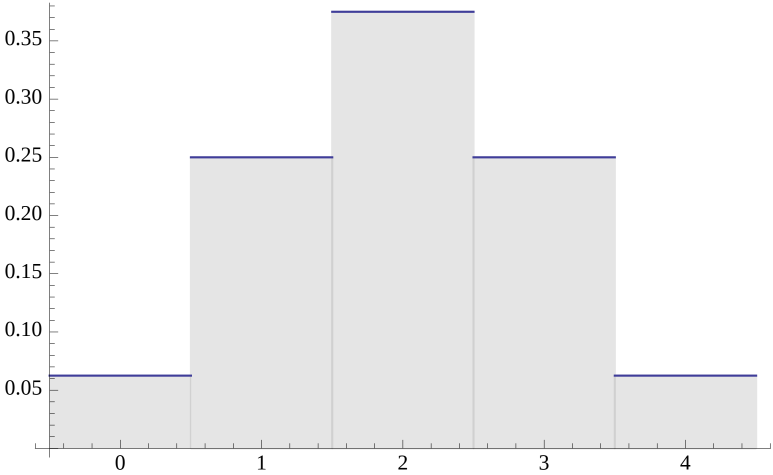
Example An experiment consists of flipping a coin 4 times and observing the sequence of heads and tails. The random variable X is the number of heads in the observed sequence. Fill in probabilities for each possible values of X in the table below.

X	0	1	2	3	4
$\mathbf{P}(X)$	?	?	?	?	?

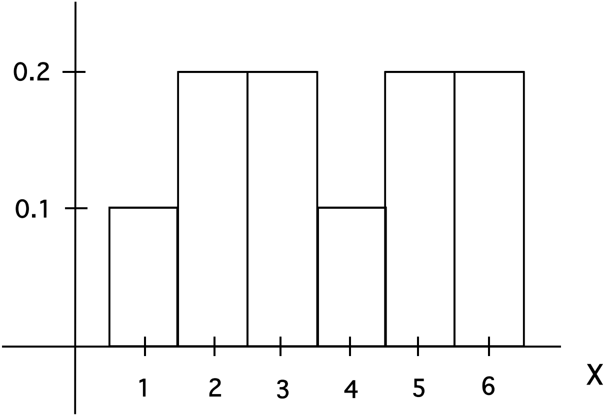
X	0	1	2	3	4
$\mathbf{P}(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

We can also represent a probability distribution for a discrete random variable with finitely many possible values graphically by constructing a bar graph. We form a category for each value of the random variable centered at the value which does not contain any other possible value of the random variable. We make each category of equal width and above each category we draw a bar with height equal to the probability of the corresponding value. if the possible values of the random variable are integers, we can give each bar a base of width 1.

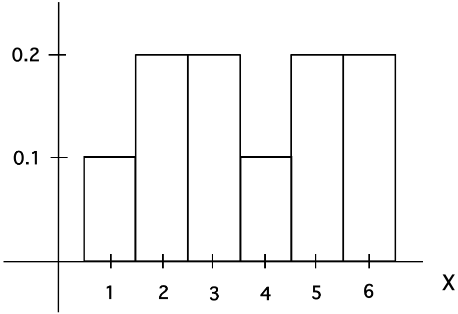
Example An experiment consists of flipping a coin 4 times and observing the sequence of heads and tails. The random variable X is the number of heads in the observed sequence. The following is a graphical representation of the probability distribution of X.



Example The following is a probability distribution histogram for a random variable X.



What is $\mathbf{P}(X \leqslant 5)$?



$\mathbf{P}(X \leqslant 5) = \mathbf{P}(X = 5) + \mathbf{P}(X = 4) + \mathbf{P}(X = 3) + \mathbf{P}(X = 1) = 0.2 + 0.1 + 0.2 + 0.2 + 0.1 = 0.8$

OR

$\mathbf{P}(X \leqslant 5) = 1 - \mathbf{P}(X = 6) = 1 - 0.2 = 0.8.$

Example In a carnival game a player flips a coin twice. The player pays \$1 to play. The player then receives \$1 for every head observed and pays \$1, to the game attendant, for every tail observed. Find the probability distribution for the random variable $X =$ the player's (net) earnings.

Example In a carnival game a player flips a coin twice. The player pays \$1 to play. The player then receives \$1 for every head observed and pays \$1, to the game attendant, for every tail observed. Find the probability distribution for the random variable $X =$ the player's (net) earnings.

There are 4 possible outcomes HH, HT, TH, TT. The return to the player in the case HH is 1, the return to the player in the case HT or TH is -1, and the return to the player in the case TT is -3. Hence $\mathbf{P}(X = 1) = \frac{1}{4}$, $\mathbf{P}(X = -1) = \frac{2}{4}$ and $\mathbf{P}(X = -3) = \frac{1}{4}$.

Example In Roulette, if you bet \$1 on red, you get your \$1 back + \$1 profit if the ball lands on red. If the ball does not land on red, the attendant keeps your money and you get nothing back. The roulette wheel (Vegas) has 18 red numbers, 18 black numbers and 2 greens. The ball is equally likely to go into any of the pockets. What is the probability distribution for your earnings for this game if you bet \$1 on red.

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There are only two outcomes: you win \$1 or you get $-\$1$.

$$\mathbf{P}(X = 1) = \frac{18}{18 + 18 + 2} = \frac{18}{38}.$$

$$\mathbf{P}(X = -1) = \frac{18 + 2}{18 + 18 + 2} = \frac{20}{38}.$$

Extras

Example: Netty's Scam Netty the Incredible runs the following scam in her spare time:

She has a business where she forecasts the gender of the unborn child for expectant couples, for a small price. The couple come for a visit to Netty's office and, having met them, Netty retires to her ante-room to gaze into her Crystal Ball. In reality, Netty flips a coin. If the result is "Heads" , she will predict a boy and if the result is "Tails", she will predict a girl. Netty returns to her office and tells the couple of what she saw in her crystal ball. She collects her fee of \$100 from the couple and promises to return \$150 if she was wrong.

What is the probability distribution for Netty's earnings per consultancy in this business?

What is the probability distribution for Netty's earnings per consultancy in this business?

Netty will win \$100 if she wins and lose \$50 if she loses. Let X be the random variable which is the amount Netty wins in one consultancy. Hence $\mathbf{P}(X = 100) = 0.5$ and $\mathbf{P}(X = -50) = 0.5$.

Example Harold and Maude play a card game as follows. Harold picks a card from a standard deck of 52 cards, and Maude tries to guess its suit without looking at it. If Maude guesses correctly, Harold gives her \$3.00; otherwise, Maude gives Harold \$1.00. What is the probability distribution for Maude's earnings for this game (assuming she is not "psychic") ?

Example Harold and Maude play a card game as follows. Harold picks a card from a standard deck of 52 cards, and Maude tries to guess its suit without looking at it. If Maude guesses correctly, Harold gives her \$3.00; otherwise, Maude gives Harold \$1.00. What is the probability distribution for Maude's earnings for this game (assuming she is not "psychic") ? If X is the random variable which is the amount Maude wins in one round. Either Maude wins \$3 or she looses \$1. Hence $\mathbf{P}(X = 3) = \frac{13}{52} = \frac{1}{4}$ and $\mathbf{P}(X = -1) = 1 - \frac{1}{4} = \frac{3}{4}$.

Example At a carnival game, the player plays \$1 to play and then rolls a pair of fair six-sided dice. If the sum of the numbers on the uppermost face of the dice is 9 or higher, the game attendant gives the player \$5. Otherwise, the player receives nothing from the attendant. Let X denote the earnings for the player for this game. What is the probability distribution for X?

Example At a carnival game, the player plays \$1 to play and then rolls a pair of fair six-sided dice. If the sum of the numbers on the uppermost face of the dice is 9 or higher, the game attendant gives the player \$5. Otherwise, the player receives nothing from the attendant. Let X denote the earnings for the player for this game. What is the probability distribution for X?

The player either wins \$5 - \$1 = \$4 or looses \$1.

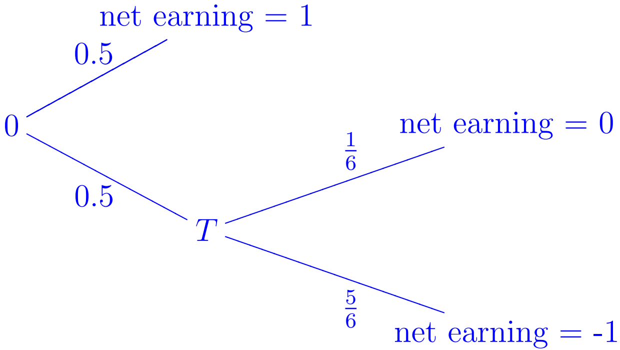
$\mathbf{P}(X = 4) = \dfrac{4+3+2+1}{36} = \dfrac{10}{36}$ and $\mathbf{P}(X = -1) = \dfrac{26}{36}$.

Example The rules of a carnival game are as follows:

1. The player pays \$1 to play the game.
2. The player then flips a fair coin, if the player gets a head the game attendant gives the player \$2 and the player stops playing.
3. If the player gets a tail on the coin, the player rolls a fair six-sided die. If the player gets a six, the game attendant gives the player \$1 and the game is over.
4. If the player does not get a six on the die, the game is over and the game attendant gives nothing to the player.

Let X denote the player's (net) earnings for this game, what is the probability distribution of X?

A tree diagram could help.



$\mathbf{P}(X=1)=0.5$; $\mathbf{P}(X=0)=\dfrac{1}{2}\cdot\dfrac{1}{6}=\dfrac{1}{12}$;

$\mathbf{P}(X=-1)=\dfrac{1}{2}\cdot\dfrac{5}{6}=\dfrac{5}{12}$.

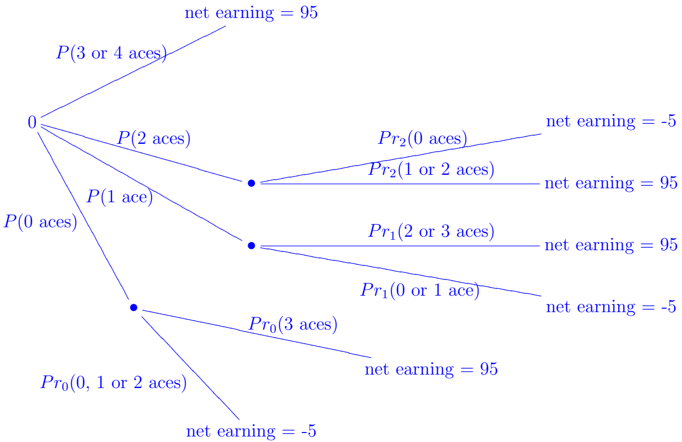
Example

The rules of a carnival game are as follows:

- The player pays \$5 to play.
- The attendant then deals a random selection of 5 cards from a standard deck of 52 cards to the player.
- If the player has at least three aces in the 5 cards he/she is dealt, the attendant gives the player \$100 and the game is over.
- Otherwise the player selects 3 cards from the 5 he/she was dealt and discards them. The attendant gives the player a random selection of 3 cards from the remaining cards in the deck to replace the ones discarded.
- The player now has 5 cards and if the player has at least 3 aces among the 5 cards, then the attendant gives the player \$100 and the game is over.
- If the player does not have at least 3 aces among the 5 cards, then the attendant gives the player nothing and the game is over.

Let X denote the player's (net) earnings for this game, what is the probability distribution for X. (Hint: use a tree diagram classifying outcomes according to the number of aces dealt.)

A tree diagram could help again.



We have calculated the needed probabilities before. The probability of getting k aces in round 1 is

$$\frac{C(4,k)\cdot C(48,5-k)}{C(52,5)}.$$

In any case, the only two outcomes are a net earning of \$95 or -\$5. There are 4 paths which end in \$95. Here are the pieces.

$$\mathbf{P}(3 \text{ or } 4 \text{ aces}) = \frac{C(4,3) \cdot C(48,2) + C(4,4) \cdot C(48,1)}{C(52,5)}$$

$$\mathbf{P}(2 \text{ aces}) = \frac{C(4,2) \cdot C(48,3)}{C(52,5)},$$

$$\mathbf{P}(1 \text{ ace}) = \frac{C(4,1) \cdot C(48,4)}{C(52,5)} \text{ or}$$

$$\mathbf{P}(0 \text{ aces}) = \frac{C(4,0) \cdot C(48,5)}{C(52,5)}.$$

$$Pr_2(1 \text{ or } 2 \text{ aces}) = \frac{C(2,1) \cdot C(45,2) + C(2,2) \cdot C(45,1)}{C(47,3)}$$

$$Pr_1(2 \text{ or } 3 \text{ aces}) = \frac{C(3,2) \cdot C(44,1) + C(3,3) \cdot C(44,0)}{C(47,3)}$$

$$Pr_0(3 \text{ aces}) = \frac{C(4,3) \cdot C(43,0)}{C(47,3)}$$