On Hrushovski's "Definability patterns and their symmetries"

Notre Dame Model Theory seminar

Anand Pillay

October 6th, 2020

More on the τ-topology I

- Before starting on the paper, I want to add something to the discussion at the end of the last (introductory) talk.
- Maybe not so irrelevant, as it seems that the various topologies are most delicate parts of the various theories and their compatibilities.
- Recall we had a saturated model M of T, the space $S_{\bar{m}}(M)$ of extensions of $tp(\bar{m}/\emptyset)$ to M where $\bar{m}$ enumerates M, considered as an $Aut(\bar{M})$-flow.
- And $E = E(S_{\bar{m}}(M))$, the Ellis semigroup of the flow, is the closure of the self maps of $S_{\bar{m}}(M)$ given by $\sigma \in Aut(M)$ in the space of all self maps of $S_{\bar{m}}(M)$.
- The semigroup operation is composition and is continuous on the left, and E naturally contains a copy of $Aut(\bar{M})$.
- $\mathcal{M}$ is a minimal left ideal of E, u is an idempotent in $\mathcal{M}$, and $G = u * \mathcal{M}$ is what we called the Ellis group attached to the original flow.

More on the τ-topology, II

- The original definition of the τ-topology on G was:
- For A a subset of G, by $u \circ A$ we mean $\{\eta \in E\colon$ there are nets $\eta_i \in E$ and $g_i \in Aut(M)$ such that $g_i \to u$, and $g_i\eta_i \to \eta\}$.
- Then $cl_\tau(A) = (u \circ A) \cap G$.
- Any τ-closed subset of G is closed in the relative topolology of G as a subset of the profinite space $\mathcal{M}$ (or E), but not vice versa in general.
- When $\mathcal{M}$ is the universal minimal $Aut(M)$-flow then there is a connection with a Galois theory of minimal flows.
- Maybe it goes through in our current situation too?? As follows:
- A "factor" of $(\mathcal{M}, u)$ is a minimal $Aut(M)$-flow X with distinguished point x_0 such that there is a (unique if it exists) surjective morphism ϕ of $Aut(M)$-flows from $\mathcal{M}$ to X with $\phi(u) = x_0$.

More on the τ-topology III

- For such a factor $\phi : (\mathcal{M}, u) \to (X, x_0)$, let $G(\phi) = \{g \in G = u * \mathcal{M} : \phi(g) = x_0)$.
- Then the statement is that (i) $G(\phi)$ is a closed subgroup of G in the τ-topology), and
- (ii) Every closed subgroup of G (in the τ-topology) arises in this fashion from a factor of $(\mathcal{M}, u)$

Hrushovski's paper I

- We now pass to Hrushovski's paper, the version from January 2020 on arXiv. (Updates??)
- Hrushovski works at a rather high level of generality. But restricted to the case of a complete theory T in language L, he considers arbitrary models M of T, and type spaces $S(M)$ (with respect to a given sort or product of sorts),
- and for a certain language (vocabulary) $\mathcal{L}$, equips each such $S(M)$ with an $\mathcal{L}$-structure,
- which, viewing the basic $\mathcal{L}$-relations as "closed" makes $S(M)$ into a "topological space".
- He then identifies, or constructs, a certain $\mathcal{L}$-structure $Core(T)$ (the core of T with respect to the chosen sorts).
- $Core(T)$ is quasi-compact and T_1 and is $\mathcal{L}$-homomorphically embedded into every $S(M)$.

Hrushovski's paper II

- He then considers the group $Aut(Core(T))$ of bijections preserving the basic relations (actually "pp-relations") which is also naturally a quasicompact T_1 topological group (maybe the group operation is only separately continuous??)
- Quotient $Aut(Core(T))$ by the normal "infinitesimal" subgroup $\mathfrak{g}$ consisting of those $\alpha \in Aut(Core(T))$ such that $\alpha(U) \cap U \neq \emptyset$ for all open subsets U of $Core(T)$.
- Then $Aut(Core(T))/\mathfrak{g}$ is a compact Hausdorff group, and is the sought after invariant of T, which maps homomorphically onto $Gal_L(T)$ (when suitable sorts are chosen at the beginning).
- In the paper $Core(T)$ is called J. There is also a variant $\bar{J}$ which is closely related to the theory exposited in my first talk, as we may see later.
- The technical aspect of the paper is complicated, where in particular $Core(T)$ is defined as a universal ec model of a certain universal theory in positive logic.

Hrushovski's paper III

- Simon's notes have a nice direct treatment of the above material (avoiding ec models of universal positive theories).
- In 3.14, and 3.15 of Hrushovski's paper, a duality is mentioned, between a certain class of maps from $S(M) \to S(M)$, and the $\mathcal{L}$-structure/topology (the "patterns language") on $S(M)$. Actually a version of 3.14 appears in Lemma 6.11 of (Lascar).
- This duality is worked out in some detail in Appendix A in the slighly different context of a saturated model M of T, the Ellis semigroup, and an "infinitary patterns" language (and $\bar{J}$).
- Krupinski may talk about the latter later.
- But in the meantine I will approach the the core of the paper via the duality suggested in 3.14/3.15 and see where it leads.

Some type spaces I

- Fix complete L-theory T and *arbitrary* model M of T (so possibly countable if T is countable).
- As in the background talk we let $\bar{m}$ be an enumeration of M, and $S_{\bar{m}}(M)$ be the space of extensions of $tp(\bar{m}/\emptyset) = p_0$ to complete types over M (in variables $\bar{x}$ say, corresponding to the tuple $\bar{m}$).
- And we let $S_{\bar{m},M}(\bar{M})$ be the space of global (over the monster model $\bar{M}$) complete types, which extend $tp(\bar{m}/\emptyset)$ and are finitely satisfiable in M.
- So there are, on the face of it, no specifically chosen group actions (flows) in the picture.
- Every $p(\bar{x}) \in S_{\bar{m},M}(\bar{M})$ is $Aut(\bar{M}/M)$-invariant, and so has a "defining schema over M": for each $\phi(\bar{x}, y) \in L$, and $b \in \bar{M}$, whether or not $\phi(\bar{x}, b) \in p$ depends (uniformly in ϕ) on $tp(b/M)$.

Some type spaces II

- Given $p(\bar{x}) \in S_{\bar{m},M}(\bar{M})$, and any subset A of $\bar{M}$, $p|A$ is just the restriction of p to A (a complete type over A extending p_0).
- Actually if N is a sufficiently saturated model containing M, then $p \in S_{\bar{m},M}(\bar{M})$ is already determined by $p|N$.
- Now suppose that $\bar{b}$ realizes p_0 in $\bar{M}$, namely that $tp(\bar{b}/\emptyset) = tp(\bar{m}/\emptyset)$.
- Then for any $p \in S_{\bar{m},M}(\bar{M})$, by $p_{\bar{b}}(\bar{x})$ we mean the image $\sigma(p)$ of p under any automorphism of $\bar{M}$ which takes $\bar{m}$ to $\bar{b}$.
- i.e., the defining schema of p over M is transported by σ to the defining schema of $p_{\bar{b}}$ over (the model) $\bar{b}$.

Semigroup action I

- We first define an "action" of $S_{\bar{m},M}(\bar{M})$ on $S_{\bar{m}}(M)$:
- For $q \in S_{\bar{m}}(M)$ and $p \in S_{\bar{m},M}(\bar{M})$, let $\bar{a}$ realize q, and let $\bar{b}$ realize $p|(M, \bar{a})$.
- Then $p(q) = f(tp(\bar{a}/\bar{b}))$ where f is the partial elementary map taking $\bar{b}$ to $\bar{m}$. i.e. $p(q)$ is $tp(\bar{a}/\bar{b})$ transported to a complete type over $\bar{m}$ (i.e. M).
- Note that this makes sense when $S_{\bar{m}}(M)$ is replaced by any type space over M in this or that sort.
- If $p = tp(\bar{m}/\bar{M})$ then clearly $p(q) = q$ for any $q \in S_{\bar{m}}(M)$.
- Actually $S_{\bar{m},M}(\bar{M})$ is closed under composition of maps, giving it a semigroup structure $*$, continuous in the first coordinate.

Semigroup action II

- So for $p, q \in S_{\bar{m},M}(\bar{M})$, what is $p * q$?
- Choose $N \geq M$ sufficiently saturated.
- Let $\bar{b}$ realize $q|N$, and let $\bar{c}$ realize $p_{\bar{b}}|(N, \bar{b})$.
- Then, as $p_{\bar{b}}$ is finitely satisfiable in $\bar{b}$, and $tp(\bar{b}/N)$ is finitely satisfiable in M, it follows that $tp(\bar{c}/N)$ is finitely satisfiable in M.
- We let $p * q$ be $tp(\bar{c}/N)$ (i.e. its unique global extension which is finitely satisfiable in M).
- Some things have to be checked, such as $(p * q)(r) = p(q(r))$ for $p, q \in S_{\bar{m},M}(\bar{M})$ and $r \in S_{\bar{m}}(M)$, as well as continuity on the left of $*$.
- One can see $*$ as composition of certain partial elementary maps: q corresponds to f_q taking $\bar{m}$ to $\bar{b}$, $p_{\bar{b}}$ to $f_{p_{\bar{b}}}$ taking $\bar{b}$ to $\bar{c}$ and $p * q$ corresponds to f_{p*q} taking $\bar{m}$ to $\bar{c}$ which is the composition $f_{p_{\bar{b}}} \circ f_q$??

Minimal left ideal I

- So we are in situation of the objects constructed by topological dynamics, but without the flow.
- In fact even in topological dynamics, I guess one can start with a flow, take its Ellis semigroup, then forget about the flow.
- Let us now write $S = S_{\bar{m}}(M)$, and $E = S_{\bar{m},M}(\bar{M})$ (even though it is not the Ellis semigroup of a flow).
- On general grounds, let $\mathcal{M}$ be a minimal left ideal (necessarily closed) in E, and let r be an idempotent in $\mathcal{M}$.
- Then $(r * \mathcal{M}, *)$ is the analogue of what we called earlier the "Ellis group", in particular it is a group, and is equipped with a T_1, quasi-compact, separately continuous topology (the τ-topology, as defined in the previous lecture).
- I will make several claims, some of which will be, or will have to be, checked later (noting that we have not even formally defined Hrushovski's pattern structure/topology on S).

Minimal left ideal II

- For $p \in E$, and $\sigma \in Aut(\bar{M})$ extending f_p, let $\hat{f}(p) =$ the image of σ in $Gal_L(T)$. Then $\hat{f}$ is a well-defined surjective semigroup homomorphism $E \to Gal_L(T)$
- Via restriction and quotienting $\hat{f}$ induces a surjective homomorphism from the compact Hausdorff group $r * \mathcal{M}/H(r * \mathcal{M})$ to $Gal_L(T)$.
- $r(S)$ is a copy of $Core(T)$, and $r * \mathcal{M}$ acting on $r(S)$ is precisely $Aut(r(S))$.
- Is the τ-topology the same as the topolology on $Aut(r(S))$ (coming from the $\mathcal{L}$-structure?
- What is the connection between the compact Hausdorff groups $r * \mathcal{M}/H(r * \mathcal{M})$ and $Aut(r(S))/\mathfrak{g})$?

The stable case I

- Before checking the claims, let us look at the case where T is stable, and the objects are considerably simplified.
- As every type has a unique coheir $S = S_{\bar{m}}(M)$ and $E = S_{\bar{m},M}(\bar{M})$ can be identified, and we already have the semigroup operation $*$ on S:
- Given p, $q \in S_{\bar{m}}(M)$. Let $\bar{b}$ realize q and $\bar{c}$ realize $p_{\bar{b}}$ independently from M over $\bar{b}$. Then $p * q = tp(\bar{c}/M)$.
- S has a unique minimal ideal $\mathcal{M}$ which is precisely the set of nonforking extensions of $p_0 = tp(\bar{m}/\emptyset)$ over M.
- $(\mathcal{M}, *)$ is already a group (so is the "Ellis group"), which is compact and Hausdorff with its existing Stone topology.
- Any $p \in \mathcal{M}$ corresponds to a partial elementary map f_p taking $\bar{m}$ to a realization $\bar{b}$ of p.

The stable case II

- Then f_p induces an elementary permutation σ_p of $acl^{eq}(\emptyset)$.
- And $(\mathcal{M}, *)$ is isomorphic to the group $Gal_{Sh}(T)$ of elementary permutations of $acl^{eq}(\emptyset)$ via $p \to \sigma_p$. (So for r the idempotent in $\mathcal{M}$, σ_r is the identity, i.e. fixes $acl^{eq}(\emptyset)$ pointwise.)
- Also $Core(T) = rS = \mathcal{M}$ follows from the definitions: If $p \in S$ and $\bar{a}$ realizes p then $r(p) = f(tp(\bar{a}/\bar{b}))$ where $\bar{b}$ realizes r independent of $\bar{a}$ over M and $f(\bar{b}) = \bar{m}$. And $\bar{a}$ is independent of $\bar{b}$ over $\emptyset$.
- And $(\mathcal{M}, *)$ is $Aut(\mathcal{M})$ as a compact space, so already equals $Aut(Core(T))/\mathfrak{g}$

Reference

- (Lascar). D. Lascar, On the Category of Models of a Complete Theory. J. Symbolic Logic 47 (1982), no. 2, 249–266.