Descriptive Set Theory and Model Theory
Second Lecture

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Thematic Program on Model Theory,
Notre Dame, June 2016

Overview of the four lectures:

1. Polish groups and ample generics (w/ A. S. Kechris)
2. Topological rigidity of automorphism groups (w/ A. S. Kechris)
3. Coarse geometry of Polish groups
4. Geometry of automorphism groups

Fraïssé classes and limits

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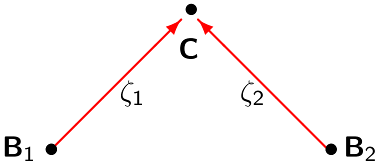

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$\mathcal{K}$ has the amalgamation property (AP) provided that, for all $A, B_1, B_2 \in \mathcal{K}$ and embeddings $\eta_i \colon A \to B_i$, there is some $C \in \mathcal{K}$ and embeddings $\zeta_i \colon B_i \to C$ so that $\zeta_1 \circ \eta_1 = \zeta_2 \circ \eta_2$.

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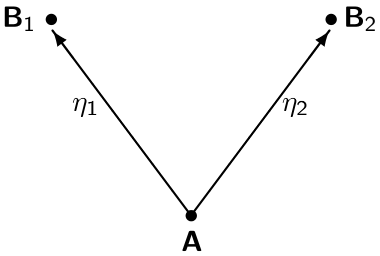

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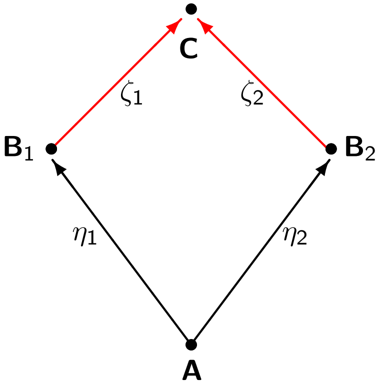

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Then $\mathcal{K}$ *is the Fraïssé class of all finite dyadic probability algebras.*

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Theorem (R. Fraïssé)

If $\mathcal{K}$ is a Fraïssé class, then there is a (unique up to isomorphism) countable ultrahomogeneous structure $\mathbb{K}$, called its Fraïssé limit, so that $\mathcal{K} = \mathrm{Age}\,\mathbb{K}$.

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- *Hall's universal locally finite group is the limit of the class of all finite groups.*

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Similarly, viewing $2^{\mathbb{N}}$ as the Cantor group $\prod_n \mathbb{Z}/2\mathbb{Z}$, the group of measure-preserving automorphisms of the clopen algebra of the Cantor group can be identified with the group

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Here, if $N_s = \{x \in 2^{\mathbb{N}} \mid s \subseteq x\}$ is a basic open set determined by a finite binary string $s \in 2^n$, we have

$$\mu(N_s) = 2^{-n}.$$

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We equip $\mathrm{Aut}(\mathbb{K})$ with the permutation group topology whose basic open sets are of the form

$$\begin{aligned}[\phi\colon B \to C] &= \{g \in \mathrm{Aut}(\mathbb{K}) \mid g \text{ extends } \phi\} \\ &= \{g \in \mathrm{Aut}(\mathbb{K}) \mid g(b) = \phi(b) \text{ for all } b \in B\},\end{aligned}$$

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Also, as $\mathbb{K}$ is ultrahomogeneous, any $\phi\colon B \to C$ defines a non-empty set $[\phi\colon B \to C]$.

A *n-system* over $\mathcal{K}$ is a tuple

$$\mathfrak{S} = \langle A,\ \phi_1\colon B_1 \to C_1,\ \ldots\ , \phi_n\colon B_n \to C_n \rangle,$$

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so that $\iota[B_i] \subseteq E_i$, $\iota[C_i] \subseteq F_i$ and the diagram commutes for all i.

$$\begin{array}{ccc} B_i & \xrightarrow{\ \phi_i\ } & C_i \\ \iota\big\downarrow & & \big\downarrow\iota \\ E_i & \xrightarrow[\ \psi_i\ ]{} & F_i \end{array}$$

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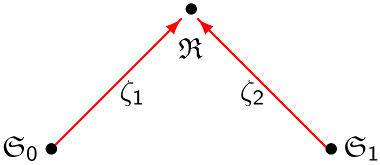

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Such an $\widetilde{\mathfrak{S}}$ is said to be an amalgamation base over $\mathfrak{S}$.

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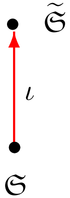

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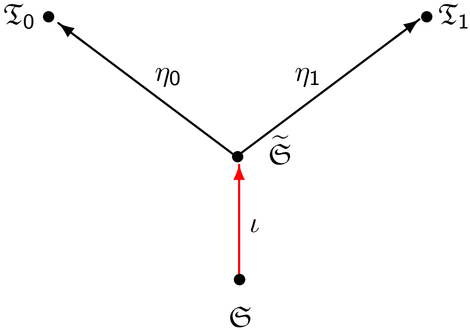

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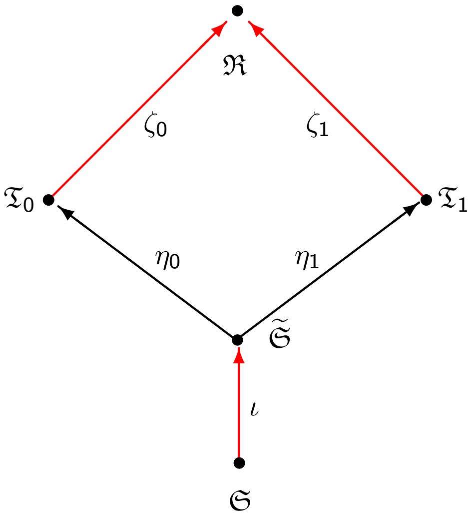

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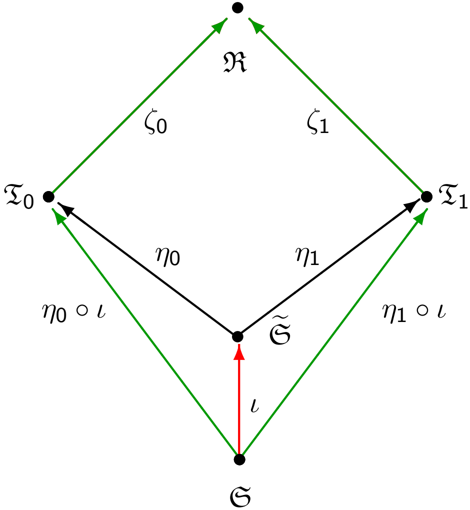

The green diagram commutes.

An amalgamation basis for dyadic measure algebras

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So let $\widetilde{A} \supseteq A$ be the algebra with atoms $\{N_s\}_{s \in 2^k}$.

Now, $b \in B_1$ and $\phi_1(b) \in C_1$ have the same measure and thus are the union of the same number of atoms N_s of $\widetilde{A}$.

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Doing this for all $\phi_1, \ldots, \phi_n$, we have a natural extension

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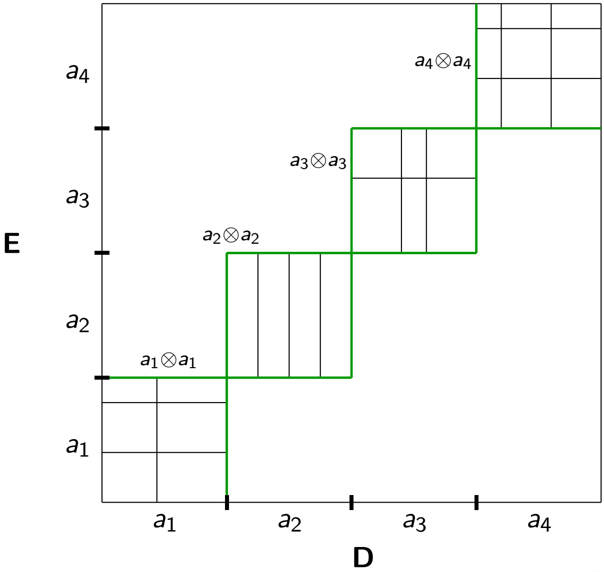
One simply uses the free product amalgamation over $\widetilde{A}$.

Free product amalgamation $D \otimes_{\widetilde{A}} E$ of two extensions D and E of $\widetilde{A}$:



Criterion for ample generics

Theorem

Let $\mathcal{K}$ be a Fraïssé class and $\mathbb{K}$ denote its corresponding Fraïssé limit. Then $\mathrm{Aut}(\mathbb{K})$ has ample generics if and only if $\mathcal{K}_p^n$ has the joint embedding and weak amalgamation properties for all n.

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First, by shrinking V and U, we may suppose that they have the form

$$V = [\phi\colon B \to C]$$

and

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Since $\mathbb{K}$ is ultrahomogeneous, we may suppose that actually

$$A \subseteq \widetilde{A} \subseteq \mathbb{K},$$

whence we have a non-empty open subset

$$W = [\widetilde{\phi}\colon \widetilde{B} \to \widetilde{C}] \subseteq V.$$

To see that the action of U on W is topologically transitive, suppose

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Then there is a system $\langle L, \alpha \colon M \to N \rangle$ with $L \subseteq \mathbb{K}$ and embeddings

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Then $g_0^{-1} g_1 \in U = [\mathrm{id} \colon A \to A]$ and $g_0^{-1} g_1[W_1] \cap W_0 \neq \emptyset$, showing that the action $U \curvearrowright W$ is topologically transitive.

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Theorem (Hodges–Hodkinson–Lascar–Shelah, Ben Yaacov–Tsankov)

Let $\mathbb{M}$ *be a countable* $\aleph_0$*-stable and* $\aleph_0$*-categorical structure.*
Then $\mathrm{Aut}(\mathbb{M})$ *has an open subgroup with ample generics.*

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On the other hand, the vertex stabiliser $\mathrm{Aut}(T_\infty, \mathbf{r})$ is an open subgroup, which consists only of elliptic elements and does have ample generics.