Descriptive Set Theory and Model Theory
Fourth Lecture

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Thematic Program on Model Theory,
Notre Dame, June 2016

Overview of the four lectures:

1. Polish groups and ample generics (w/ A. S. Kechris)
2. Topological rigidity of automorphism groups (w/ A. S. Kechris)
3. Coarse geometry of Polish groups
4. Geometry of automorphism groups

Applications to model theory

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of countable first-order structures $\mathbf{M}$.

The topology on $\mathrm{Aut}(\mathbf{M})$ is always that obtained by declaring pointwise stabilisers

$$V_{\bar{a}} = \{g \in \mathrm{Aut}(\mathbf{M}) \mid g(\bar{a}) = \bar{a}\}$$

of finite tuples $\bar{a}$ in $\mathbf{M}$ to be open.

Concepts from the previous talk

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Provided this holds, then, up to quasi-isometry,

$$\rho_S \text{ is independent of the choice of } S$$

so defines an isomorphic invariant of the group, the quasi-isometry type.

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That is, for all finite tuples $\overline{a}$ and $\overline{b}$ in **M**,

$$\mathcal{O}(\overline{a}) = \mathcal{O}(\overline{b}) \;\Leftrightarrow\; \mathrm{tp}^{\mathbf{M}}(\overline{a}) = \mathrm{tp}^{\mathbf{M}}(\overline{b}),$$

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2. similarly, provide realisations of and tools for analysing the large scale geometry of $\mathrm{Aut}(\mathbf{M})$,
3. show how the geometry of $\mathrm{Aut}(\mathbf{M})$ interacts with the structure **M**.

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Without loss of generality, we may assume that $\mathcal{S}$ consists of types of the form $p = \mathrm{tp}^{\mathbf{M}}(\overline{b}, \overline{c})$, where

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We define a graph $\mathbf{X}_{\bar{a},\mathcal{S}}$ on the set $\mathcal{O}(\bar{a})$ of realisations of $\mathrm{tp}^{\mathbf{M}}(\bar{a})$ in $\mathbf{M}$ by connecting distinct $\bar{b}, \bar{c} \in \mathcal{O}(\bar{a})$ by an edge if and only if

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Observe that, since

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for all tuples $\bar{b}$, $\bar{c}$ and automorphisms $g \in \mathrm{Aut}(\mathbf{M})$, the diagonal action of $\mathrm{Aut}(\mathbf{M})$ on $\mathcal{O}(\bar{a})$ is an action by automorphisms on the graph $\mathbf{X}_{\bar{a},\mathcal{S}}$.

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Moreover, since $\mathrm{Vert}\ \mathbf{X}_{\overline{a},\mathcal{S}} = \mathcal{O}(\overline{a})$ is a single orbit, the action

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By stipulation, we have that $\rho_{\overline{a},\mathcal{S}}(\overline{b}, \overline{c}) = \infty$ if and only if $\overline{b}$ and $\overline{c}$ lie in distinct connected components of $\mathbf{X}_{\overline{a},\mathcal{S}}$.

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We thus have a transitive isometric action $\mathrm{Aut}(\mathbf{M}) \curvearrowright \big(\mathbf{X}_{\overline{a},\mathcal{S}}, \rho_{\overline{a},\mathcal{S}}\big)$.

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1. *There is a finite set* $\mathcal{R}$ *of parameter-free types so that* $\mathbf{X}_{\bar{a},\mathcal{R}}$ *is connected, and*
2. *for every tuple* $\bar{b}$ *extending* $\bar{a}$*, there is a finite set* $\mathcal{S}$ *of parameter-free types so that*
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Condition (2), which in itself is equivalent to the pointwise stabiliser $V_{\overline{a}}$ being coarsely bounded, is clearly the most difficult to verify.

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Theorem (Milnor–Schwarz Theorem)

For $\bar{a}$ and $\mathcal{R}$ as above, the map

$$g \in \mathrm{Aut}(\mathbf{M}) \mapsto g \cdot \bar{a} \in \mathbf{X}_{\bar{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{M})$ and $\mathbf{X}_{\bar{a},\mathcal{R}}$.

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By the Milnor–Schwarz Theorem, we see that the map

$$g \in \mathrm{Aut}(T_\infty) \mapsto g(a) \in T_\infty$$

is a quasi-isometry between $\mathrm{Aut}(T_\infty)$ and $\mathbf{X}_{a,\mathcal{R}} = T_\infty$.

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Namely, T_∞ is given as the quasi-isometry type of a word metric ρ_S for some coarsely bounded generating set S.

Orbital independence relations

The verification that $\mathrm{Aut}(\mathbf{M})$ is locally bounded often relies on identifying an appropriate independence relation $\mathop{\smile\!\!\!\!\!\!|}_A$ between finite subsets of $\mathbf{M}$ relative to a fixed finite subset $A \subseteq \mathbf{M}$ or tuple $\overline{a}$ in $\mathbf{M}$.

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$$\Rightarrow \mathrm{tp}^{\mathbf{M}}(\bar{a}/B) = \mathrm{tp}^{\mathbf{M}}(\bar{b}/B).$$

When restricting our attention to ω-homogeneous structures $\mathbf{M}$, Conditions (iii) and (iv) of the definition of orbital A-independence relations can be reformulated as follows.

(iii) For all $\overline{a}$ and B, there is $\overline{b}$ with

$$\mathrm{tp}^{\mathbf{M}}(\overline{b}/A) = \mathrm{tp}^{\mathbf{M}}(\overline{a}/A) \quad \text{and} \quad \overline{b} \underset{A}{\downarrow} B.$$

(iv) For all $\overline{a}, \overline{b}$ and B,

$$\overline{a} \underset{A}{\downarrow} B \quad \& \quad \overline{b} \underset{A}{\downarrow} B \quad \& \quad \mathrm{tp}^{\mathbf{M}}(\overline{a}/A) = \mathrm{tp}^{\mathbf{M}}(\overline{b}/A)$$
$$\Rightarrow \mathrm{tp}^{\mathbf{M}}(\overline{a}/B) = \mathrm{tp}^{\mathbf{M}}(\overline{b}/B).$$

Independence notions similar the those above have recently been studied by K. Tent and M. Ziegler in connection with questions of simplicity of automorphism groups.

Theorem

Suppose $\mathbf{M}$ *is a countable structure,* $A \subseteq \mathbf{M}$ *a finite subset and* $\mathop{\smile\!\!\!\!\!\!|}_A$ *an orbital* A*-independence relation. Then the pointwise stabiliser subgroup* V_A *is coarsely bounded (relative to itself).*

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If instead, $A \neq \emptyset$, then $\mathrm{Aut}(\mathbf{M})$ is locally bounded and hence has a coarsely proper metric.

Functorial amalgamations

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Definition

Given an Fraïssé class $\mathcal{K}$ with limit $\mathbf{K}$ and a finite substructure $\mathbf{A} \subseteq \mathbf{K}$, we say that $\mathcal{K}$ satisfies functorial amalgamation over $\mathbf{A}$ if there is a way of choosing the amalgamations over $\mathbf{A}$ in the class $\mathcal{K}$ to be functorial with respect to embeddings.

In terms of arrows

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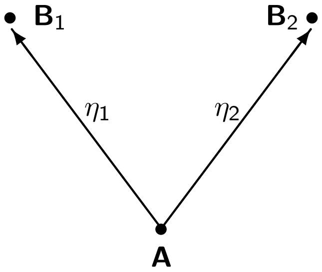

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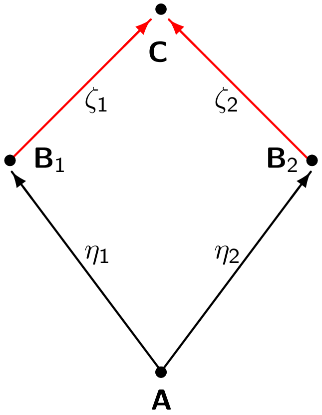

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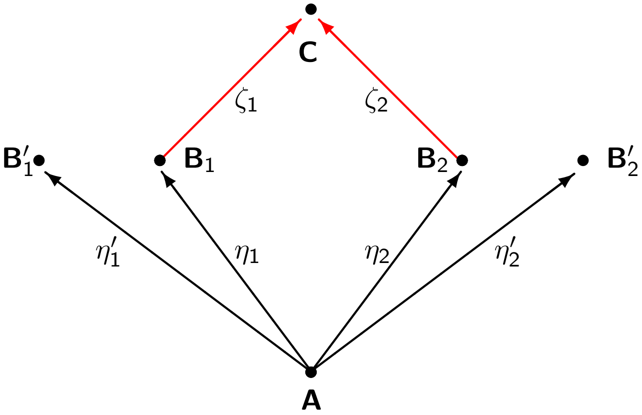

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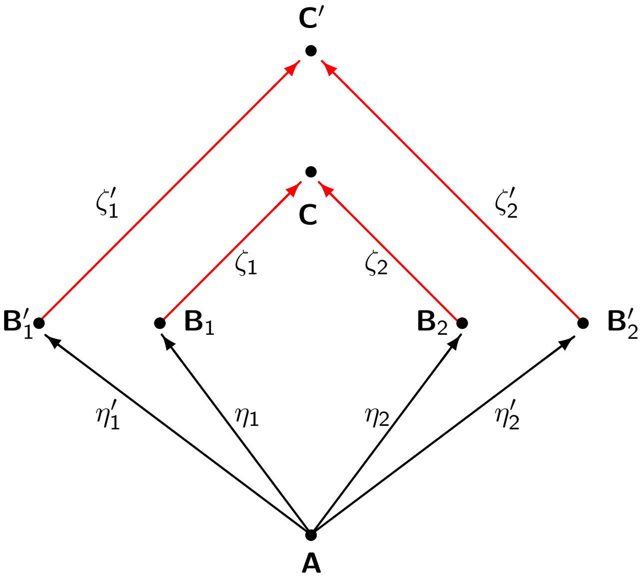

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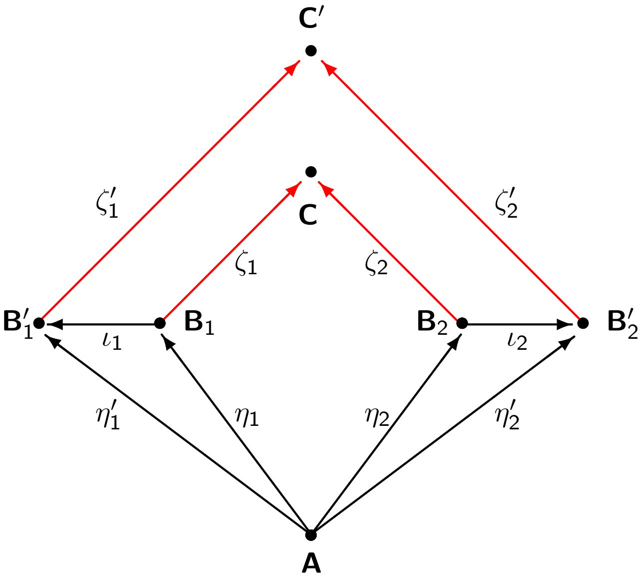

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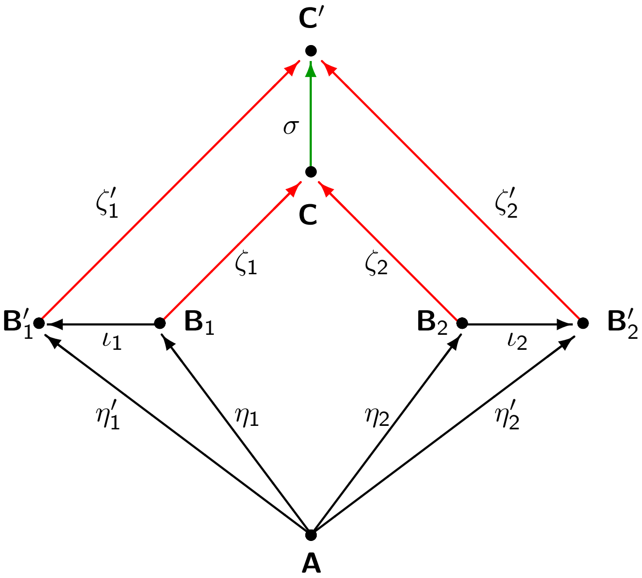

The rational Urysohn metric space

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That is, let B and C be two finite metric spaces with only a single point a in common.

The free amalgam of B and C over a is the union $B \cup C$ with

$$d(b, c) := d(b, a) + d(a, c)$$

for all $b \in B \setminus \{a\}$ and $c \in C \setminus \{a\}$.

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An important fact here is that, unless we bound the diameters of the metric spaces in question, there is no functorial amalgamation over the empty set.

Given a Fraïssé class $\mathcal{K}$ with limit $\mathbf{K}$ and a functorial amalgamation scheme over some finite $\mathbf{A} \subseteq \mathbf{K}$, we obtain an orbital $\mathbf{A}$-independence relation $\mathop{\smile\!\!\!\!\!\!|}_{\mathbf{A}}$ on $\mathbf{K}$ by setting

$$\mathbf{B} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{A}} \mathbf{C} \quad \Leftrightarrow \quad \mathbf{B}\ \&\ \mathbf{C} \text{ are functorially amalgamated over } \mathbf{A}$$

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Theorem

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Given a Fraïssé class $\mathcal{K}$ with limit $\mathbf{K}$ and a functorial amalgamation scheme over some finite $\mathbf{A} \subseteq \mathbf{K}$, we obtain an orbital $\mathbf{A}$-independence relation $\perp_{\mathbf{A}}$ on $\mathbf{K}$ by setting

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Theorem

Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ and assume that $\mathbf{A}$ is a finite substructure of $\mathbf{K}$ so that $\mathcal{K}$ admits a functorial amalgamation over $\mathbf{A}$. Then $V_{\mathbf{A}}$ coarsely bounded and thus $\mathrm{Aut}(\mathbf{K})$ is locally bounded.

Returning to $\mathbb{QU}$, this implies that the stabiliser V_a of any point $a \in \mathbb{QU}$ is coarsely bounded.

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To show that the automorphism group $\mathrm{Isom}(\mathbb{QU})$ is generated by a coarsely bounded set and to compute the quasi-isometry type, we seek a finite set $\mathcal{R}$ of parameter-free complete types, so that the graph

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For this, set $\mathcal{R} = \{d(x,y) = 1\}$ and note that any two points $x, y \in \mathbb{QU}$ can be connected by a path in $\mathbf{X}_{a,\mathcal{R}}$ of length

$$\text{at most } \lceil d(x,y) \rceil + 1, \text{ but no less than } d(x,y).$$

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Therefore, $\mathbf{X}_{a,\mathcal{R}}$ is quasi-isometric to $\mathbb{QU}$ and we conclude that the map

$$g \in \mathrm{Isom}(\mathbb{QU}) \mapsto g(a) \in \mathbb{QU}$$

is a quasi-isometry.

Groups with trivial geometry

In many familiar cases, though we are able to identify the large scale geometry of a topological group, it turns out that this is trivial.

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Let $\mathbf{M}$ *be an* $\aleph_0$*-categorical countable structure.*
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Theorem

Let $\mathbf{M}$ *be a saturated countable model of an* ω*-stable theory.*
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Tame geometry from model theoretical considerations

Recall that a structure **M** is atomic if every complete type is isolated.

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It follows that, if $\mathcal{R}$ is a finite collection of complete types, then, for every n, the relation on $\overline{b}$ and $\overline{c}$,

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Definition (J.-L. Krivine and B. Maurey)

A metric d on a set X is said to be stable if, for all d-bounded sequences (x_n) and (y_m) in X, we have

$$\lim_{n\to\infty}\lim_{m\to\infty} d(x_n,y_m) = \lim_{m\to\infty}\lim_{n\to\infty} d(x_n,y_m),$$

whenever both limits exist.

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Let T be a complete theory of a countable language $\mathcal{L}$ and let κ be an infinite cardinal number.

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Suppose $\mathbf{M}$ is a countable atomic model of a stable theory T.

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2. *if $\mathrm{Aut}(\mathbf{M})$ is generated by a coarsely bounded set, it admits a coarsely proper stable metric witnessing its quasi-isometry type.*

Noting the independence relations present in models of stable theories, one could be hopeful that the assumption that $\mathrm{Aut}(\mathbf{M})$ be locally bounded would be superfluous.

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However, this is not so.

Theorem (J. Zielinski)

There is a countable atomic model $\mathbf{M}$ *of an* ω*-stable theory so that* $\mathrm{Aut}(\mathbf{M})$ *is not locally bounded.*