Name: Date: 12/06/2018

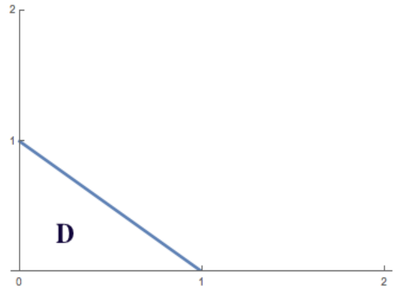
M20550 Calculus III Tutorial
Worksheet 11

1. Compute the surface integral $\iint_S (x+y+z)\ dS$, where S is a surface given by $\mathbf{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$ and $0 \le u \le 2,\ 0 \le v \le 1$.

> ***Solution:*** First, we know
>
> $$\iint_S (x+y+z)\ dS = \iint_D \Big[(u+v)+(u-v)+(1+2u+v)\Big]|\mathbf{r}_u \times \mathbf{r}_v|\, dA,$$
>
> where D is the domain of the parameters $u,\ v$ given by $0 \le u \le 2,\ 0 \le v \le 1$.
>
> We have $\mathbf{r}_u = \langle 1, 1, 2\rangle$ and $\mathbf{r}_v = \langle 1, -1, 1\rangle$. Then,
> $\mathbf{r}_u \times \mathbf{r}_v = \langle 1,1,2\rangle \times \langle 1,-1,1\rangle = \langle 3, 1, -2\rangle$. So,
>
> $$|\mathbf{r}_u \times \mathbf{r}_v| = |\,\langle 3,1,-2\rangle\,| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}.$$
>
> Thus,
>
> $$\begin{aligned}\iint_S (x+y+z)\ dS &= \int_0^1 \int_0^2 (4u+v+1)\sqrt{14}\ du\ dv \\ &= 11\sqrt{14}.\end{aligned}$$

2. Let S be the portion of the graph $z = 4 - 2x^2 - 3y^2$ that lies over the region in the xy-plane bounded by $x=0,\ y=0$, and $x+y=1$. Write the integral that computes $\iint_S \left(x^2+y^2+z\right)\ dS$.

> ***Solution:*** First, we need a parametrization of the surface S. Since S is a surface given by the equation $z = 4-2x^2-3y^2$, we can choose x and y to be the parameters. So,
>
> $$\mathbf{r}(x,y) = \left\langle x, y, 4-2x^2-3y^2\right\rangle,$$
>
> and the domain D of the parameters $x,\ y$ is given by the region in the xy-plane bounded by $x=0,\ y=0$, and $x+y=1$ (see picture below)



Now, $\mathbf{r}_x = \langle 1, 0, -4x\rangle$ and $\mathbf{r}_y = \langle 0, 1, -6y\rangle$. So, $\mathbf{r}_x \times \mathbf{r}_y = \langle 4x, 6y, 1\rangle$ and $|\mathbf{r}_x \times \mathbf{r}_y| = |\langle 4x, 6y, 1\rangle| = \sqrt{16x^2 + 36y^2 + 1}$. Thus,

$$\begin{aligned}\iint_S \left(x^2 + y^2 + z\right)\, dS &= \iint_D x^2 + y^2 + (4 - 2x^2 - 3y^2)|\mathbf{r}_x \times \mathbf{r}_y|\, dA \\ &= \int_0^1 \int_0^{-x+1} (4 - x^2 - 2y^2)\sqrt{16x^2 + 36y^2 + 1}\; dy\, dx.\end{aligned}$$

3. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is a surface given by

$$x = 2u, \quad y = 2v, \quad z = 5 - u^2 - v^2,$$

where $u^2 + v^2 \leq 1$. S has downward orientation.

Solution: We have $\mathbf{r}(u, v) = \langle 2u, 2v, 5 - u^2 - v^2\rangle$, so $\mathbf{r}_u = \langle 2, 0, -2u\rangle$ and $\mathbf{r}_v = \langle 0, 2, -2v\rangle$ and so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2, 0, -2u\rangle \times \langle 0, 2, -2v\rangle = \langle 4u, 4v, 4\rangle .$$

Note that $\mathbf{r}_u \times \mathbf{r}_v = \langle 4u, 4v, 4\rangle$ gives unit normal vectors pointing upward (z-component is positive). But, S has downward orientation so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\iint\limits_{u^2+v^2\leq 1} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\; dA.$$

Now, $\mathbf{F}\big(\mathbf{r}(u, v)\big) = \langle 2v, -2u, 5 - u^2 - v^2\rangle$. So

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \left\langle 2v, -2u, 5 - u^2 - v^2\right\rangle \cdot \langle 4u, 4v, 4\rangle = 20 - 4u^2 - 4v^2.$$

Thus,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint\limits_{u^2+v^2\le 1} (20-4u^2-4v^2)\,dA.$$
$$\overset{\text{polar}}{=} -\int_0^{2\pi}\int_0^1 (20-4r^2)r\,dr\,d\theta$$
$$= -18\pi.$$

4. Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

Solution: We want to compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the part of the cylinder $x^2+y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

First, we parametrize S: let $x = 2\cos u$, $y = 2\sin u$, $z = v$. Then

$$\mathbf{r}(u,v) = \langle 2\cos u, 2\sin u, v\rangle\,, \quad \text{domain } D \text{ is } \ 0\le u\le 2\pi,\ 0\le v\le 2.$$

Then, $\mathbf{r}_u = \langle -2\sin u, 2\cos u, 0\rangle$ and $\mathbf{r}_v = \langle 0,0,1\rangle$. So,

$$\mathbf{r}_u\times\mathbf{r}_v = \langle -2\sin u, 2\cos u, 0\rangle \times \langle 0,0,1\rangle = \langle 2\cos u, 2\sin u, 0\rangle\,.$$

Now, let's check our orientation. Let's take the point where $u = \pi/2$ and $v = 1$, ie $(x,y,z) = (0,2,1)$. At the point $(0,2,1)$, the unit normal vector points in the direction of the vector $(\mathbf{r}_u\times\mathbf{r}_v)(\pi/2,1) = \langle 0,2,0\rangle$. This means the unit normal vector is pointing away from the origin. So, our parametrization of S gives the correct orientation for S. Moving on!

Now, $\mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot(\mathbf{r}_u\times\mathbf{r}_v) = \langle 2\cos u, 2\sin u, v\rangle\cdot\langle 2\cos u, 2\sin u, 0\rangle = 4.$

Thus,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\;dA$$
$$= \iint_D 4\,dA.$$
$$= \int_0^{2\pi}\int_0^2 4\,dv\,du$$
$$= 16\pi.$$

Alternative solution using divergence theorem

We cannot apply the divergence theorem directly to the cylinder because it is not a closed surface. But we can apply the divergence theorem to the cylinder with the top and bottom added to eliminate all computations.

The divergence theorem says

$$\int\int_{\text{Top}\cup\text{Bottom}\cup\text{Side}} F \cdot dS = \int\int\int_{\text{inside of cylinder}} (\nabla \cdot F) dV$$

We want to find $\int\int_{\text{Sides}} F \cdot dS$.
The RHS is

$$\int\int\int 3dV = \int\int\int_{\text{inside of cylinder}} 3dV = 3 * (\text{volume of cylinder}) = 24\pi$$

The LHS is

$$\int\int_{\text{Top}} F \cdot ndS + \int\int_{\text{Bottom}} F \cdot ndS + \int\int_{\text{Side}} F \cdot ndS =$$

$$= \int\int_{x^2+y^2\leq 4} 2\mathbf{k} \cdot \mathbf{k} dA + \int\int_{x^2+y^2\leq 4} \mathbf{0} \cdot (-\mathbf{k}) dA + \int\int_{\text{Side}} F \cdot ndS = 8\pi + \int\int_{\text{Side}} F \cdot ndS$$

Hence $\int\int_{\text{Side}} F \cdot ndS = 24\pi - 8\pi = 16\pi$.

5. Find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, z, 1\rangle$ across the hemi-sphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with orientation away from the origin.

Solution: If we do this problem from scratch, we need to start by parametrizing the hemi-sphere:

$$x(\phi, \theta) = 2\sin\phi\cos\theta, \qquad y(\phi, \theta) = 2\sin\phi\sin\theta, \qquad z(\phi, \theta) = 2\cos\phi,$$

where $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$. Then $\mathbf{r}(\phi, \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$, where $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$. And we get

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$$

We now want to check the orientation of the surface. Let $\phi = \pi/4$ and $\theta = \pi/2$, then at the point $(0, \sqrt{2}, \sqrt{2})$, we get the vector $\mathbf{r}_\phi \times \mathbf{r}_\theta(\pi/4, \pi/2) = \langle 0, 2, 2\rangle$ points away from the origin. Thus, our parametrization gives the correct orientation of the surface.

Then, we have the flux of $\mathbf{F}$ across the given hemi-sphere H can be compute using the formula

$$\iint_H \mathbf{F}\cdot d\mathbf{S} = \iint\limits_{\substack{0\le\phi\le\pi/2\\0\le\theta\le 2\pi}} \mathbf{F}\big(\mathbf{r}(\phi,\theta)\big)\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\,dA.$$

$\mathbf{F}\big(\mathbf{r}(\phi,\theta)\big) = \langle 0, 2\cos\phi, 1\rangle$ and

$$\mathbf{F}\big(\mathbf{r}(\phi,\theta)\big)\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 8\sin^2\phi\cos\phi\sin\theta + 4\sin\phi\cos\phi$$

Thus,

$$\begin{aligned}
\iint_H \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2}\big(8\sin^2\phi\cos\phi\sin\theta + 4\sin\phi\cos\phi\big)\,d\phi\,d\theta\\
&= \int_0^{2\pi}\left(\frac{8}{3}\sin^3\phi\Big|_0^{\pi/2}\sin\theta + 2\sin^2\phi\Big|_0^{\pi/2}\right)d\theta\\
&= \int_0^{2\pi}\left(\frac{8}{3}\sin\theta + 2\right)d\theta\\
&= -\frac{8}{3}\cos\theta\Big|_0^{2\pi} + 2\cdot 2\pi\\
&= 4\pi
\end{aligned}$$

Another Solution: If you already know that for a sphere of radius 2 with orientation away from the origin, its unit normal vector is given by $\mathbf{n} = \left\langle \frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z\right\rangle$ and $|\mathbf{r}_\phi\times\mathbf{r}_\theta| = 4\sin\phi$, then we could use the definition of the flux integral to compute $\iint_H \mathbf{F}\cdot d\mathbf{S}$ as follows:

$$\begin{aligned}
\iint_H \mathbf{F}\cdot d\mathbf{S} &= \iint_H \mathbf{F}\cdot\mathbf{n}\,dS\\
&= \iint_D \langle 0, z, 1\rangle\cdot\left\langle \frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z\right\rangle\,dS\\
&= \iint_H \left(\frac{1}{2}yz + \frac{1}{2}z\right)dS\\
&= \iint\limits_{\substack{0\le\phi\le\pi/2\\0\le\theta\le 2\pi}} (2\sin\phi\cos\phi\sin\theta + \cos\phi)\,|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,dA\\
&= \int_0^{2\pi}\int_0^{\pi/2}(2\sin\phi\cos\phi\sin\theta + \cos\phi)\,4\sin\phi\,d\phi\,d\theta\\
&= \int_0^{2\pi}\int_0^{\pi/2}\big(8\sin^2\phi\cos\phi\sin\theta + 4\sin\phi\cos\phi\big)\,d\phi\,d\theta\\
&= 4\pi
\end{aligned}$$

Yet another solution using divergence theorem Again, we cannot apply the divergence theorem directly to the hemisphere because it is not a closed surface. But we can apply the divergence theorem to the hemisphere with the inside of the bottom disk added.

The divergence theorem says

$$\int\int_{\text{Hemisphere}\cup\text{Bottom}} F \cdot dS = \int\int\int_{\text{inside of hemisphere}} (\nabla \cdot F) dV$$

We want to find $\int\int_{\text{Hemisphere}} F \cdot dS$.
The RHS is 0. The LHS is

$$\int\int_{\text{Hemisphere}} F \cdot n dS + \int\int_{\text{Bottom}} F \cdot n dS$$

$$= \int\int_{\text{Hemisphere}} F \cdot n dS + \int\int_{x^2+y^2\leq 4} \mathbf{k} \cdot (-\mathbf{k}) dA = \int\int_{\text{Hemisphere}} F \cdot n dS - 4\pi$$

Hence $\int\int_{\text{Hemisphere}} F \cdot n dS = 4\pi$

Each of the problem below can be solved using one of these theorems: Green's Theorem, Stokes' Theorem, or Divergence Theorem

6. Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented upward. Let $\mathbf{F} = \langle x - y, x + y, ze^{xy} \rangle$. Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. (*Hint*: use one of the theorems you learned in class.)

Solution: This question uses Stokes' theorem: S is a surface with boundary, and we are taking the flux integral of the curl of $\mathbf{F}$.

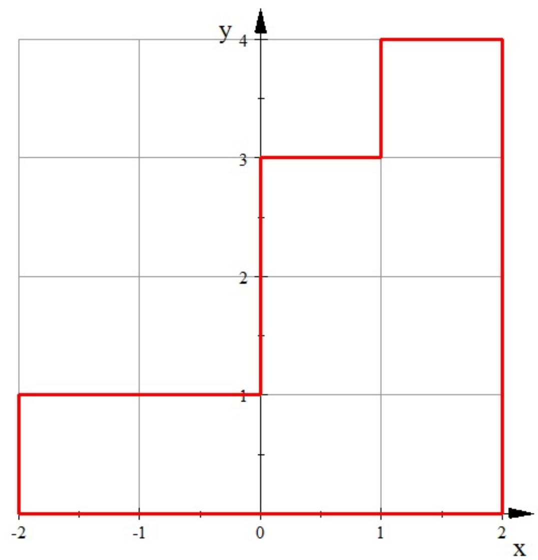
The boundary of S is given by $z = 0, 4x^2 + y^2 = 4$, and since S is oriented with upward orientation, the boundary of S has counterclockwise orientation when viewed from above. Thus, a parametrization of the boundary is given by

$$\mathbf{r}(t) = \langle \cos t, 2 \sin t, 0 \rangle, 0 \leq t \leq 2\pi.$$

Thus, by Stokes' Theorem, we have

$$\begin{aligned}
\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_{\partial S}\mathbf{F}\cdot d\mathbf{r}\\
&= \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt\\
&= \int_0^{2\pi}\langle\cos t-2\sin t,\cos t+2\sin t,0\rangle\cdot\langle-\sin t,2\cos t,0\rangle\,dt\\
&= \int_0^{2\pi}\big(-\sin t\cos t+2\sin^2 t+2\cos^2 t+4\sin t\cos t\big)\,dt\\
&= \int_0^{2\pi}(2+3\sin t\cos t)\,dt=\left(2t+\frac{3}{2}\sin^2 t\right)\Big|_0^{2\pi}\\
&= 4\pi.
\end{aligned}$$

7. Evaluate $\int_C (x^4y^5-2y)dx+(3x+x^5y^4)dy$ where C is the curve below and C is oriented in clockwise direction.



Solution: This problem uses Green's theorem. One main clue is the shape of the curve C (it has 8 pieces!). Let D be the region enclosed by the curve C. And since the orientation of C is clockwise, instead of counterclockwise, we have

$$\begin{aligned}
\int_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy &= -\iint_D \left[(3 + 5x^4y^4) - (5x^4y^4 - 2)\right] dA \\
&= -\iint_D 5\, dA \\
&= -5 \iint_D 1\, dA \\
&= -5 \cdot \text{Area}(D) \\
&= -5 \cdot 9 \\
&= -45.
\end{aligned}$$

8. Let S be the boundary surface of the region bounded by $z = \sqrt{36 - x^2 - y^2}$ and $z = 0$, with outward orientation. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} - 2yz\mathbf{k}$.

Solution: This is a closed surface, so the divergence theorem works nicely here.

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-2yz) = 1 + 2y - 2y = 1$$

Call the solid H (since it's half of a ball). So, the divergence theorem gives

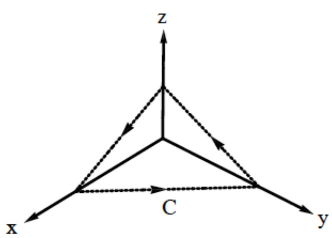
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_H 1\, dV = \text{volume}(H)$$

The solid H is half of the ball of radius 6, and so its volume is

$$\text{volume}(H) = \frac{1}{2}\left(\frac{4}{3}\pi(6)^3\right) = \frac{2}{3}(216\pi) = 144\pi.$$

9. Let C be the boundary curve of the part of the plane $x + y + 2z = 2$ in the first octant. C has counterclockwise orientation when viewing from above. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{\sin x^2}, z, 3y \rangle$.

Solution:



Note: To compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ directly, we need to do 3 integrals since C consists of 3 pieces. But, because we know C is the *boundary curve* of the surface $x+y+2z=2$ in the first octant, we can try to use Stokes' Theorem.

By Stokes' Theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S},$$

where S is the part of the plane $x+y+2z=2$ in the first octant. Since the equation $x+y+2z=2$ or $z = 1-\frac{1}{2}x-\frac{1}{2}y$ determines S, a parametrization of S is given by

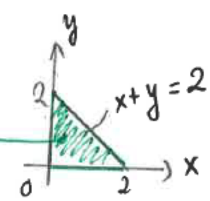
$$\mathbf{r}(x,y) = \left\langle x, y, 1-\tfrac{1}{2}x-\tfrac{1}{2}y\right\rangle, \text{ where } (x,y)\in D.$$

The domain D is given by the projection of S onto the xy-plane:

S: part of the plane $x+y+2z=2$ in the first octant

↳ $\vec{r}(x,y) = \langle x, y, 1-\frac{1}{2}x-\frac{1}{2}y\rangle$, $(x,y)\in D$

projection of S onto the xy-plane



Now, $\vec{r}_x = \langle 1, 0, -\frac{1}{2}\rangle$

$\vec{r}_y = \langle 0, 1, -\frac{1}{2}\rangle$ $\Rightarrow \vec{r}_x\times\vec{r}_y = \langle \frac{1}{2}, \frac{1}{2}, 1\rangle$

↳ upward orientation for S ✓

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{\sin x^2} & z & 3y \end{vmatrix} = \langle 2,0,0\rangle \quad \text{So, } \operatorname{curl}\vec{F}(\vec{r}(x,y)) = \langle 2,0,0\rangle$$

Thus, $\displaystyle\iint_D \operatorname{curl}\vec{F}\cdot(\vec{r}_x\times\vec{r}_y)\,dA = \iint_D \langle 2,0,0\rangle\cdot\langle \tfrac{1}{2},\tfrac{1}{2},1\rangle\,dA = \iint_D 1\,dA = \text{Area}(D) = \boxed{2}$

10. *(A Challenging Problem)* Evaluate

$$\int_C (y^3 + \cos x)dx + (\sin y + z^2)dy + x\,dz$$

where C is the closed curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$ with counterclockwise direction when viewed from above. (*Hint*: the curve C lies on the surface $z = 2xy$.)

Solution: If you rewrite this integral as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and note that the curve C lies in $\mathbb{R}^3$ and not in the plane (otherwise we'd use Green's theorem), we see that Stokes' theorem applies to it. The hint provides the surface to fill in the curve with.

First, we need to parametrize the surface $z = 2xy$:

$$\mathbf{p}(x, y) = \langle x, y, 2xy \rangle, \quad (x, y) \in D = \left\{(x, y) | x^2 + y^2 \leq 1\right\}$$

as the parametrization. C has counterclockwise orientation when viewed from above, so this means that the surface, call it S, we fill it in with must have upward orientation.

$$\begin{aligned} \mathbf{p}_x &= \langle 1, 0, 2y \rangle \\ \mathbf{p}_y &= \langle 0, 1, 2x \rangle \\ \mathbf{p}_x \times \mathbf{p}_y &= \langle -2y, -2x, 1 \rangle \end{aligned}$$

Notice that $\mathbf{p}_x \times \mathbf{p}_y$ points upward, since the $\hat{k}$-component is positive, so this is the correct choice for the orientation. Now, we need the curl of $\mathbf{F}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 + \cos x & \sin y + z^2 & x \end{vmatrix} = \left\langle -2z, -1, -3y^2 \right\rangle$$

Finally, we apply Stokes' Theorem

$$
\begin{aligned}
\int_C (y^3+\cos x)dx + (\sin y + z^2)dy + x\,dz &= \iint_S (\text{curl }\mathbf{F})\cdot d\mathbf{S} \\
&= \iint_D (\text{curl }\mathbf{F})\cdot(\mathbf{p}_x \times \mathbf{p}_y)dA \\
&= \iint_D \left\langle -4xy, -1, -3y^2 \right\rangle \cdot \langle -2y, -2x, 1\rangle \, dA \\
&= \iint_D \left(8xy^2 + 2x - 3y^2\right) dA \\
&= \int_0^{2\pi}\int_0^1 \left(8r^3\cos\theta\sin^2\theta + 2r\cos\theta - 3r^2\sin^2\theta\right) r \;\, drd\theta \\
&= \int_0^{2\pi} \left(\frac{8}{5}r^5\cos\theta\sin^2\theta + \frac{2}{3}r^3\cos\theta - \frac{3}{4}r^4\sin^2\theta\right)\Bigg|_0^1 d\theta \\
&= \int_0^{2\pi} \left(\frac{8}{5}\cos\theta\sin^2\theta + \frac{2}{3}\cos\theta - \frac{3}{4}\sin^2\theta\right) d\theta \\
&= \int_0^{2\pi} \left(\frac{8}{5}\cos\theta\sin^2\theta + \frac{2}{3}\cos\theta - \frac{3}{4}\left(\frac{1-\cos 2\theta}{2}\right)\right) d\theta \\
&= \left(\frac{8}{15}\sin^3\theta + \frac{2}{3}\sin\theta - \frac{3\theta}{8} + \frac{3}{16}\sin 2\theta\right)\Bigg|_0^{2\pi} \\
&= -\frac{3}{4}\pi
\end{aligned}
$$