Name: Date: 09/13/2018

M20550 Calculus III Tutorial
Worksheet 3

1. Find an equation of the tangent line to the space curve $\mathbf{r}(t) = \langle e^t, 3t, \sin t\rangle$ at the point $(e^\pi, 3\pi, 0)$.

> ***Solution:*** First, we want to find t corresponding to the point $(e^\pi, 3\pi, 0)$. The value of t corresponding to $(e^\pi, 3\pi, 0)$ must satisfy the equations
>
> $$e^t = e^\pi, \quad 3t = 3\pi, \quad \sin t = 0.$$
>
> From the second equation, we know $t = \pi$.
>
> Next, we want to find $\mathbf{r}'(\pi)$, the tangent vector at $t = \pi$. The derivative of $\mathbf{r}(t)$ is given by $\mathbf{r}'(t) = \langle e^t, 3, \cos t\rangle$. So the tangent vector at $t = \pi$ is $\mathbf{r}'(\pi) = \langle e^\pi, 3, -1\rangle$.
>
> Then, the vector equation of the tangent line at $(e^\pi, 3\pi, 0)$ is
>
> $$\langle x, y, z\rangle = \langle e^\pi, 3\pi, 0\rangle + t\langle e^\pi, 3, -1\rangle.$$

2. Find the distance from the point $(1, 0, 0)$ to the space curve given by $\mathbf{r}(t) = \langle e^t, \sin t, \cos t\rangle$.

> ***Solution:*** The distance from the point to the curve can be thought of as a function of t in that at each time we can compute the distance from the point to $\mathbf{r}(t)$. This we can write as $D(t) = \sqrt{(e^t-1)^2 + \sin^2 t + \cos^2 t}$. We would like to minimize this quantity, which we can do by looking for critical points using its derivative. We also note that minimizing $D(t)$ also minimizes $D(t)^2$ and vice versa, so we compute
>
> $$\frac{d}{dt}(D(t)^2) = \frac{d}{dt}((e^t-1)^2 + 1) = 2e^t(e^t - 1)$$
>
> Since there is only one critical point, and the distance is certainly unbounded as t gets large, we reach the minimum distance at $t = 0$ and see
>
> $$D(0) = \sqrt{(e^0-1)^2 + \sin^2 0 + \cos^2 0} = \sqrt{1} = 1$$
>
> is the distance from the point to the curve.

3. Find $\mathbf{r}(t)$ if $\mathbf{r}''(t) = 2\sec^2 t \tan t\mathbf{i}$, $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, and $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution:

$$\mathbf{r}'(t) = \int \mathbf{r}''(t)\,dt = \int \left\langle 2\sec^2 t \tan t, 0, 0 \right\rangle\,dt = \left\langle \sec^2 t, 0, 0 \right\rangle + \ \mathbf{c}.$$

To find $\mathbf{c}$, we use the information $\mathbf{r}'(0) = \langle 1,1,1\rangle$. From the above, we have $\mathbf{r}'(0) = \langle \sec^2(0), 0, 0\rangle + \mathbf{c}$. So, $\langle \sec^2(0), 0, 0\rangle + \mathbf{c} = \langle 1,1,1\rangle \implies \mathbf{c} = \langle 1,1,1\rangle - \langle 1,0,0\rangle = \langle 0,1,1\rangle$. Thus, we get

$$\mathbf{r}'(t) = \left\langle \sec^2 t, 0, 0 \right\rangle + \langle 0,1,1\rangle \implies \mathbf{r}'(t) = \left\langle \sec^2 t, 1, 1 \right\rangle.$$

Then

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \left\langle \sec^2 t, 1, 1 \right\rangle dt = \langle \tan t, t, t\rangle + \mathbf{d}.$$

To find $\mathbf{d}$, we use the information $\mathbf{r}(0) = \langle 2,3,2\rangle$. We have $\mathbf{r}(0) = \langle \tan 0, 0, 0\rangle + \mathbf{d} = \langle 2,3,2\rangle$. So, $\mathbf{d} = \langle 2,3,2\rangle - \langle \tan 0, 0, 0\rangle = \langle 2,3,2\rangle$.

Finally, we get

$$\mathbf{r}(t) = \langle \tan t, t, t\rangle + \langle 2,3,2\rangle \implies \mathbf{r}(t) = \langle \tan t + 2, t+3, t+2\rangle.$$

4. Find the unit tangent vector, the principal unit normal vector, and the unit binormal vectors to the curve $\mathbf{r}(t) = \langle \cos 3t, \sin 2t, t^3\rangle$ at $t = \pi$.

Solution: We have $\mathbf{r}(t) = \langle \cos 3t, \sin 2t, t^3\rangle$. So

$$\mathbf{r}'(t) = \left\langle -3\sin 3t, 2\cos 2t, 3t^2 \right\rangle \implies \mathbf{r}'(\pi) = \left\langle 0, 2, 3\pi^2 \right\rangle.$$

$$\mathbf{r}''(t) = \langle -9\cos 3t, -4\sin 2t, 6t\rangle \implies \mathbf{r}''(\pi) = \langle 9, 0, 6\pi\rangle.$$

Also,

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \left\langle 0, 2, 3\pi^2 \right\rangle \times \langle 9, 0, 6\pi\rangle = \left\langle 12\pi, 27\pi^2, -18 \right\rangle.$$

Then,

$$\mathbf{T}(\pi) = \frac{\mathbf{r}'(\pi)}{|\mathbf{r}'(\pi)|} = \frac{\langle 0, 2, 3\pi^2\rangle}{|\langle 0, 2, 3\pi^2\rangle|} = \frac{1}{\sqrt{4 + 9\pi^4}} \left\langle 0, 2, 3\pi^2 \right\rangle.$$

$$\mathbf{B}(\pi) = \frac{\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)}{|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)|} = \frac{\langle 12\pi, 27\pi^2, -18\rangle}{|\langle 12\pi, 27\pi^2, -18\rangle|}$$

$$= \frac{1}{3\sqrt{36 + 16\pi^2 + 81\pi^4}} \left\langle 12\pi,\ 27\pi^2,\ -18 \right\rangle.$$

$$\begin{aligned}\mathbf{N}(\pi) &= \mathbf{B}(\pi)\times\mathbf{T}(\pi)\\ &= \frac{1}{3\sqrt{36+16\pi^2+81\pi^4}}\frac{1}{\sqrt{4+9\pi^4}}\left\langle 12\pi,\ 27\pi^2,\ -18\right\rangle \times \left\langle 0,2,3\pi^2\right\rangle\\ &= \frac{1}{3\sqrt{36+16\pi^2+81\pi^4}}\frac{1}{\sqrt{4+9\pi^4}}\left\langle 36+81\pi^4, -36\pi^3, 24\pi\right\rangle.\end{aligned}$$

5. Find the equation for the normal and osculating planes to the curve $\mathbf{r}(t) = \arctan t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$ at the point $(0,0,1)$. **Challenge:** Without graphin' software, sketch the curve. Can you describe the limit as $t \to \pm\infty$?

Solution: First, we note that the point $(0,0,1)$ corresponds to $t=0$ since $\arctan(0) = 0$.

A normal vector of the normal plane at $t=0$ is $\mathbf{r}'(0)$. We have

$$\mathbf{r}'(t) = \left\langle (1+t^2)^{-1}, \cos t, -\sin t\right\rangle \implies \mathbf{r}'(0) = \langle 1,1,0\rangle.$$

So, the normal plane at the point $(0,0,1)$ is given by

$$\langle 1,1,0\rangle \cdot \langle x,y,z\rangle = \langle 1,1,0\rangle \cdot \langle 0,0,1\rangle \implies x+y=0.$$

A normal vector of the osculating plane at $t=0$ is $\mathbf{r}'(0)\times\mathbf{r}''(0)$. We have, $\mathbf{r}''(t) = \left\langle \frac{-2t}{(1+t^2)^2}, -\sin t, -\cos t\right\rangle$ and so $\mathbf{r}''(0) = \langle 0,0,-1\rangle$. Then,

$$\mathbf{r}'(0)\times\mathbf{r}''(0) = \langle 1,1,0\rangle \times \langle 0,0,-1\rangle = \langle -1,1,0\rangle.$$

So, we can take $\langle -1,1,0\rangle$ to be a normal vector for this osculating plane. And the equation is

$$\langle -1,1,0\rangle \cdot \langle x,y,z\rangle = \langle -1,1,0\rangle \cdot \langle 0,0,1\rangle \implies y-x=0.$$

6. Find the length of the curve $\mathbf{r}(t) = \langle \sin t, \cos t, 2t\rangle$ from $(0,1,0)$ to $(0,1,4\pi)$.

Solution:

First, we need the derivative:

$$\mathbf{r}'(t) = \langle \cos t, -\sin t, 2\rangle$$

and its magnitude

$$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + 4} = \sqrt{1+4} = \sqrt{5}.$$

And now, the point $(0,1,0)$ corresponds to $t=0$ and the point $(0,1,4\pi)$ correspond to $t=2\pi$. Then, we have the length of $\mathbf{r}$ is

$$L = \int_{t=0}^{t=2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{5}dt = 2\sqrt{5}\pi.$$

7. A particle moves with position function $\mathbf{r}(t) = \left\langle \sin t, \cos t, \sin^2 t \right\rangle$. Find the tangential and normal components of acceleration when $t = \pi/4$.

Solution: We have

$$\mathbf{r}'(t) = \langle \cos t, -\sin t, 2\sin t\cos t\rangle = \langle \cos t, -\sin t, \sin(2t)\rangle$$

$$\implies \mathbf{r}'(\pi/4) = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1 \right\rangle,$$

$$\mathbf{r}''(t) = \langle -\sin t, -\cos t, 2\cos(2t)\rangle \implies \mathbf{r}''(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle.$$

And so

$$a_T = \frac{\mathbf{r}'(\pi/4) \bullet \mathbf{r}''(\pi/4)}{|\,\mathbf{r}'(\pi/4)\,|} = 0.$$

We know $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. Since $a_T = 0$, we get $\mathbf{a} = a_N\mathbf{N}$. So,

$$|\mathbf{a}| = a_N|\mathbf{N}| = a_N \cdot 1 = a_N.$$

Thus,

$$\begin{aligned} a_N = |\mathbf{a}| &= |\,\mathbf{r}''(\pi/4)\,| \\ &= \left| \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle \right| \\ &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\ &= 1 \end{aligned}$$