Name: Date: 11/01/2018

M20550 Calculus III Tutorial
Worksheet 7

1. Using spherical coordinates, compute the volume, $V(R)$ of a sphere of radius R.

Solution: This is equivalent to just computing

$$\iiint_{Sphere} dV$$

(intuitively, we are summing up the volumes of infinitely many infinitesimally small boxes of volume "dV" inside the sphere.) Recall that the standard spherical coordinates are

$$(x, y, z) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

for $(\rho, \theta, \phi) \in [0, R] \times [0, 2\pi) \times (0, \pi)$ and the volume element of the sphere with respect to these coordinates is given by $dV = \rho^2 sin\phi d\theta d\phi d\rho$. So,

$$\begin{aligned} V(R) = & \int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 sin\phi d\theta d\phi d\rho \\ =& 2\pi \int_0^R \int_0^\pi \rho^2 sin\phi d\phi d\rho \\ =& 4\pi \int_0^R \rho^2 d\rho \\ =& \frac{4}{3}\pi R^3 \end{aligned}$$

2. Now compute the surface area, $A(R)$, of a sphere of radius R. Hint: Recall the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

And recall the common problem from single variable calculus where you have to find the volume of a water tank of height h by integrating the cross sectional area, $A(y)$, over the height.

$$Volume(Tank) = \int_0^h A(y)dy$$

We have a similar formula for the volume of the sphere;

$$V(R) = \int_0^R A(\rho)d\rho.$$

Solution: Let $A(\rho)$ be the surface area of the sphere of radius ρ, we wish to find $A(R)$. Observe

$$\int_0^R A(\rho)d\rho = V(R) = \frac{4}{3}\pi R^3$$

So by the fundamental theorem of calculus, we get

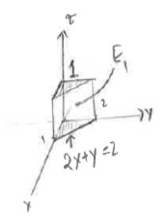
$$A(R) = \frac{d}{dR}\left[\int_0^R A(\rho)d\rho\right] = \frac{dV(R)}{dR} = 4\pi R^2.$$

Another way to solve this problem is to realize through geometric intuition or by reasoning similar to the argument above that

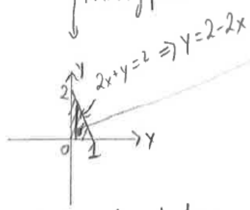
$$A(R) = \int_0^{\pi}\int_0^{2\pi} R^2 sin\phi d\theta d\phi.$$

3. (a) Let E_1 be the solid that lies under the plane $z = 1$ and above the region in the xy-plane bounded by $x = 0$, $y = 0$, and $2x + y = 2$. Write the triple integral $\iiint_{E_1} xz\, dV$ but do not evaluate it.

Solution:



in the xy-plane



With the order $dy\,dx$:

$0 \le y \le 2-2x$

$0 \le x \le 1$

We'll use rectangular coordinates to write $\iiint_{E_1} xz\, dV$.

$$\iiint_{E_1} xz\, dV = \iint_R \left(\int_{z=0}^{z=1} xz\, dz\right) dA$$

(now write $dA = dy\,dx$, and the limits for y and x comes from the picture of R)

$$= \boxed{\int_{x=0}^{x=1}\int_{y=0}^{y=2-2x}\int_{z=0}^{z=1} xz\, dz\, dy\, dx}$$

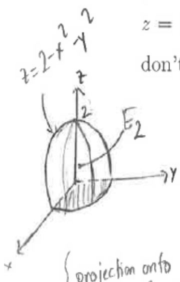
$\left(\text{Another answer is } \int_{y=0}^{y=2}\int_{x=0}^{x=\frac{2-y}{2}}\int_{z=0}^{z=1} xz\, dz\, dx\, dy\right)$

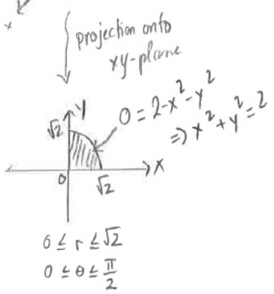
(b) Let E_2 be the solid region in the first octant that lies under the paraboloid

$z = 2 - x^2 - y^2$. Write the triple integral $\iiint_{E_2} xz\,dV$ in cylindrical coordinates (you don't need to evaluate it).

Solution:

(b) Let E_2 be the solid region in the first octant that lies under the paraboloid $z = 2 - x^2 - y^2$. Write the triple integral $\iiint_{E_2} xz\,dV$ in cylindrical coordinates (you don't need to evaluate it).





In cylindrical coordinates, $dV = r\,dz\,dr\,d\theta$

From the picture of E_2, we see that $0 \le z \le 2 - x^2 - y^2 = 2 - r^2$

To get the bounds for r and θ, we look at the projection of the solid E_2 onto the xy-plane. We see that

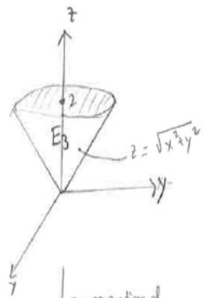
$0 \le r \le \sqrt{2}$ and $0 \le \theta \le \frac{\pi}{2}$.

So, $$\iiint_{E_2} xz\,dV = \int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{z=0}^{z=2-r^2} \underbrace{(r\cos\theta)}_{\substack{= \\ x \text{ in cylindrical coord.}}} z\, r\,dz\,dr\,d\theta$$

$$= \boxed{\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_0^{2-r^2} z r^2 \cos\theta\,dz\,dr\,d\theta}$$

(c) Let E_3 be the solid region that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$. Write the triple integral $\iiint_{E_3} xz\,dV$ in spherical coordinates (you don't need to evaluate it).

Solution:



In spherical coord., $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ and $x = \rho \sin\phi \cos\theta$

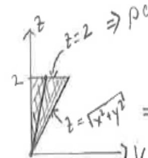
$z = \rho\cos\phi$

We can easily see that $0 \le \theta \le 2\pi$. Now, in order to get the bounds for ρ and ϕ, we can look at the cross section of the solid E_3 with the positive yz-plane (see picture on left). Converting the approriate equations to spherical coord., we get

$$0 \le \rho \le \frac{2}{\cos\phi} \quad \text{and} \quad 0 \le \phi \le \frac{\pi}{4}$$

(recall, ϕ is measured from the positive z-axis)

cross section of E_3 with the positive yz-plane



$z = 2 \Rightarrow \rho\cos\phi = 2$

$\Rightarrow \rho = \frac{2}{\cos\phi}$

$z = \sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = \rho\sin\phi$

$\Rightarrow \cos\phi = \sin\phi$

$\Rightarrow \phi = \frac{\pi}{4}$

$0 \le \rho \le \frac{2}{\cos\phi}$

$0 \le \phi \le \frac{\pi}{4}$

Thus,

$$\iiint_{E_3} xz \, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\frac{2}{\cos\phi}} (\rho\sin\phi\cos\theta)(\rho\cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \boxed{\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\frac{2}{\cos\phi}} \rho^4 \sin^2\phi \cos\phi \cos\theta \, d\rho \, d\phi \, d\theta}$$

4. Write the integral that computes the volume of the part of the solid cylinder $x^2 + y^2 \le 1$ that lies between the planes $z = 0$ and $z = 2 - y$.

Solution: This is best done in cylindrical coordinates,

$$\iiint_R dV = \int_0^1 \int_0^{2\pi} \int_0^{2 - r\sin\theta} r dz d\theta dr.$$

5. Find the mass of the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ whose density is $\delta(x, y, z) = x^2 + y^2 + z^2$.

Solution: Let E be the solid in consideration. Now, to find the mass, we simply integrate the density function over the entire solid to get;

$$\begin{aligned}\int_1^2\int_0^\pi\int_0^{2\pi}\delta(\rho)\rho^2 sin\phi d\theta d\phi d\rho &= \int_1^2 A(\rho)\delta(\rho)d\rho\\ &= \int_1^2 4\pi\rho^2\rho^2 d\rho\\ &=4\pi\frac{\rho^5}{5}\bigg|_1^2\\ &=4\pi(\frac{32}{5}-\frac{1}{5})\\ &=\frac{124\pi}{5}.\end{aligned}$$

Note: The fact that the density only depended on ρ simplified our work here.

6. Find the center of mass of the solid S bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$ if S has constant density 1 and total mass $\frac{\pi}{2}$. (Hint: $\overline{x}$ and $\overline{y}$ can be found by symmetry of the solid being considered).

Solution: Since the density is constantly 1, we just need to compute the average values of x, y and z inside this solid. Because the solid is rotationally symmetric about the z-axis, we get $\overline{x} = \overline{y} = 0$. Now we compute

$$\begin{aligned}\overline{z} =&\frac{2}{\pi}\int_0^1\int_0^{2\pi}\int_0^{\sqrt{z}} zrdrd\theta dz\\ =&\frac{2}{\pi}\int_0^1\int_0^{2\pi} z\frac{r^2}{2}\bigg|_0^{\sqrt{z}} d\theta dz\\ =&\frac{2}{\pi}\int_0^1\int_0^{2\pi}\frac{z^2}{2}d\theta dz\\ =&2\int_0^1 z^2 dz\\ =&\frac{2}{3},\end{aligned}$$

so the center of mass is given by $(0, 0, \frac{2}{3})$.

7. In this problem, we are going to calculate the same integral in two different ways by changing coordinates. Compute the following integral;

$$\int_0^1\int_0^1 x^3 y dx dy$$

first, by making the coordinate change $u = x^2, v = xy$, and then as you normally would. (Don't forget to multiply by the Jacobian!)

Solution:

We first compute the Jacobian;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ \frac{-v}{u^{\frac{3}{2}}} & \frac{1}{\sqrt{u}} \end{vmatrix} = \frac{1}{2u}$$

(note: u is always positive so we don't need to take the absolute value) now, we know by the change of variables formula that

$$\int_0^1 \int_0^1 x^3 y dx dy = \int_0^1 \int_0^{\sqrt{u}} uv \frac{1}{2u} dv du = \int_0^1 \frac{v^2}{4}\bigg|_{v=0}^{v=\sqrt{u}} du = \int_0^1 \frac{u}{4} du = \frac{1}{8}.$$

If we compute this integral in the usual way, we get;

$$\int_0^1 \int_0^1 x^3 y dx dy = \int_0^1 \frac{y}{4} dy = \frac{1}{8}.$$