Math 20580 Name:______________________________

Midterm 2 Instructor:______________________________

October 26, 2017 Section:______________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. a b ~~c~~ d e
2. a b c ~~d~~ e
3. a b ~~c~~ d e
4. a b c d ~~e~~
5. ~~a~~ b c d e
6. a ~~b~~ c d e
7. a b c ~~d~~ e
8. a b c d ~~e~~

____

Multiple Choice.

9.

10.

11.

12.

____

Total.

Part I: Multiple choice questions (7 points each)

1. Find the determinant of the matrix

$$\begin{bmatrix} 1 & 11 & 0 & 2 \\ 0 & -3 & 0 & 0 \\ 4 & -9 & 6 & 12 \\ 2 & -20 & 0 & 3 \end{bmatrix}$$

(a) +25 (b) -0 (c) 18 (d) -25 (e) -18

$$(-3) \cdot \det \begin{bmatrix} 1 & 0 & 2 \\ 4 & 6 & 12 \\ 2 & 0 & 3 \end{bmatrix} = (-3) \cdot 6 \cdot \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= (-3) \cdot 6 \cdot (-1) \cdot$$

$$= 18$$

2. Find the matrix of change of coordinates $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ between the following bases of $\mathbb{R}^2$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

(a) $\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3. Consider the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ and let H be the subspace of $C(\mathbb{R})$ spanned by the functions $\{1, \sin^2 t, \cos^2 t, \sin t \cos t, \sin 2t, \cos 2t\}$. What is the dimension of H?

(a) 1 (b) 2 (c) 3 (d) 4 (e) H is infinite-dimensional

Hint. You may use the trig identities: $\sin 2t = 2 \sin t \cos t$ $\cos 2t = 2\cos^2 t - 1$.

$\{1, \sin^2 t, \sin t \cos t\}$ are linearly independent

and span H: • $\cos^2 t = 1 - \sin^2 t$

• $\sin 2t = 2 \cdot (\sin t \cos t)$

• $\cos 2t = 1 - 2 \cdot \sin^2 t$

So dim $= 3$

4. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

(a) 0,0,2 (b) 0,1,2 (c) 2,2,2 (d) 2,4,6 (e) 0,2,4

$$\det \begin{bmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda) \cdot \det \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left((2-\lambda)^2 - 4\right)$$

$$= (2-\lambda)\left(\cancel{4} - 4\lambda + \lambda^2 - \cancel{4}\right)$$

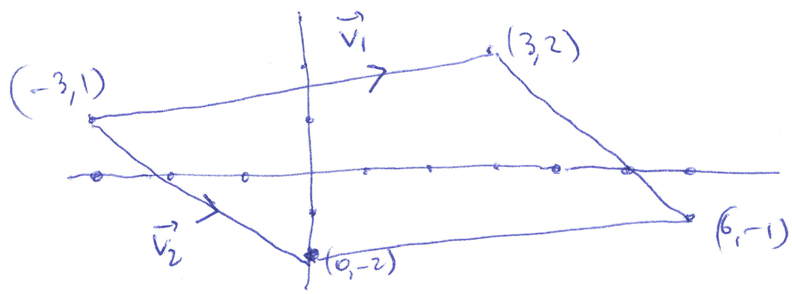
$$= (2-\lambda) \cdot \lambda \cdot (\lambda - 4)$$

is $= 0$ when $\lambda = 0, 2, 4$

5. Find the area of the parallelogram whose vertices are

$$(0,-2),\ (6,-1),(-3,1),(3,2).$$

(a) 21 (b) 15 (c) 12 (d) 3 (e) 6



$\vec{V}_1 = \begin{bmatrix} 3-(-3) \\ 2-1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$\vec{V}_2 = \begin{bmatrix} 0-(-3) \\ -2-1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$$\text{Area} = \left| \det \begin{bmatrix} 6 & 3 \\ 1 & -3 \end{bmatrix} \right| = |-21| = 21$$

6. Which of the following statements are always true?

× I. Row-equivalent matrices have the same characteristic equations.

✓ II. Similar matrices have the same eigenvalues.

× III. The determinant of a square matrix is equal to the product of the diagonal entries.

(a) I. is true but II. and III. are false
(b) II. is true but I. and III. are false
(c) III. is true but I. and II. are false
(d) All of them are true
(e) None of them are true

I. fails because: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to 2\cdot R_1} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

are row equivalent

but eigenvalues 1,0 respectively 2,0.

III fails because $\det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3$

product of diagonal entries is $2\cdot 2 = 4$

II holds: if $B = PAP^{-1}$ then

$B - \lambda \cdot I = P(A-\lambda I)P^{-1} \Rightarrow \det(B-\lambda I) = \cancel{\det P} \cdot \det(A-\lambda I) \cdot \cancel{\det P^{-1}}$

Same characteristic equation $\Rightarrow$ same eigenvalues.

7. The vector $\vec{v} = \begin{bmatrix} -1+3i \\ 2 \end{bmatrix}$ is a complex eigenvector of the matrix $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$.
What is the corresponding complex eigenvalue?

(a) $3+2i$ (b) $3-4i$ (c) 2 (d) $4+3i$ (e) $5+5i$

$$A \cdot \vec{v} = \begin{bmatrix} 3(-1+3i) - 5\cdot 2 \\ 2(-1+3i) + 5\cdot 2 \end{bmatrix} = \begin{bmatrix} \cdots \\ 8+6i \end{bmatrix} = \lambda \cdot \begin{bmatrix} -1+3i \\ 2 \end{bmatrix}$$

$$\Rightarrow 2\lambda = 8+6i$$

$$\Rightarrow \lambda = 4+3i$$

8. Consider the following basis of $\mathbb{R}^3$ consisting of orthogonal vectors:

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}_{\vec{u}_2}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}_{\vec{u}_3} \right\}$$

Find the $\mathcal{B}$-coordinate vector $[\vec{v}]_{\mathcal{B}}$ where $\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1/5 \\ -2/5 \\ 1/5 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} 4/9 \\ -7/9 \\ -4/9 \end{bmatrix}$

$$\vec{v} = \frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} \cdot \vec{u}_1 + \frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} \vec{u}_2 + \frac{\vec{v}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} \vec{u}_3$$

$$= \frac{4}{9} \vec{u}_1 + \frac{(-7)}{9} \vec{u}_2 + \frac{(-4)}{9} \vec{u}_3$$

So $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 4/9 \\ -7/9 \\ -4/9 \end{pmatrix}$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 1 & 1 & 0 & 0 \\ -3 & -3 & 4 & 4 \end{bmatrix}$$

(a) Find a basis for Row(A) (the row space of A).

Reduced echelon form of A is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Row(A) is

$\{(1,1,0,0), (0,0,1,1)\}$

(b) Determine the rank of A and the dimension of the null space of A.

$\text{rank } A = \dim \text{Row } A = 2$

$$\begin{aligned} \dim(\text{Nul } A) &= \#\text{ columns} - \text{rank } A \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

(c) Give an example of a non-zero unit vector which is orthogonal to Row(A).

$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ is in Nul A, so $\vec{v}$ is orthogonal to Row(A)

but it is NOT a unit vector.

Take $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{2}}\vec{v} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$.

10. Consider the vector space $\mathbb{P}_2$ of polynomials of degree at most two, and the transformation $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ given by

$$T(p(t)) = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}.$$

(a) Show that $\mathcal{B} = \{1 + t^2, 2 - t, (1 + t)^2\}$ is a basis of $\mathbb{P}_2$.

Relative to the basis $\mathcal{C} = \{1, t, t^2\}$ of $\mathbb{P}_2$ we have

$$[1+t^2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad [2-t]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad [(1+t)^2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

so it suffices to check that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = 4 \neq 0$$ So indeed we have a basis.

(b) Find the matrix of T relative to the basis $\mathcal{B}$ of $\mathbb{P}_2$ from part (a) and the standard basis of $\mathbb{R}^3$ (you may use that T is a linear transformation without explaining why).

$$T(1+t^2) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \qquad T(2-t) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad T((1+t)^2) = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$$

So the matrix is

$$M = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$$

(c) Suppose that $p(t)$ is a polynomial whose $\mathcal{B}$-coordinate vector is $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Find $p(t)$ and $T(p(t))$.

We have
$$\begin{aligned} p(t) &= 1\cdot(1+t^2) - 1\cdot(2-t) + 1\cdot(1+t)^2 \\ &= 3t + 2t^2 \end{aligned}$$

$$T(p(t)) = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -1 & 4 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

Can also use $T(p(t)) = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix} = \begin{bmatrix} 3+2 \\ 3+4 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$.

11. Consider the matrix $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$. Determine whether A is diagonalizable or not.

Eigenvalues

$$\det \begin{bmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda) - (-1)$$
$$= 9 - 6\lambda + \lambda^2$$
$$= (3-\lambda)^2$$

So $\lambda = 3$ is the only eigenvalue, with multiplicity 2.

Eigenspaces

$$E_3 = \mathrm{Nul}(A - 3I) = \mathrm{Nul} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$
$$= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Since $\dim E_3 = 1 < 2$

the matrix A is NOT diagonalizable.

12. Consider the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find the orthogonal projection of $\vec{v}$ onto $L = \text{Span}\{\vec{u}\}$.

$$\hat{v} = \text{proj}_L \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u} = \frac{7}{14} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 3/2 \end{bmatrix}$$

(b) Find the distance from $\vec{v}$ to L.

$$\vec{v} - \hat{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -1 \\ -1/2 \end{bmatrix}$$

$$\|\vec{v} - \hat{v}\| = \sqrt{(7/2)^2 + (-1)^2 + (-1/2)^2}$$

$$= \sqrt{\frac{49}{4} + 1 + \frac{1}{4}}$$

$$= \sqrt{\frac{54}{4}}$$

$$= \frac{3\sqrt{6}}{2}$$