Math 20580 Name:____________________

Midterm 1 Instructor:____________________

September 19, 2017 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. a b ~~c~~ d e
2. a b c ~~d~~ e
3. a ~~b~~ c d e
4. a b ~~c~~ d e
5. a b ~~c~~ d e
6. ~~a~~ b c d e
7. a b ~~c~~ d e
8. a ~~b~~ c d e

____

Multiple Choice.

9.

10.

11.

12.

____

Total.

Part I: Multiple choice questions (7 points each)

1. Consider the linear system

$$\begin{cases} x_1 - 3x_2 = 5 \\ x_2 + x_3 = 0 \end{cases}$$

Which of the following (x_1, x_2, x_3) is a solution of the system?

(a) $(-3,-1,1)$ and $(2,-1,1)$ (b) $(-1,-2,2)$ and $(-3,-1,1)$

(c) $(2,-1,1)$ and $(-1,-2,2)$ (d) $(-5,0,0)$ and $(3,1,-1)$

(e) none of the above

$$\begin{cases} 2-3(-1)=5 \\ (-1)+1=0 \end{cases} \qquad \begin{cases} (-1)-3(-2)=5 \\ (-2)+2=0 \end{cases}$$

2. For which values of h and k is the matrix below in reduced echelon form?

$$A = \begin{bmatrix} 1 & 2 & h & 1 \\ 0 & 0 & k & -2 \end{bmatrix}$$

(a) $h = 1$ and $k = 0$ (b) $h = 1$ and any k

(c) $k = 1$ and any h (d) $h = 0$ and $k = 1$

(e) none of the above

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

3. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$$

Which of the following statements are true?

A. $\{\vec{v}_1, \vec{v}_2\}$ are linearly dependent.

B. $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ are linearly independent.

C. $\vec{v}_4$ is in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(a) A,B only (b) A,C only (c) B,C only (d) A,B,C (e) A only

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 2 & 4 \\ 3 & 6 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{v_2} = 2\vec{v_1}$

So (A) holds

no pivot

So (C) holds

$$[\vec{v_2} \ \vec{v_3} \ \vec{v_4}] \to \dots \to \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

no pivot

So linearly dependent (B) fails

4. Find the product AB where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1\cdot1 - 1\cdot1 + 2\cdot1 & 1\cdot0 - 1\cdot1 + 2\cdot0 \\ 1\cdot1 + 2\cdot1 + 1\cdot1 & 1\cdot0 + 2\cdot1 + 1\cdot0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$$

5. Which of the following matrices is invertible?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$$

(a) A only (b) A,B,C only (c) A,B only (d) D only (e) B, C only

$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix}$ invertible

$B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ invertible

C NOT Square, so NOT invertible

$D = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ NOT invertible

no pivot

6. Consider the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, and a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with the property that

$$T(\vec{u}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\vec{v}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The image of the vector $\vec{u} + 2\vec{v}$ under the transformation T is

(a) $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} 9 \\ 6 \end{bmatrix}$ (e) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$T(\vec{u} + 2\vec{v}) = T(\vec{u}) + 2 \cdot T(\vec{v})$$
$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

7. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ be a basis for a subspace of H in $\mathbb{R}^4$ where

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \\ -1 \end{bmatrix}.$$

If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is the coordinate vector (relative to $\mathcal{B}$) of some element $\vec{x}$ in H then

(a) $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$ (b) $\vec{x}$ is in $\mathbb{R}^2$ (c) $\vec{x} + \vec{b}_2 = 2\vec{b}_1$

(d) $\vec{x}, \vec{b}_1, \vec{b}_2$ are linearly independent (e) none of the above.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ means $\vec{x} = 2\vec{b}_1 - \vec{b}_2$

which is the same as $\vec{x} + \vec{b}_2 = 2\vec{b}_1$

8. The ranks of the matrices

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & 7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $C = 3$

are given by

(a) $\text{rank}(A) = 2$, $\text{rank}(B) = 3$, $\text{rank}(C) = 3$

(b) $\text{rank}(A) = 1$, $\text{rank}(B) = 2$, $\text{rank}(C) = 3$.

(c) $\text{rank}(A) = 2$, $\text{rank}(B) = 3$, $\text{rank}(C) = 4$.

(d) $\text{rank}(A) = 1$, $\text{rank}(B) = 3$, $\text{rank}(C) = 3$.

(e) $\text{rank}(A) = 2$, $\text{rank}(B) = 2$, $\text{rank}(C) = 3$.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $A = 1$

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so rank $B = 2$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ t \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -t \end{bmatrix}.$$

Find the values of t for which $\vec{v}_1$ is contained in $\operatorname{Span}\{\vec{v}_2, \vec{v}_3\}$.

$$\left[\begin{array}{cc|c} 2 & 0 & 1 \\ 1 & 1 & 0 \\ t & -t & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 1 \\ t & -t & 1 \end{array}\right] \longrightarrow$$

$$\xrightarrow[R_3 \to R_3 - tR_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2t & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - tR_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1-t \end{array}\right]$$

$\vec{v}_1$ is in $\operatorname{Span}\{\vec{v}_2, \vec{v}_3\}$ precisely when the last column contains no pivot, that is $1-t=0$, or equivalently

$$\boxed{t=1}$$

10. Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by

$$T(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ -x_1 \\ 2x_1 + 3x_2 \\ x_1 + 2x_2 \end{bmatrix}.$$

(a) Find the standard matrix of T.

$$T(x_1,x_2) = x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$ so the standard matrix is

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

(b) Write down four distinct vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ that are in the range of T.

$$\vec{v}_1 = T(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = T(1,0) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = T(0,1) = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \qquad \vec{v}_4 = T(1,1) = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 3 \end{bmatrix}$$

(c) Is the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in the range of T? $\Longleftrightarrow$ Is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in $Col(A)$?

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & -2 \end{array}\right]$$

pivot in the last column, so NO!

$\vec{v}$ is NOT in the range of T.

11. Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} -1 & 0 & -2 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_3 \to R_3 + 3R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & -2 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_3} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 7 & 0 & -5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_1 \to -R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 \end{array}\right]$$

A^{-1}

$$A^{-1} = \begin{bmatrix} -7 & 0 & 5 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

12. Find a basis for Col(A) and a basis for Nul(A) where

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 & 1 \\ 2 & -4 & 1 & 4 & 1 \\ -1 & 2 & -1 & -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \textcircled{1} & -2 & 0 & 1 & 1 \\ 2 & -4 & 1 & 4 & 1 \\ -1 & 2 & -1 & -3 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} \textcircled{1} & -2 & 0 & 1 & 1 \\ 0 & 0 & \textcircled{1} & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix} \to$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} \textcircled{1} & -2 & 0 & 1 & 1 \\ 0 & 0 & \textcircled{1} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col(A) $\longleftrightarrow$ pivot columns, namely 1^{st} and 3^{rd}

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Basis for Nul A solve $A\vec{x} = \vec{0}$.

x_2, x_4, x_5 free variables

$x_2 = s$
$x_4 = t$
$x_5 = u$

$x_3 + 2x_4 - x_5 = 0 \Rightarrow x_3 = u - 2t$

$x_1 - 2x_2 + x_4 + x_5 = 0 \Rightarrow x_1 = 2s - t - u$

Parametric vector form

$$\vec{x} = \begin{bmatrix} 2s-t-u \\ s \\ u-2t \\ t \\ u \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$ so

basis for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$