Math 20580 Name:____________________
Midterm 3 Instructor:____________________
November 14, 2017 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c d e
2. a b c d e
3. a b c d e
4. a b c d e
5. a b c d e
6. a b c d e
7. a b c d e
8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

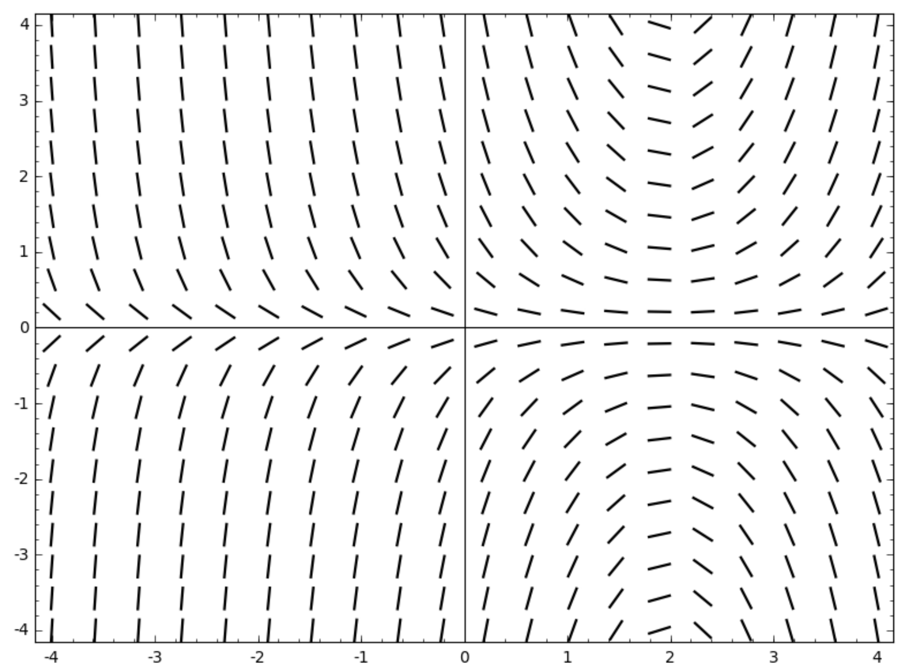
Part I: Multiple choice questions (7 points each)

1. Which of the following functions is a solution of the initial value problem

$$(y' - \sin x)^2 = 1 + x^2 - y^2, \quad y(0) = 1$$

(a) $-\sin x$ (b) $x \sin x + \cos x$ (c) $\cos x$ (d) $x \cos x - \sin x$ (e) $\sin x - \cos x$

2. Determine $f(t, y)$ if the differential equation $y' = f(t, y)$ has direction field (the value of t is measured on the horizontal axis, and the value of y on the vertical axis)



(a) $\sin(t) + y$ (b) $y + t^2$ (c) $t \sin(y)$ (d) $ty - 2y$ (e) $e^y(t - 1)$

3. Consider the orthogonal vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ and let $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. The matrix of the projection onto V is

(a) $\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1/9 & -2/9 & 0 \\ 2/9 & 2/9 & 0 \\ 2/9 & -1/9 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -2 \\ 2 & 2 \\ 2 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 5/9 & -2/9 & 4/9 \\ -2/9 & 8/9 & 2/9 \\ 4/9 & 2/9 & 5/9 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & 1 \end{bmatrix}$

4. Let A be an $m \times n$ matrix with linearly independent columns and let $\vec{b}$ in $\mathbb{R}^m$ be a vector which is not in $Col(A)$. Which of the following statements may be false?

(a) There exists a vector $\vec{x}$ in $\mathbb{R}^n$ with $A\vec{x} - \vec{b}$ perpendicular to $Col(A)$.

(b) $\det(A^T A) \neq 0$.

(c) $m > n$.

(d) The vector $\vec{b}$ is not the zero vector.

(e) $\det(AA^T) \neq 0$.

5. Find the solution to the initial value problem

$$t\frac{dy}{dt} + 3y = \frac{t}{1+t^4}, \qquad y(1) = 0.$$

(a) $y = \ln\left(\frac{1+t^4}{2t^3}\right)$ (b) $y = t^3 - 1$ (c) $y = \frac{1}{4t^3} \cdot \ln\left(\frac{1+t^4}{2}\right)$

(d) $y = \frac{1}{2} \cdot \arctan(t^2) - \pi/8$ (e) $y = \frac{4t^3 - 4}{1+t^4}$

6. Which of the following functions can be used as an integrating factor for the equation $y' + ty = \cos t$?

(a) t (b) $e^{t^2/2}$ (c) $t^2/2$ (d) e^t (e) $e^{\cos t}$

7. The ordinary differential equation

$$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

is

(a) linear (b) autonomous (c) separable (d) an equation of order 2
(e) none of the above.

8. The solution of the initial value problem

$$x \cdot \frac{dy}{dx} = y + xy, \qquad y(1) = 2$$

is the function

(a) $y = \dfrac{e^x}{2(x+1)}$ (b) $y = \ln(x) + 2x$ (c) $y = x^2 + x$ (d) $y = 2$ (e) $y = 2xe^{x-1}$.

Part II: Partial credit questions (11 points each). Show your work.

9. Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 4 \\ -2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -3 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis for W.

(b) Find the QR decomposition of the matrix A with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

10. Let $A = \begin{bmatrix} -1 & 1 \\ -6 & 4 \\ 2 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

(b) Find the vector in the column space of A which is closest to $\vec{b}$.

11. A tank initially contains 50 liters of water and 20 grams of salt. Water containing a salt concentration of 2 g/L enters the tank at the rate of 5 L/min, and the well-stirred mixture leaves the tank *at the same rate*.

(a) Find an expression for the amount of salt in the tank at any time t.

(b) How long does it take for the amount of salt to reach 60 grams.

(c) Find the approximate amount of salt after 100 years.

12. (a) Find, in terms of y_0, the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = t \cdot e^y \\ y(0) = y_0 \end{cases}$$

(b) Find the maximal interval on which the solution to the initial value problem above exists, and explain how this interval depends on y_0.