Department of Mathematics
University of Notre Dame
Math 20580 – Fall 2014.

Name: ____________________

Instructor: ______________
& Section

Exam 3

November 18, 2014

This exam is in 2 parts on 8 pages and contains 11 problems worth a total of 96 points. An additional 4 points will be awarded for following the instructions. You have 1 hour and 15 minutes to work on it. No calculators, books, notes, or other aids are allowed. Be sure to write your name on this page and to put your initials at the top of every page in case pages become detached. Good luck!

Honor Pledge: As a member of the Notre Dame community, I will not participate in nor tolerate academic dishonesty.

Signature:________________________________

You must record here your answers to the multiple choice problems.
Place an $\times$ through your answer to each problem.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)

Please do not write below this line.

This space is for grading.

Answers to the partial credit problems should be circled on the page with the problem.
Any work you do not want considered should be crossed out.

MC.	________
9.	________
10.	________
11.	________
Tot.	________

Multiple Choice

1. (6 pts.) Let $\phi(x)$ be a solution to $\frac{dy}{dx} = \frac{1-x^2}{y}$ that satisfying $\phi(0) = 4$. Find $\phi(3)$.

(a) 1 (b) $\sqrt{3}$ (c) $\sqrt{2}$

(d) Not a real number (e) 2

2. (6 pts.) Find an integrating factor for the differential equation

$$y' - 2xy = xe^{-2x}.$$

(a) $-x^2$ (b) $-2x$ (c) e^{-2x^2} (d) e^{-x^2} (e) e^{-2x}

3. (6 pts.) Find all the stable equilibrium solutions of the autonomous system

$$\frac{dy}{dt} = y - y^3.$$

(a) $y = 0$ (b) $y = 0, y = -1$ (c) $y = 0, y = 1$

(d) $y = 1, y = -1$ (e) $y = 0, y = 1, y = -1$

4. (6 pts.) Which of the following initial value problems is guaranteed to have a unique solution?

(a) $y' = (y-1)^{1/5}, y(0) = 1.$

(b) $y' = y^{1/5}, y(0) = 0$

(c) $y' = (y-1)^{1/5}, y(1) = 0$

(d) $y' = y^{1/5}, y(1) = 0$

(e) $y' = (y-1)^{1/5}, y(1) = 1$

5. (6 pts.) Determine a maximum interval where the solution to the initial value problem is guaranteed to exist:

$$(9-t^2)y' = \ln(2-t) + y\sqrt{t+4}, \quad y(-2) = 0.$$

(a) No such interval exists

(b) $-3 < t < 2$

(c) $t < 2$

(d) $-4 < t < -1$

(e) $-3 < t$

6. (6 pts.) Find a least squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}.$$

(a) $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 216 \\ 36 \end{bmatrix}$

(c) $\begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$

(d) $\begin{bmatrix} 9 \\ 12 \end{bmatrix}$

(e) $\begin{bmatrix} 27 \\ 288 \end{bmatrix}$

7. (6 pts.) Which of the following statements is NOT true of the solutions to the differential equation

$$y' + 2y = 0 \ ?$$

(a) $e^{-2t} + c$, where c is an arbitrary scalar, is the general solution

(b) It is a first order differential equation

(c) The solutions form a vector space

(d) e^{-2t} is a solution

(e) It is a linear differential equation

8. (6 pts.) Solve the differential equation $y' = 3 - \frac{1}{2}y$ subject to the condition $y(0) = 1$.

(a) $y = 6 + 5e^{-\frac{1}{2}t}$

(b) $y = e^{-\frac{1}{2}t}$

(c) $y = 6 + e^{-\frac{1}{2}t}$

(d) $y = 6 - 5e^{-\frac{1}{2}t}$

(e) $y = 6 - 5e^{\frac{1}{2}t}$

Partial Credit

You must show your work on the partial credit problems to receive credit!

9. (16 pts.) Use the method of least squares to find the coefficients a_0 and a_1 so that the line $y = a_0 + a_1 x$ is the best fit to the (x, y) data points $(-2, 2)$, $(0, 3)$, $(2, 1)$.

10. (16 pts.) A tank initially contains 120 L of pure water. A mixture containing a concentration of $(2t + 4)$ g/L of salt enters the tank at a rate of 60 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression for the amount of salt in the tank at any time t.

11. (16 pts.) Given the following autonomous system:

$$\frac{dy}{dx} = y(2-y).$$

(a) Find a general solution for this differential equation.
(b) Find all the stable equilibrium solutions of the autonomous system.
(c) Solve the initial value problem provided that $y(0) = 1$.

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Exam 3D solutions

Multiple choice.

(1) Separating variables in $\frac{dy}{dx} = \frac{1-x^2}{y}$ gives $y\,dy = (1-x^2)\,dx$ so $\int y\,dy = \int (1-x^2)\,dx$, $\frac{1}{2}y^2 = x - \frac{1}{3}x^3 + C$ and $y = \pm\sqrt{2x - \frac{2}{3}x^3 + 2C}$. When $x=0$, $y(0) = 4 = \pm\sqrt{2C}$ so $C = 8$ and the sign is "+". Thus, $\phi(x) = \sqrt{2x - \frac{2}{3}x^3 + 16}$. So $\phi(3) = \sqrt{6 - 18 + 16} = \sqrt{4} = 2$.

(2) $e^{\int -2x\,dx} = e^{-x^2}$

(3) Note $f(y) = y - y^3 = y(1-y)(1+y)$. So the critical points are $y = -1, 0, 1$. For $y < -1$ (e.g. $y = -2$) or $0 < y < 1$ (e.g. $y = \frac{1}{2}$) one has $f(y) > 0$. For $-1 < y < 0$ (e.g. $y = -\frac{1}{2}$) or $y > 1$ (e.g. $y = 2$), one has $f(y) < 0$. The stable equilibria occur at critical points c where $f(y) > 0$ for $y < c$ and $f(y) < 0$ for $y > c$ i.e. at $y = 1$ and $y = -1$.

(4) The solution for the IVP $y' = f(x,y)$, $y'(x_0) = y_0$ will be unique provided f and $\frac{\partial f}{\partial y}$ are defined and continuous on an open rectangle containing (x_0, y_0). For the equation $y' = (y-1)^{1/5}$ with $y(1) = 0$, one has $f(x,y) = (y-1)^{1/5}$, $\frac{\partial f}{\partial y} = \frac{1}{5}(y-1)^{-\frac{4}{5}}$, and $(x_0, y_0) = (1, 0)$, which satisfies these conditions. For the other equations, the partial derivative with respect to y does not exist at (x_0, y_0) so uniqueness is not guaranteed.

(5) The IVP is $y' - \frac{\sqrt{t+4}}{9-t^2}y = \frac{\ln(2-t)}{9-t^2}$ with $y(-2) = 0$. This is $y' + p(y)y = g(t)$ where $p(t) = -\frac{\sqrt{t+4}}{9-t^2}$ and $q(t) = \frac{\ln(2-t)}{9-t^2}$. The solution will exist on any open interval containing -2 on which $p(t)$ and $g(t)$ are defined and continuous i.e. not containing any point t with $t \le -4$, $t^2 = 9$ (i.e. $t = \pm 3$) or $t \ge 2$. The maximum such interval is $-3 < t < 2$.

(6) A least squares solution is given by solving $A^TAx = A^Tb$. Here, $A^TA = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ Since A^TA is invertible, the unique least squares solution is $x = (A^TA)^{-1}(A^Tb) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{24} \end{bmatrix} \begin{bmatrix} 9 \\ 24 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$.

(7) If $y = e^{-2t} + c$, then $y' = -2e^{-2t}$ and $y' + 2y = 2c$ which is not zero for arbitrary c. The other parts are all true.

(8) The IVP is $y' + \frac{1}{2}y = 3$. An integrating factor is $e^{\int \frac{1}{2}dt} = e^{\frac{1}{2}t}$. Multiplying by $e^{\frac{1}{2}t}$ gives $\frac{d}{dt}(e^{\frac{1}{2}t}y) = 3e^{\frac{1}{2}t}$. Integrating, $e^{\frac{1}{2}t}y = \int 3e^{\frac{1}{2}t}\,dt =$

$6e^{\frac{1}{2}t} + C$ and $y(t) = 6 + Ce^{-\frac{1}{2}t}$. So $1 = y(0) = 6 + C$, $C = -5$ and $y(t) = 6 - 5e^{-\frac{1}{2}t}$.

(9) Substituting the points in the equation of the line gives equations $2 = a_0 - 2a_1$, $3 = a_0$ and $1 = a_0 + 2a_1$. These are inconsistent so we calculate a least squares solution. The system is $Au = b$ where $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$, $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. We calculate $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$, $A^T b = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $u = (A^T A)^{-1}(A^T b) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{4} \end{bmatrix}$. So $a_0 = 2$, $a_1 = -\frac{1}{4}$.

(10) Let $V = 120$ denote the volume, $Q(t)$ denote quantity of salt. Concentration of salt at time t is $Q/120$, so the ODE is $\frac{dQ}{dt} = 60(2t + 4) - \frac{Q}{120}60$ i.e. $\frac{dQ}{dt} + \frac{1}{2}Q = 120(t+2)$. Mutiplying by integrating factor $e^{\int \frac{1}{2}\,dt} = e^{\frac{1}{2}t}$ gives $\frac{d}{dt}(e^{\frac{1}{2}t}Q) = 120(t+2)e^{\frac{1}{2}t}$. Integrating, $e^{\frac{1}{2}t}Q = \int 120(t+2)e^{\frac{1}{2}t}\,dt$. Using integration by parts, the right hand side is $120[(t+2)2e^{\frac{1}{2}t} - \int 2e^{\frac{1}{2}t}\,dt] = 120[2(t+2)e^{\frac{1}{2}t} - 4e^{\frac{1}{2}t}] + C = 240te^{\frac{1}{2}t} + C$. Hence $Q(t) = 240t + Ce^{-\frac{1}{2}t}$. Putting $t = 0$, $0 = Q(0) = 0 + C$ and $C = 0$. So $Q(t) = 240t$.

(11)(a) Separating variables, $-\int \frac{1}{y(y-2)}\,dy = \int dx + c$. The integrand on the left is of the form $\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{(y-2)}$. So $1 = A(y-2) + By$. Putting $y = 2$ gives $B = \frac{1}{2}$. Putting $y = 0$ gives $A = -\frac{1}{2}$. So $-\int \frac{1}{y(y-2)}\,dy = \int -\frac{1}{2(y-2)} + \frac{1}{2y}\,dy = \frac{1}{2}(\ln|y| - \ln|y-2|) = \frac{1}{2}\ln|\frac{y}{y-2}|$. So the solution is $\frac{1}{2}\ln|\frac{y}{y-2}| = x + c$, $\ln|\frac{y}{y-2}| = 2x + 2c$ or $|\frac{y}{y-2}| = e^{2x+2c}$. So $\frac{y}{y-2} = \pm e^{2c}e^{2x} = Ce^{2x}$ where $C = \pm e^{2c}$ is another constant. That is $\frac{y-2}{y} = Ce^{-2x}$, $1 - \frac{2}{y} = Ce^{-2x}$, $\frac{y}{2} = \frac{1}{1-Ce^{-2x}}$ and $y = \frac{2}{1-Ce^{-2x}}$.

(b) $f(y) = y(2-y) = 0$ when $y = 0, 2$. So the equilibrium solutions are $y = 0$, $y = 2$. For $y < 0$ or $y > 2$, $f(y) < 0$, while for $0 < y < 2$, $f(y) > 0$. Hence only $y = 2$ is stable.

(c) $y(0) = 1 = \frac{2}{1-Ce^0}$ so $C = -1$ and the solution is $y = \frac{2}{1+e^{-2x}}$.