1. (6pts) Let A be an $n \times n$ matrix satisfying $A^T A = I$. Let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbb{R}^n$ such that $\mathbf{u} \cdot \mathbf{v} = 4$. Find $(A\mathbf{u}) \cdot (A\mathbf{v})$.

(a) $1/4$ (b) $-1/4$ (c) 0 (d) -4 (•) 4

Solution:
$A^T A = I$ means A is unitary so $(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = 4$.

2. (6pts) Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$. Compute $\text{proj}_W \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(a) $\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (•) $\frac{1}{2}\begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ (e) $\frac{1}{2}\begin{bmatrix} 1 \\ 3 \\ -3 \\ -1 \end{bmatrix}$

Solution:
Note that the vectors in W are orthogonal so

$$\text{proj}_W \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} =$$

$$\frac{-2}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \frac{0}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{-4}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

3.(6pts) Classify the differential equation $\dfrac{dy}{dx} = \dfrac{\sin(x)y}{\cos(x)+y}$.

(a) *2nd order* (b) *autonomous* (c) *separable* (•) *exact* (e) *linear*

Solution:

It appears to be neither linear, separable or autonomous. It is first order, not second. We can write it as

$$\sin(x)y\,dx + \big(-\cos(x)\big)\,dy = 0$$

But $\dfrac{\partial \sin(x)y}{\partial y} = \sin(x)$ and $\dfrac{\partial - \cos(x)}{\partial x} = \sin(x)$ so it is exact.

4.(6pts) Solve the differential equation $y' + 3\sqrt{t}\,y = \sqrt{t}$.

(a) $y = 2t^{3/2} + C$ (b) $y = Ct^{-3/2}$ (c) $y = \dfrac{1}{3} + C$ (•) $y = \dfrac{1}{3} + Ce^{-2t\sqrt{t}}$

(e) $y = C\sqrt{t}e^{2t\sqrt{t}}$

Solution:

Equation is linear 1st order in standard form. $\displaystyle\int 3\sqrt{t}dt = 3\frac{t^{3/2}}{3/2} + C$ so $\mu = e^{2t^{3/2}}$ is a choice of integrating factor. Need to do $\displaystyle\int \sqrt{t}e^{2t^{3/2}}dt$. Substitute $u = 2t^{3/2}$ so $du = 3\sqrt{t}dt$ so $\displaystyle\int \sqrt{t}e^{2t^{3/2}}dt = \frac{1}{3}\int e^u du = \frac{e^u}{3} + C = \frac{e^{2t^{3/2}}}{3} + C$ and the solution is $y = \dfrac{\frac{e^{2t^{3/2}}}{3} + C}{e^{2t^{3/2}}}$.

5.(6pts) Let $\phi(x)$ be a solution to $\dfrac{dy}{dx} = \dfrac{1+y^2}{x^2}$ that satisfies $\phi(1) = 0$. Find $\phi(2)$.

(a) $\dfrac{1}{1-\tan(1/2)}$ (•) $\tan(1/2)$ (c) $\dfrac{1}{1-\tan^{-1}(2)}$ (d) $\tan^{-1}(2)$

(e) $\tan(2)$

Solution:

Equation separates as $\dfrac{dy}{1+y^2} = \dfrac{dx}{x^2}$ so $\arctan(y) = -x^{-1} + C$. The initial condition is $y(1) = 0$ so $\arctan(0) = -1 + C$ so $C = 1$ and the solution is $\arctan(y) = \dfrac{x-1}{x}$. Hence $y = \tan\left(\dfrac{x-1}{x}\right)$ and $y(2) = \tan(1/2)$.

6.(6pts) Find the general solution to $3y'' + y' - 2y = 0$.

(a) $y = c_1e^{-t} + c_2e^{3t/2}$ (b) $y = c_1e^{-t/3} + c_2e^{t/2}$ (c) $y = c_1e^{t/2} + c_2e^{-3t/2}$

(d) $y = c_1e^{t/2} + c_2e^{-2t/3}$ (•) $y = c_1e^{-t} + c_2e^{2t/3}$

Solution:

This equation is 2nd order linear with constant coefficients so e^{rt} is a solution whenever $3r^2 + r - 2 = 0$ or $(3r-2)(r+1) = 0$ so the roots are -1 and $\dfrac{2}{3}$. The general solution is $c_1e^{-t} + c_2e^{\frac{2t}{3}}$

7.(6pts) Determine an interval where the solution to the initial value problem is guaranteed to exist.

$$(t^2-4)y' = \sqrt{3-t}\,y + \ln(1+t), \qquad y(0)=0$$

(a) $-1<t<3$ (b) $-1<t$ (•) $-1<t<2$ (d) $t<3$ (e) $-2<t$

Solution:
The equation is linear and the standard form is

$$y' + \frac{-\sqrt{3-t}}{t^2-4}\,y = \frac{\ln(1+t)}{t^2-4}, \qquad y(0)=0$$

The problem asks for the biggest open interval containing 0 over which the two functions of t are continuous. We need $t \leqslant 3$ for the square root; $t>-1$ for the log function; and $t \neq 2$, -2 for the division by t^2-4. Hence $-1<t<2$.

8.(6pts) Find all the *stable* equilibrium solutions of the autonomous system

$$\frac{dy}{dt} = 3y - 4y^2 + y^3$$

(•) $y=1$ (b) $y=0$, $y=-4$ (c) $y=0$, $y=3$

(d) $y=1$, $y=3$, $y=-4$ (e) $y=3$

Solution:
The equilibria occur at solutions to $3y-4y^2+y^3=0$ or $y(y^2-4y+3)=y(y-1)(y-3)$ or $y=0$, 1 and 3. For a stable equilibrium at y_0, $\dfrac{dy}{dt}>0$ changes sign from positive to negative as y crosses y_0.

Crossing 0, $y(y-1)(y-3)$ changes from negative to positive so this equilibrium is unstable. The same thing happens at 3, but crossing 1 two terms are negative for y a bit less than 1, and only one term is negative if y is a bit bigger than 1 so 1 is stable.

9.(6pts) A large tank contains 500 gallons of a water/sugar mixture. Liquid is entering the tank at a rate of 15 gallons/minute and contains 1 pound of sugar per gallon. The mixture is kept well stirred and drains off the tank at a rate of 10 gallons/minute.

If the tank initially has 100 pounds of sugar, determine a differential equation satisfied by $s(t)$, the amount of sugar in pounds in the tank at time t (at least until the tank is full).

(a) $\dfrac{ds}{dt} = 30 - \dfrac{s}{500 + 20t}$ (•) $\dfrac{ds}{dt} = 15 - \dfrac{2s}{100 + t}$ (c) $\dfrac{ds}{dt} = 500 - \dfrac{s}{20}$

(d) $\dfrac{ds}{dt} = 15 - \dfrac{s}{50}$ (e) $\dfrac{ds}{dt} = 15 - \dfrac{s}{500 + 20t}$

Solution:

$\dfrac{ds}{dt}$ measures the change in the amount of sugar. If time is measured from the beginning of the process, $s(0) = 100$. The amount of sugar is changing because of two things. Liquid is entering at a constant rate if 15 gals/min which adds 1 lbs/gal $\times$ 15 gals/min = 15 lbs/min. of sugar to the tank.

Liquid is draining out at a rate of 10 gals/min so sugar is leaving at a rate of 10 gals/min $\times$ $s(t)/V(t)$ lbs/gal where $V(t) = 500 + 5t$ is the volume of the liquid in gallons. Hence sugar is leaving at a rate of $\dfrac{10s(t)}{600 + 5t}$ lbs/min.

Hence $\dfrac{ds}{dt} = 15 - \dfrac{10s}{500 + 5t} = 15 - \dfrac{2s}{100 + t}$.

10.(14pts) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$.

(a) (10pts) Use the Gram-Schmidt process to find an orthogonal basis for col(A).
(b) (4pts) Use the result of (a) to find the Q in the QR-decomposition of A, $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular matrix. DO NOT find R.

Solution:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$$\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} =$$

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Hence $Q = \dfrac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \end{bmatrix}$.

You were told not to find R but if you had been required to find it, proceed as follows. Since $R = Q^T A$,

$$R = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -2 \\ \sqrt{3} & -\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\ 0 & 6 & 6 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

Check

$$\frac{1}{6} \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\ 0 & 6 & 6 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

11.(14pts) If $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 5 \end{bmatrix}$ find the least squares solution to $A\mathbf{x} = \mathbf{b}$.

Solution:

$A^T = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ so $A^TA = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$. Hence the least squares solution is the vector $\hat{\mathbf{x}}$ which satisfies

$$\begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 3 & 6 \\ 3 & 3 & 9 \end{array}\right] \quad \left[\begin{array}{cc|c} 6 & 3 & 6 \\ 1 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 6 & 3 & 6 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -12 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right]$$

so $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ is the least squares solution.

12.(14pts) Determine an explicit solution to $(e^x + e^{-y})\,dx + e^x\,dy = 0$ that satisfies $y(0) = 0$.
(a) (7pts) Find an integrating factor.
(b) (7pts) Give an implicit solution to the original initial value problem.

Solution:

$M = e^x + e^{-y}$, $N = e^x$ so $M_y - N_x = -e^{-y} - e^x$ so $\dfrac{M_y - N_x}{M} = -1$ so $\dfrac{d\mu}{dy} = -\mu$ or $\mu = e^y$.

Check $(e^{x+y}+1)\,dx + e^{x+y}\,dy = 0$ and $\dfrac{\partial e^{x+y}+1}{\partial y} = e^{x+y} = \dfrac{\partial e^{x+y}}{\partial x}$ so $(e^{x+y}+1)\,dx + e^{x+y}\,dy = 0$ is exact.

$\dfrac{\partial \psi}{\partial x} = e^{x+y} + 1$ so $\psi = e^{x+y} + x + g(y)$.

$\dfrac{\partial \psi}{\partial y} = e^{x+y} + g'(y) = e^{x+y}$ so $g(y)$ is a constant and the solutions are the level curves of $\psi = e^{x+y} + x$. The curve passes through $(0,0)$ so $e^{x+y} + x = 1$ is the implicit form of the solution.

Explicitly, $e^{x+y} = 1 - x$, $x + y = \ln(1-x)$ so $y = \ln(1-x) - x$.