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M20580 L.A. and D.E. Tutorial
Worksheet 3
Sections 1.5, 1.7–1.9

1. Determine if the system has a non-trival solution. If yes, describe all solutions in parametric form.

$$(a)\quad \begin{aligned} x_1 - 2x_2 + x_3 &= 0\\ 2x_1 + 4x_2 + x_3 &= 0\\ 3x_1 + 2x_2 + x_3 &= 0 \end{aligned} \qquad (b)\quad \begin{aligned} x_1 - 2x_2 &= 0\\ 2x_1 + 4x_2 + x_3 &= 0\\ 3x_1 + 2x_2 + x_3 &= 0 \end{aligned} \qquad (c)\quad x_1 + x_2 + x_3 = 0$$

(a) $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 4 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix}$ => It has only trivial Solution

(b) $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ => x_3 is free, $x_2 = -\frac{1}{8}x_3$, $x_1 = -\frac{1}{4}x_3$ => Solutions are $x_3\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}$ => x_2 is free, x_3 is free, $x_1 = -x_2 - x_3$ => Solutions are $\begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

2. Describe all solutions of $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

Solve $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 4 & 1 & 2 \\ 3 & 2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{4} & 1 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

x_3 is free. $x_2 = -\frac{1}{8}x_3$, $x_1 = 1 - \frac{1}{4}x_3$

So Solutions are $\begin{bmatrix} 1-\frac{1}{4}x_3 \\ -\frac{1}{8}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$.

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3. Determine if the vectors are linearly independent.

$$(a)\quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\4\end{bmatrix} \qquad (b)\quad \begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}2\\7\\5\end{bmatrix}, \begin{bmatrix}3\\10\\7\end{bmatrix}$$

(a) Solve $\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 2 & 0 & 0 & 0\\0 & 0 & 3 & 0 & 0\\0 & 0 & 0 & 4 & 0\end{bmatrix}$

There is only trivial Solution, So they are linearly independent.

(b) Solve $\begin{bmatrix}1 & 2 & 3 & 0\\3 & 7 & 10 & 0\\2 & 5 & 7 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$.

Since x_3 is a free variable. So they are not linearly independent

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4. (a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(\mathbf{x}) \doteq A\mathbf{x}$. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

(b) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation. If $T(\mathbf{u}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $T(\mathbf{w}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Find $T(\mathbf{x})$, where $\mathbf{x} = 2\mathbf{u} + 3\mathbf{v} + 4\mathbf{w}$.

$$T(\vec{x}) = T(2\vec{u} + 3\vec{v} + 4\vec{w}) = 2T(\vec{u}) + 3T(\vec{v}) + 4T(w)$$

$$= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

(c) Continuing from part (b), if we know $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, find a matrix A such that $T(x) = Ax$ for any x in $\mathbb{R}^3$.

$$A = \begin{bmatrix} A\vec{u} & A\vec{v} & A\vec{w} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}.$$

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5. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation given by:

$$(a)\quad T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \qquad (b)\quad T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}, \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

Find the standard matrix for the linear transformation T, i.e., find a 2×2 matrix A such that $Tx = Ax$.

(a) Let $x_1 = 1$ $x_2 = 0$. $\Rightarrow T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then, Let $x_1 = 0$, $x_2 = 1$ $\Rightarrow T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So $A = \left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) Let $x_1 = 1$, $x_2 = 0$ $\Rightarrow T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Let $x_1 = 0$ $x_2 = 1$ $\Rightarrow T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

So $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

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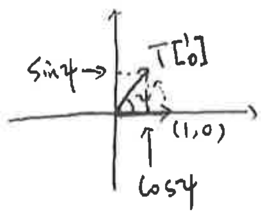
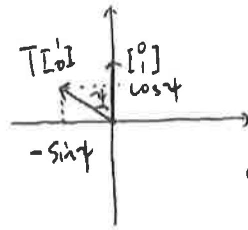
6. (a) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle ψ, with counterclockwise rotation for a positive angle. We know that T is linear. Find the standard matrix A for this transformation. (Hint: draw the unit circle and think about what T does to the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.)

For vector $\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\psi\\ \sin\psi\end{bmatrix}$





For $\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\psi\\ \cos\psi\end{bmatrix}$

So $A = \begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}$

(b) Now let $T(\mathbf{x}) = A\mathbf{x}$ be a rotation of $\frac{\pi}{4}$ in counter-clock wise direction on the Euclidean plane $\mathbb{R}^2$, where A is a 2×2 matrix. Then the matrix A is equal to

(a) $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ (c) $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

From part (a), $A = \begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}$, Set $\psi = \frac{\pi}{4}$,

we get $A = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

(c) Let $A = \begin{bmatrix}\cos(\psi) & -\sin(\psi)\\ \sin(\psi) & \cos(\psi)\end{bmatrix}$ and T is a linear transformation: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Prove that for any angle ψ, the map $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one and onto.

Method 1: One-to-one: Suppose $T\vec{u} = T\vec{v}$, then $T\vec{u} - T\vec{v} = 0$

$T(\vec{u}-\vec{v}) = 0$. Since T is rotation, which preserves the length, so $\vec{u}-\vec{v} = 0$

$\Rightarrow \vec{u} = \vec{v}$.

Onto: For $\forall \vec{u} \in \mathbb{R}^2$, rotate clock-wisely through the angle ψ, and get a vector $\tilde{u}$, then $T\tilde{u} = \vec{u}$.

Method 2:

One-to-one: By theorem 12 in Section 1.9, it suffices to show that $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$ are linearly independent.

We argue by contradiction, suppose they are linearly dependent, then $\exists \alpha \in \mathbb{R}$, such that

$$\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} = \alpha \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

$\Rightarrow \cos\varphi = -\alpha \sin\varphi$ and $\sin\varphi = \alpha \cos\varphi$

So $\cos\varphi = -\alpha\sin\varphi = -\alpha^2 \cos\varphi$

if $\cancel{\alpha}$ $\cos\varphi \neq 0$, then $\alpha^2 = -1$, impossible.

if $\cos\varphi = 0$, then $\sin\varphi \neq 0$, and its clear that $\begin{bmatrix} 0 \\ \sin\varphi \end{bmatrix}$ and $\begin{bmatrix} -\sin\varphi \\ 0 \end{bmatrix}$ are not multiple of each othe

So they are linearly independent.

On to: Just need to show $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$ and $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$ span $\mathbb{R}^2$.

From above, they are not multiple of each other, so they span $\mathbb{R}^2$.