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Name: SOLUTIONS

M20580 L.A. and D.E. Tutorial
Worksheet 4
Sections 2.1–2.3, 2.8

1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Compute $(A+B)(A-B)^T$?

> ***Solution:*** First, $A - B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so $(A-B)^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
>
> Then $(A+B)(A-B)^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. Which of the following equations involving 3×3-matrices A, B, C and I_3 (the identity matrix) *could* be *false* for some such matrices A, B, C ?

   (a) $(A+B)^2 = A^2 + 2AB + B^2$
   (b) $(A+B)C = AC + BC$
   (c) $(AB)C = A(BC)$
   (d) $A + B = B + A$
   (e) $(I_3 + A)(I_3 - A) = I_3 - A^2$

> ***Solution:*** (b), (c), (d) are correct from properties of matrix opreations (theorems 1 and 2, section 2.1).
>
> Let's check (e): $(I_3 + A)(I_3 - A) = I_3(I_3 - A) + A(I_3 - A) = I_3I_3 - I_3A + AI_3 - AA = I_3 - A + A - A^2 = I_3 - A^2$ so (e) is true.
>
> Now, for (a), $(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2$ could be different from $A^2 + 2AB + B^2$ if $AB \neq BA$, which can happen for some A, B. So (a) could be false.

3. Let A be an $n \times n$ square matrix. Suppose that for some $\mathbf{b}$ in $\mathbb{R}^n$, the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent. Which of the following statements must be true?

   (a) A is invertible.
   (b) The linear system $A\mathbf{x} = \mathbf{c}$ has more than one solution for some $\mathbf{c}$ in $\mathbb{R}^n$.
   (c) A has a pivot in every column.
   (d) The linear map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

(e) There is an $n \times n$-matrix B with $AB = \mathrm{I}_n$.

(f) The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solution: Since $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$, the linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is not onto. If T is not onto, by the Invertible Matrix Theorem [IMT] (theorem 8, section 2.3), A is not invertible (eliminate (a)), and T is not one-to-one either. So $A\mathbf{x} = \mathbf{c}$ has at least two solutions for some $\mathbf{c}$ in $\mathbb{R}^n$. Thus, (b) is the correct answer. And since A is not invertible, again by the IMT, (c), (d), (e), (f) are not true in this case.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations with

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad S(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad S(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad S(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find the standard matrix for the transformation ST.

Solution: The standard matrix for T is $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and the standard matrix for S is $B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

So, the standard matrix for ST is $BA = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

5. (a) Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ -3 & -2 & -6 \\ -1 & -1 & -2 \end{bmatrix}$.

(b) Find the solution to the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

Solution: (a) Row-reduce $\begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ -3 & -2 & -6 & 0 & 1 & 0 \\ -1 & -1 & -2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$. The inverse is $\begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) $A\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \implies \mathbf{x} = (A^{-1}) \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is $\mathbf{u}$ in Nul A? Is $\mathbf{u}$ in Col A? Justify your answers.

Solution: Nul A is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Since $A\mathbf{u} = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix} \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$, $\mathbf{u}$ is a solution to the above homogeneous equation. Hence, $\mathbf{u}$ is in Nul A.

Col A is the span of all columns of A. To see if $\mathbf{u}$ is in Col A, we check if $A\mathbf{x} = \mathbf{u}$ is consistent (i.e. $\mathbf{u}$ can be written as a linear combination of the columns of A): Row reducing $\begin{bmatrix} 1 & -1 & 5 & -7 \\ 2 & 0 & 7 & 3 \\ -3 & -5 & -3 & 2 \end{bmatrix}$ into echelon form $\begin{bmatrix} 1 & -1 & 5 & -7 \\ 0 & 2 & -3 & 17 \\ 0 & 0 & 0 & 49 \end{bmatrix}$. We see that the equation $A\mathbf{x} = \mathbf{u}$ has no solution (inconsistent). So, $\mathbf{u}$ is not in Col A.

7. If V and W are subspaces of $\mathbb{R}^n$, define $V \cap W$ to be the subset of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ such that $\mathbf{v}$ is in V and $\mathbf{v}$ is in W. Show that $V \cap W$ is a subspace.

Solution: To show $V \cap W$ is a subspace of $\mathbb{R}^n$, we need to check three properties:

(i). **The zero vector is in** $V \cap W$: Since we know V is a subspace, $\mathbf{0}$ must be in V. Similarly, since W is a subspace, $\mathbf{0}$ is also in W. Thus, $\mathbf{0}$ is in $V \cap W$.

(ii). $V \cap W$ **is closed under vector addition**: Let $\mathbf{x}$ and $\mathbf{y}$ be two arbitrary vectors in $V \cap W$. We want to show $\mathbf{x} + \mathbf{y} \in V \cap W$. Since $\mathbf{x}, \mathbf{y} \in V \cap W$, we have $\mathbf{x}, \mathbf{y}$ are in V. So, $\mathbf{x} + \mathbf{y} \in V$ since V is a subspace. Similarly, $\mathbf{x}, \mathbf{y} \in V \cap W$ implies $\mathbf{x}, \mathbf{y}$ are in W. So, $\mathbf{x} + \mathbf{y} \in W$ since W is a subspace. So, we get $\mathbf{x} + \mathbf{y}$ are both in V and W. This implies $\mathbf{x} + \mathbf{y} \in V \cap W$.

(iii). $V \cap W$ **is closed under scalar multiplication**: Let $\mathbf{x}$ be in $V \cap W$ and c is a scalar. We want to show $c\mathbf{x}$ is in $V \cap W$. Since $\mathbf{x} \in V \cap W$, we get $\mathbf{x} \in V$ and $\mathbf{x} \in W$. And since both V and W are subspaces, we must have $c\mathbf{x} \in V$ and $c\mathbf{x} \in W$. So, $c\mathbf{x} \in V \cap W$.

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8. The row-reduced echelon form of the 3×6 matrix $A = \begin{bmatrix} 0 & 2 & 4 & 1 & 5 & 6 \\ 0 & 1 & 2 & -1 & 7 & -5 \\ 0 & -1 & -2 & -2 & 2 & 0 \end{bmatrix}$ is given by $B = \begin{bmatrix} 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. (You may assume this; you do not have to check it.)

(a) Determine a basis for the null space null(A).

(b) Determine a basis for the column space col(A).

Solution: From the reduced echelon form of A, we see that the pivot columns are 2, 4, 6; so x_1, x_3 and x_5 are free variables. So the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_3 - 4x_5 \\ x_3 \\ 3x_5 \\ x_5 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, a basis for Nul A is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Since the reduced echelon form of A tells us that columns 2, 4, 6 are pivot columns, these columns in matrix A form a basis of Col A.

So, a basis for Col A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix} \right\}$.