Name: SOLUTIONS Date: 02/20/20

M20580 L.A. and D.E. Tutorial
Worksheet 5

Warm Up (with solutions)

1. Recall from lectures the ways to rewrite determinants of these $n \times n$ matrices in terms of other determinant(s) and scalars.
   (a) $\det(AB) = \det(A)\det(B)$
   (b) B is A but with one row multiplied by k. $\det(B) = k\det(A)$
   (c) $\det\left(A^T\right) = \det(A)$

2. Write down a basis for the vector space $\mathbb{P}_3$, the set of real degree (at most) 3 polynomials. What is the dimension of $\mathbb{P}_3$?

> ***Solution:*** $\{1, t, t^2, t^3\}$ The basis has 4 elements so $\dim(\mathbb{P}_3) = 4$.

Main Questions

3. Let A be an invertible matrix. Using properties of determinants, show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

> ***Solution:*** Using "$\det(AB) = \det(A)\det(B)$" we can write $\det(I_n) = \det(A)\det(A^{-1})$ since $AA^{-1} = I_n$. Now we can recognise that $\det(I_n) = 1$ and rearrange the equation to what we wanted.

4. Find the determinant of the matrix:

$$A = \begin{bmatrix} -7 & 2 & 6 & 15 \\ -3 & 0 & 4 & 0 \\ -12 & 0 & -7 & -5 \\ 2 & 0 & -3 & 0 \end{bmatrix}$$

> ***Solution:*** We use the row/col expansion formula repeatedly, first targeting the second column.
>
> $$\det(A) = (-1)^{1+2} 2 \begin{vmatrix} -3 & 4 & 0 \\ -12 & -7 & -5 \\ 2 & -3 & 0 \end{vmatrix} = (-2)(-1)^{3+2}(-5) \begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix} = -10$$

5. Let A and B be 4×4 matrices, with $\det(A) = 2$ and $\det(B) = -3$. Compute:
   (a) $\det(5A)$
   (b) $\det(A^T BA)$
   (c) $\det(B^5)$
   (d) $\det(B^{-1}A)$ (B is invertible)

Solution:

(a) We can view $5A$ as A with each of the 4 rows multiplied by the scalar 5. Using Question 1 part b, this means the determinant is multiplied by 5 for *every* row we change this way. Hence $\det(5A) = 5^4 \det(A) = 1250$.

(b) Using the rules from Q1 in turn we get $\det(A^T BA) = \det(A^T)\det(B)\det(A) = \det(A)\det(B)\det(A) = (\det(A))^2 \det(B) = -12$.

(c) $\det(B^5) = \det(BBBBB) = (\det(B))^5 = -243$.

(d) $\det(B^{-1}A) = \det(B^{-1})\det(A) = \frac{\det(A)}{\det(B)} = -\frac{2}{3}$ using Question 2.

6. Use Cramer's rule to compute the solutions of the following system.

$$\begin{aligned} 2x_1 - 4x_2 \qquad &= 8 \\ -x_1 \qquad - 2x_3 &= 0 \\ 7x_2 + 5x_3 &= 3 \end{aligned}$$

Solution: The matrix for this system is

$$A = \begin{bmatrix} 2 & -4 & 0 \\ -1 & 0 & -2 \\ 0 & 7 & 5 \end{bmatrix} \qquad \det(A) = 2\begin{vmatrix} 0 & -2 \\ 7 & 5 \end{vmatrix} + (-1)(-4)\begin{vmatrix} -1 & -2 \\ 0 & 5 \end{vmatrix} = 28 - 20 = 8$$

Cramer's Rule says the solution is given by $x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$.

$$\det(A_1(\mathbf{b})) = \begin{vmatrix} 8 & -4 & 0 \\ 0 & 0 & -2 \\ 3 & 7 & 5 \end{vmatrix} = (-1)(-2)\begin{vmatrix} 8 & -4 \\ 3 & 7 \end{vmatrix} = 2(56 + 12) = 136$$

$$\det(A_2(\mathbf{b})) = \begin{vmatrix} 2 & 8 & 0 \\ -1 & 0 & -2 \\ 0 & 3 & 5 \end{vmatrix} = 2\begin{vmatrix} 0 & -2 \\ 3 & 5 \end{vmatrix} - 8\begin{vmatrix} -1 & -2 \\ 0 & 5 \end{vmatrix} = 2(0 + 6) - 8(-5 + 0) = 52$$

$$\det(A_3(\mathbf{b})) = \begin{vmatrix} 2 & -4 & 8 \\ -1 & 0 & 0 \\ 0 & 7 & 3 \end{vmatrix} = -(-1)\begin{vmatrix} -4 & 8 \\ 7 & 3 \end{vmatrix} = -12 - 56 = -68$$

Therefore the solution is $x_1 = \frac{136}{8} = 17,\ x_2 = \frac{52}{8} = \frac{13}{2},\ x_3 = \frac{-68}{8} = -\frac{17}{2}$.

7. Find a basis for the space spanned by the given vectors $\mathbf{v}_1, \cdots, \mathbf{v}_5$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

Solution: We want a maximal linearly independent subset of these vectors, equivalently, we may choose the pivot columns of $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5]$

$$= \begin{bmatrix} \boxed{1} & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -3 & 1 & 2 \\ 0 & \boxed{1} & -4 & -3 & 1 \\ 0 & 2 & -8 & -5 & 0 \\ 0 & -3 & 12 & 5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 0 & -3 & 1 & 2 \\ 0 & \boxed{1} & -4 & -3 & 1 \\ 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -3 & 1 & 2 \\ 0 & \boxed{1} & -4 & -3 & 1 \\ 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This row echelon form shows that we have 3 pivot columns and hence a linearly independent (pivot columns are independent) set which spans $\operatorname{Col} A$ (a free column is always in the span of the ones to the left of it) is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. Alternative choices include any two of the first 3 and either of the last two.

8. Consider the vector space $\mathbb{P}_3$ of real polynomials of degree (at most) 3. We can define a transformation D by

$$\begin{aligned} D : \mathbb{P}_3 &\longrightarrow \mathbb{P}_3 \\ f(t) &\longmapsto f'(t) \end{aligned}$$

For example $D(t^3 - 1) = 3t^2$.

(a) Prove or explain why this is a *linear* transformation.

> ***Solution:*** It satisfies the two axioms:
>
> 1. $D(f+g) = D(f) + D(g)$ since $\frac{\mathrm{d}}{\mathrm{d}t}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\mathrm{d}g}{\mathrm{d}t}$.
> 2. $D(\lambda f) = \lambda D(f)$, again by the usual rule for derivatives.

(b) *Without* writing down a matrix in terms of a basis, describe the *kernel* of this transformation, $\ker(D)$ a.k.a. the *null space* of D.
Hint: What defines this subspace? Solve it.

> ***Solution:*** The kernel is defined by the vectors (a.k.a. polynomials) f where $D(f) = f' = 0$. We know this is precisely the *constant polynomials* i.e. $f(t) = C$. As a subspace this could be written $\operatorname{span}\{1\}$.

(c) Again, without a matrix, write down the *range* of D, perhaps using notation from lectures. Give a basis for this.

> ***Solution:*** $D(at^3 + bt^2 + ct + d) = 3at^2 + 2bt + c$ therefore, thinking about the fact that a, b, c are arbitrary, the range is $\mathbb{P}_2$, the degree (at most) 2 polynomials. A good basis is $\{1, t, t^2\}$.

(d) Compute the matrix which represents D in a basis of your choice. Does this matrix explain the previous two parts? *Hint: The 'nice'/'usual' basis is recommended.*

> ***Solution:*** Let's use the basis $\{1, t, t^2, t^3\}$. Then $D(t^3) = 3t^2$, $D(t^2) = 2t$, $D(t) = 1$, $D(1) = 0$. This gives a nice matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and in fact it is in row echelon form. The pivots show that the first variable (for basis vector 1) is free, in fact 1 is the basis for $\ker D = \operatorname{Nul} A$. Also it is clear that the column span (the range of D a.k.a. $\operatorname{Col} A$) is $\operatorname{span}\{1, t, t^2\}$ with the row of zeros at the bottom.

Reminder: the kernel, $\ker D$ is the generalized idea of "$\operatorname{Nul} A$" if D had the matrix A. Similarly, range of D replaces "$\operatorname{Col} A$". The benefit here is that we focus on a 'matrix-free' definition, which is sometimes easier as above.