Name: SOLUTIONS Date: 02/20/20

M20580 L.A. and D.E. Tutorial
Vector spaces, subspaces and linear transformations

1. The next 3 problems concern the spanning set theorem(4.3) in abstract vector spaces. Consider the space spanned by the given polynomials $p_1, ...p_5$, inside the vector space of polynomials with highest power≤ 3.

$$p_1 = 1-3t^2+2t^3, p_2 = t+2t^2-3t^3, p_3 = -3-4t+t^2+6t^3, p_4 = 1-3t-8t^2+7t^3, p_5 = 2+t-6t^2+9t^3$$

   1. Find a basis for the space span$\{p_1, ..., p_5\}$ [1].
   2. Are the polynomials p_2, p_3, p_4 linearly dependent?
   3. Explain why the polynomials p_1, p_3, p_5 form a basis of this space.

Solution: $\mathbb{P}_3$ is *isomorphic* to $\mathbb{R}^4$: A polynomial is determined by its coefficients, and the collection of coefficients associated to a polynomial is a vector in $\mathbb{R}^4$. Another way of saying the same thing - the set $\{1, t, t^2, t^3\}$ is a basis of $\mathbb{P}_3$. The coordinates associated to this basis are elements of $\mathbb{R}^4$ and lets us regard our ambient space of polynomials as $\mathbb{R}^4$.

[1] It might be helpful to look at last week's problem:
Find a basis for the space spanned by the given vectors $\mathbf{u}_1, \cdots, \mathbf{u}_5$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

Solution: We want a maximal linearly independent subset of these vectors. We may choose the pivot columns of $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5]$.

$$= \begin{bmatrix} \boxed{1} & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -3 & 1 & 2 \\ 0 & \boxed{1} & -4 & -3 & 1 \\ 0 & 2 & -8 & -5 & 0 \\ 0 & -3 & 12 & 5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 0 & -3 & 1 & 2 \\ 0 & \boxed{1} & -4 & -3 & 1 \\ 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -3 & 0 & 4 \\ 0 & \boxed{1} & -4 & 0 & -5 \\ 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This row echelon form shows that we have 3 pivot columns and hence a linearly independent (pivot columns are independent) set which spans Col A (a free column is always in the span of the ones to the left of it) is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. Alternative choices include any two of the first 3 and either of the last two.

1. The columns u_1, u_2, u_4 form a basis for the column space span$\{u_1, u_2, u_4\}$ because they are the pivot columns in this matrix.
2. u_2, u_3, u_4 are linearly independent - if you make a matrix just from the columns u_2, u_3, u_4 and row reduce that matrix you will get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

. Since there is a pivot in every column(in the matrix we just created), the vectors are linearly independent.
3. The columns u_1, u_3, u_5 are linearly independent because if we make a matrix from just those columns and row reduce that matrix, you will get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. There are 3 linearly independent vectors here and the basis we found in the first part has 3 elements. This means that these 3 linearly independent vectors u_1, u_3, u_5 actually form a basis.

2. Let V be a a vector space. Let $\{v_1, v_2, v_3, v_4\}$ be a linearly independent subset of V. Consider the subspace of V spanned by the vectors

$$w_1 = v_1 - 3v_3 + 2v_4, w_2 = v_2 + 2v_3 - 3v_4, w_3 = -3v_1 - 4v_2 + v_3 + 6v_4$$

$$w_4 = v_1 - 3v_2 - 8v_3 + 7v_4, w_5 = 2v_1 + v_2 - 6v_3 + 9v_4$$

1. Find a basis of $Span\{w_1, ...w_5\}$
2. Is $\{w_2, w_3, w_4\}$ a linearly dependent subset of V?
3. Explain why the vectors w_1, w_3, w_5 form a basis of $Span\{w_1, ..., w_5\}$
4. Do w_1, w_3, w_5 form a basis for V?

Solution: This problem is word for word the same as the last one. v_1, v_2, v_3, v_4 is a basis for span $v_1, ..., v_4$. The coordinates associated to this basis are elements of $\mathbb{R}^4$ and lets us regard span $v_1, ..., v_4$ as $\mathbb{R}^4$. Under this correspondence $w_1, ...w_5$ will correspond to $u_1, ...u_5$. The problem is finished just as above.

Let V be a vector space. Recall that a sub**set** W, of V is called a sub**space** of V if it satisfies some other nice properties:

- W has the zero vector of V.
- The sum of any two vectors in W is still in W.
- If I scale a vector's length by a positive or negative scalar, the vector is still in W.

3. (Homework question 4.1.4) Let $V = \mathbb{R}^2$ and W' be a line not going through the origin. Why is the subset W', not a subspace of $\mathbb{R}^2$? How many of the above requirements are violated?([2])

Solution: The origin plays the role of the zero vector in $\mathbb{R}^2$. W' doesn't go through the origin. So W' doesn't contain the zero vector. The sum of two vectors in W' will lie in a line parallel(and hence not intersecting) W'. Same thing happens if we scale a vector by a scalar(that isn't $= 1$). In particular there is some scalar which makes things go bad.

4. Let W be a subspace of V and let v be a fixed vector not in W. Let W' be the subset of vectors of the form $v + w$ where w is in W. (W' is like the line not going through the origin above). Why is the subset W' not a subspace of $\mathbb{R}^2$? How many of the above requirements are violated?

Solution: We imitate the reasoning above. All the requirements miserably fail again:

- W' doesn't go through the origin: Here's a proof by contradiction. Suppose a vector of the form $v + w$ is $= 0$. Then $v = -w$. $-w$ is in W because w is in W. So v is in W. Contradiction.

  This confirms our intuition that W' looks like a plane that doesn't go through the zero vector in V.

- Suppose I have a vector of the form $v + w_1$ and another vector $v + w_2$. The sum will be $2v + w_1 + w_2$. Suppose that $2v + w_1 + w_2$ is $=$ to something of the form $v + w_3$. Then $v = w_3 - w_2 - w_1$. Hence v is in W. Contradiction.

- We'll show that when we scale a vector $v + w_1$ in W' Suppose that $c(v + w_1) = v + w_2$. Then $(c-1)v = w_2 - cw_1$. Let c be any scalar not equal to 1(so we can divide out $(c - 1)$. Then $v = (w_2 - cw_1)/(c - 1)$ so v is in W. Contradiction.

5. (Homework question 4.1.15) Let $V = \mathbb{R}^3$. Let W be the sub**set** of all vectors of the form $a\begin{pmatrix}3\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\-5\end{pmatrix}$. Let W' be the sub**set** of all vectors of the form $\begin{pmatrix}0\\4\\0\end{pmatrix} + a\begin{pmatrix}3\\0\\1\end{pmatrix} +$

[2]Hint: All of them

$b\begin{pmatrix}1\\0\\-5\end{pmatrix}$. Why isn't W' a subspace of $\mathbb{R}^3$?

> ***Solution:*** This is true because of the general fact above. Here W is the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix}0\\4\\0\end{pmatrix}$ and $\begin{pmatrix}3\\0\\1\end{pmatrix}$. v is the vector $\begin{pmatrix}0\\4\\0\end{pmatrix}$.

6. Fun fact: The last two requirements of being a subspace imply the first requirement -

   If w is a random vector[3] in W, $-w = (-1)w$ is in W.(**WHY?**). $w + (-w)$ is also in W (**WHY?**). $w + -w = 0$. Hence 0 is automatically contained in W.

[3] A technicality is that the subset W can't be the empty subset. Don't worry about it.

Now let W and V be two random vector spaces. Recall that a linear transformation T from W to V is a map of sets from W to V [4], satisfying some other nice properties:

- $T(0) = 0$
- $T(v_1 + v_2) = T(v_1) + T(v_2)$
- $T(cv) = cT(v)$

7. Let $W = \mathbb{R}^2$ and $V = \mathbb{R}^3$. Consider the map sending $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. It is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$. Why is this map a linear transformation?

Solution: The map is a linear transformation because matrix multiplication is a linear transformation. Matrix multiplication is linear transformation because of the distributive properties of the real numbers.

8. Now consider the map sending $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$. Why isn't this map a linear transformation. How many of the above requirements are violated?

Solution: All of them!

1.
$$\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2.
$$\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \neq \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

3.
$$c \left(\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) \neq \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -5 \end{pmatrix} c \begin{pmatrix} x \\ y \end{pmatrix}$$

9. Show that the first requirement for linear transformations is implied by the other two requirements. *Hint: This is similar to question 6*

[4] notation - $T : W \to V$

Solution: We have that $T(0) = T(0+0) = T(0) + T(0)$. Subtracting $T(0)$ from both sides gives $T(0) = 0$.

10. Let W and V be random vector spaces again and let T be a linear transformation from W to V. Let v be a nonzero vector. Let $T' = T + v$. Is T' a linear transformation?

Solution: No it isn't. All the properties of being a linear transformation fail.

1. $T'(0) = v + T(0) = v + 0 = v$. So $T'(0) \neq 0$
2. $T'(w_1 + w_2) = v + T(w_1 + w_2) = v + T(w_1) + T(w_2)$. On the other hand $T'(w_1) + T'(w_2) = 2v + T(w_1) + T(w_2)$. These expressions differ by a v so they are not the same.
3. $T'(cw) = v + T(cw) = v + cT(w)$. On the other hand, $cT'(w) = cv + cT(w)$. The difference of these two expressions is $(c-1)v$. If I pick $c \neq 1$ then this is not zero, so these two things are not the same.

(Fun remark) You might have noticed that there is a lot in common between the notion of a linear transformation and the notion of a subspace. Here's the reason.

11. Let T be a linear transformation from W to V.

    1. Let w_1, w_2 be two vectors in W. Show that if I add two vectors of the form $T(w_1)$, $T(w_2)$ that the result is $T(\textit{something})$.

       Solution: This is a definition

    2. Show that $T(cw) = cT(w)$

       Solution: This is a definition

    3. Conclude that the image/columnspace[5] of T is a subspace of V.

       Solution: These two requirements are the requirements of being a subspace.

[5] these are the vectors that are $T(\textit{something})$