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M20580 L.A. and D.E. Tutorial
Worksheet 7
Sections 5.3-5.5, 6.1-6.3

1. Let the matrix

$$A = \begin{bmatrix} 1 & 0 & -4 \\ -6 & -1 & 12 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -4 \\ -6 & -1-\lambda & 12 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (-1-\lambda)(1-\lambda)(-1-\lambda)$$
$$= (1-\lambda)(1+\lambda)^2$$

So eigenvalues are $\lambda=1$, $\lambda=-1$, $\lambda=-1$

(b) Find a basis for each eigenspace corresponding to each eigenvalue which you found in part (a). **Make sure you indicate which eigenvalue each subspace basis corresponds to.**

For $\lambda=1$, Solve $\left[\begin{array}{ccc|c} 0 & 0 & -4 & 0 \\ -6 & -2 & 12 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

the eigenvectors are $x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$, so a basis is given

by $\left\{ \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda=-1$, Solve $\left[\begin{array}{ccc|c} 2 & 0 & -4 & 0 \\ -6 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Solutions are $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. So a basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or if none such exists, explain why. Note: You do **not** need to compute P^{-1}.

$$P = \begin{bmatrix} -\frac{1}{3} & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

2. Let

$$A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}.$$

(a)Find all eigenvalues of A.

For $A - \lambda I = \begin{bmatrix} -3-\lambda & 12 \\ -2 & 7-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (-3-\lambda)(7-\lambda) + 24 = (\lambda-1)(\lambda-3)$.

So all eigenvalues are $\lambda = 1$, $\lambda = 3$.

(b)Suppose you know that $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ are eigenvectors of A, where A is as above. Use this information to diagonalize A, i.e., find matrices P and D such that $A = PDP^{-1}$.

First check which eigenvalues correspond to $\vec{v}_1$ and $\vec{v}_2$.

$A\vec{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow A\vec{v}_1 = 3\vec{v}_1$

$A\vec{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow A\vec{v}_2 = \vec{v}_2$

Let $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$

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3. Let $T : \mathbb{P}_2 \to \mathbb{P}_1$ be the linear transformation given by

$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t, \qquad \text{where } a_0, a_1, a_2 \text{ are scalars.}$$

(Note that T is the differentiation operator.)

Let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$ and $\mathcal{C} = \{1, t\}$ be the standard basis for $\mathbb{P}_1$.

Find the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$, i.e., find a matrix M such that $[T(\mathbf{p})]_{\mathcal{C}} = M[\mathbf{p}]_{\mathcal{B}}$, for $\mathbf{p} \in \mathbb{P}_2$.

$$M = \begin{bmatrix} [T(1)]_e & [T(t)]_e & [T(t^2)]_e \end{bmatrix}$$

$$= \begin{bmatrix} [0]_e & [1]_e & [2t]_e \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4. Let A be the matrix

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Find two complex eigenvectors of A.

$$A - \lambda I = \begin{bmatrix} \cos(\theta) - \lambda & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (\cos(\theta) - \lambda)^2 + \sin^2(\theta)$$

$$= \lambda^2 - 2\cos(\theta)\lambda + 1.$$

Solve $\det(A - \lambda I) = 0 \Rightarrow \lambda = \cos(\theta) \pm i\sin(\theta)$.

For $\lambda = \cos(\theta) + i\sin(\theta)$

Solve $\begin{bmatrix} -i\sin(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & -i\sin(\theta) & 0 \end{bmatrix} \sim \left[\begin{array}{cc|c} -i\sin(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 \end{array}\right].$

So eigenvectors are $x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

For $\lambda = \cos(\theta) - i\sin(\theta)$, since it's complex conjugate of $\cos(\theta) + i\sin(\theta)$.

$\Rightarrow$ eigenvectors are $x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$.

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5. Which of the following is NOT an orthogonal set? (Using the standard inner product)

1. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$
2. $\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-6\\4\end{bmatrix}$
3. $\begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix}, \begin{bmatrix}\sin(t)\\-\cos(t)\end{bmatrix}$
4. $\begin{bmatrix}3\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$
5. $\begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\3\end{bmatrix}$

6. Given $\mathbf{u} \neq 0$ in $\mathbb{R}^n$, Let $L = \text{Span}\{\mathbf{u}\}$. Show that the mapping $\mathbf{x} \to \text{proj}_L\mathbf{x}$ is a linear transformation.

By definition, $\text{proj}_L\vec{x} = \dfrac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$.

① for any $\vec{x}_1, \vec{x}_2$,

$$\text{proj}_L(\vec{x}_1+\vec{x}_2) = \frac{(\vec{x}_1+\vec{x}_2)\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{\vec{x}_1\cdot\vec{u}+\vec{x}_2\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \frac{\vec{x}_1\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} + \frac{\vec{x}_2\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \text{proj}_L\vec{x}_1 + \text{proj}_L\vec{x}_2$$

② for c scalar, $\vec{x}$.

$$\text{proj}_L(c\vec{x}) = \frac{(c\vec{x})\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = c\,\frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = c\,\text{proj}_L(\vec{x}).$$

Thus, $\vec{x} \to \text{proj}_L\vec{x}$ is a linear transformation.