Mixed Counting Problems

We have studied a number of counting principles and techniques since the beginning of the course and when we tackle a counting problem, we may have to use one or a combination of these principles. The counting principles we have studied are:

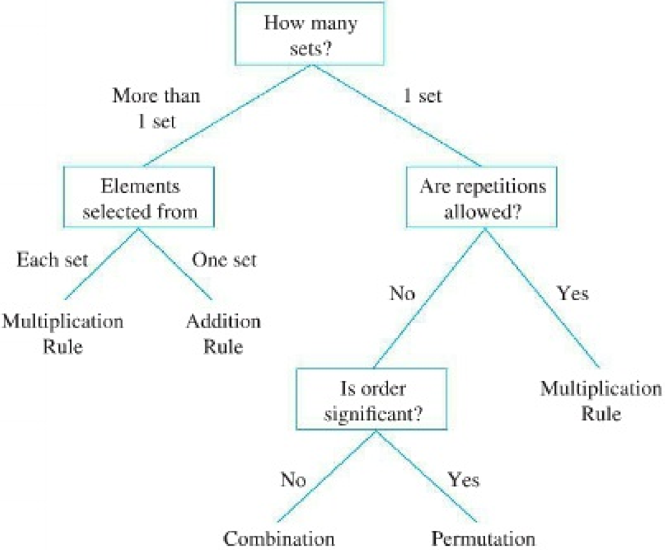
- **Inclusion-exclusion principle**: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- **Complement Rule** $n(A') = n(U) - n(A)$.
- **Multiplication principle**: If I can break a task into r steps, with m_1 ways of performing step 1, m_2 ways of performing step 2 (no matter what I do in step 1), $\dots$, m_r ways of performing step r (no matter what I do in the previous steps), then the number of ways I can complete the task is
$$m_1 \cdot m_2 \cdots m_r.$$
(This also applies if step i of task amounts to selecting from set A_i with m_i elements.)
- **Addition principle**: If I must choose exactly one activity to complete a task from among the (disjoint) activities $A_1, A_2, \dots, A_r$ and I can perform activity 1 in m_1 ways, activity 2 in m_2 ways, $\dots$, activity r in m_r ways, then I can complete the task in
$$m_1 + m_2 + \cdots + m_r$$
ways. (This also applies if task amounts to selecting one item from r disjoint sets $A_1, A_2, \dots, A_r$ with $m_1, m_2, \dots, m_r$ items respectively.)
- **Permutations**: The number of arrangements of n objects taken r at a time is
$$\mathbf{P}(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}.$$
- **Permutations of objects with some alike:**
  - The number of different permutations (arrangements), where order matters, of a set of n objects (taken n at a time) where r of the objects are identical is
  $$\frac{n!}{r!}.$$
  - Consider a set of n objects which is equal to the disjoint union of k subsets, $A_1, A_2, \dots, A_k$, of objects in which the objects in each subset A_i are identical and the objects in different subsets A_i and A_j, $i \neq j$ are not identical. Let r_i denotes the number of objects in set A_i, then the number of different permutations of the n objects (taken n at a time) is
  $$\frac{n!}{r_1! r_2! \dots r_n!}.$$

  This can also be considered as an application of the technique of **"overcounting"** where we count a larger set and then divide.
- **Combinations:** The number of ways of choosing a subset of (or a sample of) r objects from a set with n objects, where order does not matter, is
$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}.$$
Note this was also an application of the technique of "overcounting".

Mixed Counting Problems

Problem Solving Strategy: You may be able to solve a counting problem with a single principle or a problem may be a multilevel problem requiring repeated application of one or several principles. When asked to count the number of objects in a set, it often helps to think of how you might complete the task of constructing an object in the set. It also helps to keep the technique of "overcounting" in mind. The following flowchart from your book may help you decide whether to use the multiplication principle, the addition rule, the formula for the number of permutations or the formula for the number of combinations for a problem or a problem part requiring one of these.

Mixed Counting Problems



Mixed Counting Problems

Example An experiment consists of rolling a 20 sided die three times. The number on top of each die is recorded. The numbers are written down in the order in which they are observed. How many possible ordered triples of numbers can result from the experiment? (Note the triple $(17, 10, 3)$ is not the same result as the triple $(3, 10, 17)$.)

There are 20 ways each throw can come up and the order is important so the answer is $20 \cdot 20 \cdot 20 = 20^3 = 8000$.

Mixed Counting Problems

Example (Hoosier Lottery) When you buy a Powerball ticket, you select 5 different white numbers from among the numbers 1 through 59 (order of selection does not matter), and one red number from among the numbers 1 through 35. How many different Powerball tickets can you buy?

If you check out the Powerball web site you will see that you need to select 5 distinct white numbers, so you can do this $\mathbf{C}(59, 5) = 5,006,386$ ways. Then you can pick the red number $\mathbf{C}(35, 1) = 35$ ways so the total number of tickets is $\mathbf{C}(59, 5) \cdot \mathbf{P}(35, 1) = 5,006,386 \cdot 35 = 175,223,510.$

Mixed Counting Problems

Often problems fit the model of pulling marbles from a bag. For example many of our previous problems involving poker hands fit this model. Polling a population to conduct an observational study also fit this model.

Example: An bag contains 15 marbles of which 10 are red and 5 are white. 4 marbles are selected from the bag.

There's ambiguity here: e.g., if on one draw I select four red marbles, and on another draw I select a *different* four red marbles, are these considered the same sample or not? We'll assume that they are *not* the same sample. For example, we could imagine that the marbles are numbered, each with a different number, so that we can tell marbles of the same color apart. This way of thinking will be very useful for calculating probabilities later, when we try to set up an "equally likely sample space"

Mixed Counting Problems

$\boxed{\text{10 red, 5 white, numbered marbles}}$

(a) How many (different) samples (of size 4) are possible?

The order does not matter but the numbers do so we are selecting 4 elements from a set of $10 + 5$ elements. Hence the answer is $\mathbf{C}(15, 4) = 1,365$.

(b) How many samples (of size 4) consist entirely of red marbles?

The order does not matter but the numbers do so we are selecting 4 elements from a set of 10 elements. Hence the answer is $\mathbf{C}(10, 4) = 210$.

Mixed Counting Problems

$$\boxed{\text{10 red, 5 white, numbered marbles}}$$

(c) How many samples have 2 red and 2 white marbles?
We can select 2 numbered red marbles in $\mathbf{C}(10, 2)$ ways and 2 numbered white marbles in $\mathbf{C}(5, 2)$ ways. Neither choice affects the other so the answer is
$\mathbf{C}(10, 2) \cdot \mathbf{C}(5, 2) = 45 \cdot 10 = 450.$

(d) How many samples (of size 4) have exactly 3 red marbles? We can select 3 numbered red marbles in $\mathbf{C}(10, 3)$ ways and 1 numbered white marble in $\mathbf{C}(5, 1)$ ways. Neither choice affects the other so the answer is
$\mathbf{C}(10, 3) \cdot \mathbf{C}(5, 1) = 120 \cdot 5 = 600.$

Mixed Counting Problems

$\boxed{\text{10 red, 5 white, numbered marbles}}$

(e) How many samples (of size 4) have at least 3 red?

The answer is the number of samples with 3 red plus the number of samples with 4 red. We can select 4 numbered red marbles in $\mathbf{C}(10, 4)$ ways and 0 numbered white marbles in $\mathbf{C}(5, 0)$ ways. Neither choice affects the other so the answer is $\mathbf{C}(10, 4) \cdot \mathbf{C}(5, 0) = 210 \cdot 1 = 210$.
From the last example, there are 600 ways to select samples with exactly 3 red marbles so our answer is
$600 + 210 = 810$.

Mixed Counting Problems

10 red, 5 white, numbered marbles

(f) How many samples (of size 4) contain at least one red marble?

One answer is"the number with exactly 1" + "the number with exactly 2" ... "the number with exactly 4". This is

$$\mathbf{C}(10,1)\cdot\mathbf{C}(5,3)+\mathbf{C}(10,2)\cdot\mathbf{C}(5,2)+\mathbf{C}(10,3)\cdot\mathbf{C}(5,1)+\mathbf{C}(10,4)\cdot\mathbf{C}(5,0)$$

which is

$$10\cdot10+45\cdot10+120\cdot5+210\cdot1=100+450+600+210=1,360$$

It is also the total number of samples $(1,365)$ minus the number of samples with no red marbles which is $\mathbf{C}(10,0)\cdot\mathbf{C}(5,4)=5$.

Mixed Counting Problems

Example: Recall that a standard deck of cards has 52 cards. The cards can be classified according to suits or denominations. There are 4 suits, hearts, diamonds, spades and clubs. There are 13 cards in each suit. There are 13 denominations, Aces, Kings, Queens, ... ,Twos, with 4 cards in each denomination. A poker hand consists of a sample of size 5 drawn from the deck. Poker problems are often like urn problems, with a hitch or two.

(a) How many poker hands consist of 2 Aces and 3 Kings?

You can pick aces in $\mathbf{C}(4,2)$ ways and kings in $\mathbf{C}(4,3)$ ways. Neither choice affects the other so the answer is $\mathbf{C}(4,2) \cdot \mathbf{C}(4,3) = 6 \cdot 4 = 24$.

Mixed Counting Problems

(b) How many poker hands consist of 2 Aces, 2 Kings and a card of a different denomination?

You can pick the 2 aces, 2 kings in $\mathbf{C}(4,2) \cdot \mathbf{C}(4,2) = 6 \cdot 6 = 36$ ways. You can pick the remaining card in any of $52 - 8 = 44$ ways so the answer is $36 \cdot 44 = 1,584$.

(c) How many Poker hands have three cards from one denomination and two from another (a full house)?

There are 13 ways to pick the first denomination. Then are then $\mathbf{C}(4,3)$ways to pick 3 cards of that denomination. There are 12 ways to pick the second denomination and then $\mathbf{C}(4,2)$ ways to pick 2 cards of that denomination. Hence there are
$13 \cdot \mathbf{C}(4,3) \cdot 12 \cdot \mathbf{C}(4,2) = 13 \cdot 4 \cdot 12 \cdot 6 = 3,744$.

Mixed Counting Problems

(d) A royal flush is a hand consisting of an Ace, King, Queen, Jack and Ten, where all cards are from the same suit. How many royal flushes are possible?

There is exactly 1 way to pick a royal flush in each suit so there are 4 of them.

(e) A flush is a hand consisting of five cards from the same suit. How many different flushes are possible?

There are $\mathbf{C}(13, 5)$ ways to get all cards of the same suit so there are $\mathbf{C}(13, 5) \cdot \mathbf{C}(4, 1) = 1,287 \cdot 4 = 5,148$ flushes.

Mixed Counting Problems

Another useful model to keep in mind is that of repeatedly flipping a coin. This is especially useful for counting the number of outcomes of a given type when the experiment involves several repetitions of an experiment with two outcomes. We will explore probabilities for experiments of this type later when we study the Binomial distribution. We have already used this model in taxi cab geometry.

Example: Coin Flipping Model If I flip a coin 20 times, I get a sequence of Heads (H) and tails (T).

(a) How many different sequences of heads and tails are possible?

There are 2 ways the first flip can come up; 2 more for the second and so on. Hence $2 \cdots 2 = 2^{20} = 1,048,576$.

Mixed Counting Problems

(b) How may different sequences of heads and tails have exactly five heads?

Now we want to keep track of how many heads/tails there are in our sequence. This problem is similar to the taxi cab problem. There are 20 positions which need to be filled with either an 'H' or a 'T'. If we want exactly h heads in the sequence the answer if $\mathbf{C}(20, h)$.

To see we are on the right track recall

$$\boxed{2^n = \mathbf{C}(n, 0) + \mathbf{C}(n, 1) + \mathbf{C}(n, 2) + \mathbf{C}(n, 3) + \cdots + \mathbf{C}(n, n)}$$

so the number of sequences with 0 heads plus the number of sequences with 1 head plus ... plus the number of sequences with 20 heads is all the sequences so should be 2^{20} as in part (a). The actual answer to our problem is $\mathbf{C}(20, 5) = 15,504$.

Mixed Counting Problems

(c) How many different sequences have at most 2 heads?

We did the work in part (b). The answer is

$$\mathbf{C}(20,0) + \mathbf{C}(20,1) + \mathbf{C}(20,2) = 1 + 20 + 190 = 211$$

(d) How many different sequences have at least three heads?

$\mathbf{C}(20,3) + \mathbf{C}(20,4) + \cdots + \mathbf{C}(20,19) + \mathbf{C}(20,20)$.

OR

$2^{20} - \big(\mathbf{C}(20,0) + \mathbf{C}(20,1) + \mathbf{C}(20,2)\big) = 1,048,576 - 211 = 1,048,365$

Mixed Counting Problems

Example To make a non-vegetarian fajita at Lopez's Grill, you must choose between a flour or corn tortilla. You must then choose one type of meat from 4 types offered. You can then add any combination of extras (including no extras). The extras offered are fajita vegetables, beans, salsa, guacamole, sour cream, cheese and lettuce. How many different fajitas can you make?

Think of this from the point of view of the kitchen. An order comes in and you need to assemble it. First you select the tortilla: 2 choices. Then you add the meat: 4 choices. So far there are $2 \cdot 4 = 8$ possibilities. Now you need to add the extras. There are 7 extras and the order can be any subset of them. Hence your choices are any subset of this set with 7 elements so $2^7 = 128$. Hence the total possible is $8 \cdot 128 = 1024$.

Extra Problems

Example (a) How many different words (including nonsense words) can you make by rearranging the letters of the word

EFFERVESCENCE

E$\mapsto$ 5; F$\mapsto$ 2; R$\mapsto$ 1; V$\mapsto$ 1; S$\mapsto$ 1; C$\mapsto$ 2; N$\mapsto$ 1. Hence there are $5+2+1+1+1+2+1=13$ letters total and so there are

$$\frac{13!}{2!\cdot 5!\cdot 2!\cdot 1!\cdot 1!\cdot 1!\cdot 1!}=\frac{\mathbf{P}(13,5)}{4}=\frac{51,891,840}{4}=12,972,960$$

words.

Extra Problems

(b) How many different 4 letter words (including nonsense words) can be made from the letters of EFFERVESCENCE, if letters cannot be repeated?

There are 7 distinct letters so if repetitions are not permitted the answer is $\mathbf{P}(7,4) = 840$.
(c) How many different 4 letter words (including nonsense words) can be made from the letters of the above word, if letters can be repeated?

Answer: 7^4. Do not confuse this with the MUCH harder problem of given 13 tiles with the letters in EFFERVESCENCE, how many 4 letter words can be produced? So for example, you could use F twice but not 3 times.

Extra Problems

Example The Notre Dame Model UN club has 20 members. Five are seniors, four are juniors, two are sophomores and nine are freshmen.

(a) In how many ways can the club select a president, a secretary and a treasurer if every member is eligible for each position and no member can hold two positions?

20 members, 3 officers so $\mathbf{P}(20, 3)$. Note you are selecting an ordered subset of 3 distinct elements.

(b) In how many ways can the club choose a group of 5 members to attend the next Model UN meeting in Washington.

Answer: $\mathbf{C}(20, 5)$. This time you need a subset of all the members which has 5 elements but the order isn't important.

Extra Problems

(c) In how many ways can the club choose a group of 5 members to attend the next Model UN meeting in Washington if all members of the group must be freshmen?

Answer: $\mathbf{C}(9, 5)$ since you now must select your subset from the set of 9 freshmen.

Extra Problems

(d) In how many ways can the group of five be chosen if there must be at least one member from each class?

There are 5 ways to select a senior, 4 ways to select a junior, 2 ways to select a sophomore and 9 ways to select a freshman. This gives $5 \cdot 4 \cdot 2 \cdot 9 = 360$ ways to select a subset with 4 elements containing one member of each class. When you have done this there are $20 - 4 = 16$ members left and you may choose any one of these to round out the group. Hence the answer is $\dfrac{360 \cdot 16}{2} = 2,880$. You must divide by 2 because each set of 5 elements selected by this procedure occurs twice.

Extra Problems

Here is another approach. Because there are 5 members in the subset and 4 classes, exactly one class occurs twice. If there are 2 seniors, these can be selected in $C(5,2)$ ways and the set filled out with 1 junior, 1 sophomore and 1 freshman, hence in $C(5,2)\cdot 4\cdot 2\cdot 9 = 720$ ways. If there are 2 juniors, these can be selected in $C(4,2)$ ways and the set filled out with 1 senior, 1 sophomore and 1 freshman, hence in $5\cdot C(4,2)\cdot 2\cdot 9 = 540$ ways. If 2 sophomores, $5\cdot 4\cdot C(2,2)\cdot 9 = 180$. If 2 freshmen, $5\cdot 4\cdot 2\cdot C(9,2) = 1,440$. Hence the answer is $720 + 540 + 180 + 1,440 = 2,880$.

Extra Problems

Example Harry Potter's closet contains 12 numbered brooms, of which 8 are Comet Two Sixty's (numbered 1 - 8) and 4 are Nimbus Two Thousand's (Numbered 9-12). Harry, Ron, George and Fred want to sneak out for a game of Quidditch in the middle of the night. They don't want to turn on the light in case Snape catches them. They reach in the closet and pull out a sample of 4 brooms.

Extra Problems

(a) How many different samples are possible?

This is not a well-defined question. Do you want to know how many different sets of brooms you can get or do you want to know how many ways there are if we keep track of which broom Harry gets, which one Ron gets, and so on.

In other words, do you want **subsets** or **ordered subsets**?

For subsets, the answers is $\mathbf{C}(12, 4) = 495$; for ordered subsets the answer is $\mathbf{P}(12, 4) = 11,880$.

Extra Problems

(b) How many samples have only Comet Two Sixty's in them?

Replace the 12 in the answers for part (a) with 8.

(c) How many samples have exactly one Comet Two Sixty in them?
The unordered version solution is familiar. There are $\mathbf{C}(8,1) = 8$ ways to pick the Comet Two Sixty and $\mathbf{C}(4,3) = 4$ ways to pick the rest so the answer is $8 \cdot 4 = 32$.

To do the ordered version, observe that once you have an unordered set of 4 elements, there are $4! = 24$ ways to order it. Hence the ordered answer is $32 \cdot 24 = 768$.

Extra Problems

(d) How many samples have at least 3 Comet Two Sixty's?

Figure out how many samples there are with exactly 3; then figure out how many there are with exactly 4 and then add the two answers.
For exactly k Comet Two Sixty's we have
$\mathbf{C}(8, k) \cdot C(4, 4-k)$ unordered subsets and therefore
$\mathbf{C}(8, k) \cdot C(4, 4-k) \cdot 4!$ ordered ones.