EXACT SUBCATEGORIES OF TRIANGULATED CATEGORIES

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Abstract. A full, extension-closed additive subcategory D of a triangulated category C in which $\mathrm{Hom}_C^{-1}(M,N) = 0$ for all M, N in D has a natural structure of exact category, with $\mathrm{Ext}_D^1(M,N) \cong \mathrm{Hom}_C^1(M,N)$ naturally.

Introduction

The heart of a t-structure on a triangulated category C is a full abelian subcategory D of C, closed under extensions in C, and satisfying $\mathrm{Hom}_C^n(M,N) = 0$ for all M, N in D and integers $n < 0$. Important examples of abelian categories arising this way include categories of perverse sheaves [1]. The purpose of this note is to prove the following fact.

Theorem. *Let C be a triangulated category and D be an extension closed subcategory of D such that $\mathrm{Hom}_D^{-1}(M,N)$ for all M, N in D. Then C has a natural structure of exact category (with short exact sequences obtained by suppressing the arrows of degree 1 in the distinguished triangles of C with vertices in D). Moreover, there are natural isomorphisms $\mathrm{Ext}_C^i(M,N) \cong \mathrm{Hom}_D^i(M,N)$ for $0 \le i \le 1$.*

Remarks. For any class S of objects of C such that $\mathrm{Hom}_C^{-1}(M,N) = 0$ for all M, N in S, the smallest extension closed full subcategory D of C containing S and a zero object of C satisfies the conditions of the theorem.

We collect for the readers convenience the relevant definitions and background in Section 1, and then provide a proof of the theorem in Section 2.

1. Exact and Triangulated Categories

1.1. **Exact categories.** Exact categories (in the sense of Quillen [9]) arise naturally as full, extension closed subcategories of abelian categories, equipped with the class of "short exact sequences" comprised of those short exact sequences of the abelian category involving objects and maps of the exact category. For our purposes here, an axiomatic description is more useful. We recall below the simplification of Quillen's exact category axioms due to Keller [5]; see also [3].

Let D be an additive category endowed with a class E of sequences

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0 \tag{1.1.1}$$

of objects and maps of D, to be called short exact sequences. One calls the maps i (resp., j) occurring in some member of E an admissible monomorphism (resp., admissible epimorphism). (Some references e.g. [5] call the maps i inflations, the

Date: Oct 21,2005.
1991 *Mathematics Subject Classification.* 18E10, 18E30.
Partially supported by the N.S.F..

maps j deflations and pairs (i, j) conflations). We say that D is an exact category (with short exact sequences E) if the following self-dual system of axioms holds:

(i) Any sequence (1.1.1) of objects and maps in D which is isomorphic to a sequence in E is in E.
(ii) For M, M' in D, the split exact sequence $0 \to M \to M \oplus M' \to M' \to 0$ is in E.
(iii) For (1.1.1) in D, $i = \ker\ j$ and $j = \operatorname{coker}\ i$ in D.
(iv) Admissible epimorphisms are closed under composition. Dually for admissible monomorphisms.
(v) Admissible epimorphisms are closed under base change by arbitrary maps in D. Dually for admissible monomorphisms.

By [5, Appendix A], these axioms are equivalent to those of Quillen [9]. For any exact category (in which the classes of extensions form sets e.g. a small exact category) one may define Yoneda Ext^i-groups with the usual properties (the arguments are essentially the same as for abelian categories, for which see [6]). See also [7].

1.2. **Triangulated categories.** We recall the definition of triangulated categories [10]; see also [4], [1], [2].

Let C be an additive category with an automorphism $M \mapsto M[1]\colon C \to C$, called translation. We write $M \mapsto M[n]$ for the n-th iterate of the translation, for any integer n and write $\operatorname{Hom}^n(M, N) := \operatorname{Hom}_C(M, N[n])$. A triangle T in C is a sequence of morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

We write this triangle as $T = (X, Y, Z; u, v, w)$. If $T' = (X', Y', Z'; u', v', w')$ is another triangle, a morphism of triangles $T \to T'$ is a triple $(f, g, h,)$ where $f \in \operatorname{Hom}(X, X')$, $g \in \operatorname{Hom}(Y, Y')$ and $h \in \operatorname{Hom}(Z, Z')$ satisfy $gu = u'f$, $hv = v'g$ and $f[1]w = w'h$. With the evident composition of morphisms, this defines the additive category of triangles of C.

1.3. A triangulated category is an additive category with translation and a family of triangles of C called the distinguished triangles of C, satisfying the following axioms:

(TR1) (a) Any triangle in C isomorphic to a distinguished triangle is distinguished.
(b) For any $u \in \operatorname{Hom}_C(X, Y)$, there is a distinguished triangle $(X, Y, Z; u, v, w)$.
(c) F or any X in C, $(X, X, 0; \operatorname{Id}_X, 0, 0)$ is a distinguished triangle.

(TR2) $(X, Y, Z; u, v, w)$ is a distinguished triangle iff $(Y, Z, X[1]; v, w, -u[1])$ is a distinguished triangle.

(TR3) If $T = (X, Y, Z; u, v, w)$ and $T' = (X', Y', Z'; u', v', w')$ are distinguished triangles and $f \in \operatorname{Hom}(X, X')$ and $Y \in \operatorname{Hom}(Y, Y')$ satisfy $gu = u'f$, there exists $h \in \operatorname{Hom}(Z, Z')$ such that (f, g, h) is a morphism of triangles $T \to T'$.

(TR4) For any triangles $T_1 = (X, Y, Z'; u, v, w)$, $T_2 = (Y, Z, X'; u', v', w')$ and $T_3 = (X, Z, Y'; u'', v'', w'')$ such that $u'' = u'u$, there exist morphisms $f\colon Z' \to Y'$ and $g\colon Y' \to X'$ in C such that $(Z', Y', X'; f, g, v[1]w')$ is a distinguished triangle and $(\operatorname{Id}_X, u', f)\colon T_1 \to T_3$ and $(u, \operatorname{Id}_Z, g)\colon T_3 \to T_2$ are morphisms of triangles.

The axiom (TR4) is called the octahedral axiom, and we call (TR2) the axiom for turning triangles. We note that the triangle in (TR1)(b) is unique up to (possibly non-unique) isomorphism

1.4. Let C be a triangulated category. An additive functor $F\colon C \to A$ where A is an abelian category is called a cohomological functor on C if each triangle $(X, Y, Z; u, v, w)$ induces a long exact sequence

$$\cdots \to F(X[p]) \xrightarrow{F(u[p])} F(Y[p]) \xrightarrow{F(v[p])} F(Z[p]) \xrightarrow{F(w[p])} F(X[p+1]) \to \cdots$$

of abelian groups. The following lemma summarizes some well-known additional facts about triangulated categories.

Lemma. (a) *The opposite category C^{op} has a structure of triangulated category with translation $M \mapsto M[-1]$ and a distinguished triangle $(X, Y, Z; u, v, w)$ in C^{op} for each distinguished triangle $(Z, Y, X; w, v, u)$ in C (here, we identify the objects and morphisms in C^{op} with those of C in the usual way).*

(b) *In a distinguished triangle $(X, Y, Z; u, v, w)$ we have $vu = 0$ (so by turning triangles, $wv = 0$, and $u[1]w = 0$).*

(c) *For M in C, $\mathrm{Hom}(M, ?)\colon C \to \mathbb{Z}\mathrm{Mod}$ (resp., $\mathrm{Hom}(?, N)\colon C^{\mathrm{op}} \to \mathbb{Z}\mathrm{Mod}$) is a cohomological functor.*

(d) *$(X, Y, Z; u, v, 0)$ is a distinguished triangle iff $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is a split exact sequence in the additive category C.*

(e) *The direct sum (in the category of triangles of C) of two distinguished triangles is a distinguished triangle.*

(f) *Let $T = (X, Y, Z; u, v, w)$ be a distinguished triangle with $\mathrm{Hom}^{-1}(X, Z) = 0$. If $T' = (X, Y, Z'; u, v', w')$ is any distinguished triangle, there is a unique isomorphism $(\mathrm{Id}_x, \mathrm{Id}_Y, h)\colon T \to T'$; moreover, if $v' = v$ then $T' = T$.*

Proof. For (a)–(c), see e.g. [4]. For (d), see [8]. For (f), see [1, 1.1.9–1.1.10]. We provide a proof of (e). If $T_i = (X_i, Y_i, Z_i; u_i, v_i, w_i)$ are triangles for $i = 1, 2$ then their direct sum is

$$T_1 \oplus T_2 := (\oplus_i X_i, \oplus_i Y_i, \oplus_i Z_i; \oplus_i u_i, \oplus_i v_i, \oplus_i w_i).$$

To prove (e), we must show that $T_1 \oplus T_2$ is distinguished if T_1 and T_2 are distinguished. The proof is similar to that of the standard fact that if (f, g, h) is a morphism of distinguished triangles in which f and g are isomorphisms, then h is an isomorphism (see [4]). Choose a distinguished triangle $T = (\oplus X_i, \oplus Y_i, E; \oplus u_i, v, w)$. Letting $\pi_{i,X}$ and $\pi_{i,Y}$ denote the projections for $\oplus X_i$ and $\oplus Y_i$ respectively, we may choose a morphism $(\pi_{i,X}, \pi_{i,Y}, t_i)\colon T \to T_i$ for each $i = 1, 2$. These induce a morphism $f = (\mathrm{Id}_{\oplus X_i}, \mathrm{Id}_{\oplus Y_i}, t)\colon T \to T_1 \oplus T_2$ where $t = \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right)$. Now any (covariant or contravariant) cohomological functor F on C gives long exact sequences corresponding to $T_1 \oplus T_2$ and T, and F induces a morphism between these long exact sequences. By the 5-lemma, $F(t)$ is an isomorphism for any such F. Taking in turn $F = \mathrm{Hom}(\oplus Z_i, ?)$ and $F = \mathrm{Hom}(?, E)$ shows t has right and left inverses, so t is an isomorphism. Hence f is an isomorphism so $T_1 \oplus T_2 \cong T$ is distinguished. $\square$

1.5. A subcategory D of the triangulated category C is said to closed under extensions (see [1, 1.2.6]) if for every triangle $(X, Y, Z; u, v, w)$ with X and Z in D, one has Y in D also. A full extension closed subcategory of C containing a zero object is additive (by 1.4(d)) and strict (i.e. contains all isomorphs in C of any of its objects).

2. Proof of the Theorem

Throughout this section, we write "triangle" to mean "distinguished triangle."

2.1. We restate in fuller detail the main result of this note.

Theorem. *Let D be a full additive subcategory of the triangulated category C, with D closed under extensions in C. Assume that $\mathrm{Hom}_C^{-1}(M, N) = 0$ for all M, N in D. Then*

(a) *D has a natural structure of exact category in which $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is a short exact sequence of C iff there is a triangle $(X, Y, Z; u, v, w)$ for some $w\colon Z \to X[1]$.*
(b) *$\mathrm{Ext}_D^i(M, N) \cong \mathrm{Hom}_C^i(M, N)$ naturally for $0 \le i \le 1$ and M, N in D.*
(c) *For a distinguished triangle $T = (X, Z, Y'; u'', v'', w'')$ in C with X, Y', Z in D, and corresponding short exact sequence $E\colon 0 \to X \xrightarrow{u''} Z \xrightarrow{v''} Y' \to 0$ in C, the natural squares*

$$\begin{tikzcd}
\mathrm{Hom}_C(M, Y') \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}_C^1(M, X) \arrow[d, "\cong"] \\
\mathrm{Hom}_D(M, Y') \arrow[r] & \mathrm{Hom}_D^1(M, X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}_C(X, M) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}_C^1(Y', M) \arrow[d, "\cong"] \\
\mathrm{Hom}_D(Y', M) \arrow[r] & \mathrm{Hom}_D^1(Y', M)
\end{tikzcd}$$

commute for any M in C i.e. the connecting homomorphisms (horizontal maps in the squares) induced by T and E may be naturally identified under the isomorphisms of (b).

2.2. **Proof of 2.1(a).** We verify the exact category axioms given in 1.1. The condition $\mathrm{Hom}_{C^{\mathrm{op}}}^{-1}(M, N) = 0$ holds for M, N in the subcategory D^{op} of C^{op}, so we may argue using duality. First note that by 1.4(f), for any short exact sequence $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ in C, there is a unique morphism $w\colon Z \to X[1]$ so $T = (X, Y, Z; u, v, w)$ is a triangle; moreover, any triangle $T' = (X, Y, Z'; u, v', w')$ is isomorphic to T, and so in particular, Z' is in D. Obviously, 1.1(i) follows from (TR1)(a). The fact that D is additive, together with 1.1(ii), follows from 1.4(d). Applying $\mathrm{Hom}_C(M, ?)$ and $\mathrm{Hom}_C(?, M)$ to a triangle T as above with X, Y, Z in D, for M in D, shows that $u = \ker v$ and $v = \operatorname{coker} u$ in D, by 1.4(c) and our assumption $\mathrm{Hom}_C^{-1}(N, N') = 0$ for N, N' in D (cf [1, 1.2]). This establishes 1.1(iii).

If one supposes in (TR4) that u and u' are admissible monomorphisms in D, then X, Y, Z', Z and X' are in D, so Y' is in D since D is closed under extensions in C and therefore $u'u$ is an admissible monomorphism in D. Together with the dual argument, this establishes 1.1(iv).

Now we show admissible monomorphisms are closed under cobase change by arbitrary maps in D. Suppose that $u'' : X \to Z$ is an admissible monomorphism in

D and $a\colon X \to W$ is an arbitrary map in D. We must construct a pushout square

$$\begin{tikzcd} X \arrow[r, "u''"] \arrow[d, "a"] & Z \arrow[d, "c_1"] \\ W \arrow[r, "c_2"] & Z' \end{tikzcd}$$

in D in which c_2 is an admissible monomorphism. The axioms (TR1)–(TR3) assure that in the octahedral axiom we may take $Y = Z \oplus W$, $X' = W[1]$, $u = \begin{pmatrix} u'' \\ -a \end{pmatrix}$, $u' = (\mathrm{Id}_Z, 0)$, $v' = 0$ and $w' = \begin{pmatrix} 0 \\ -\mathrm{Id}_{W[1]} \end{pmatrix}$; here, the triangle $(Y, Z, X'; u', v', w')$ arises by turning the triangle

$$(W, Z \oplus W, Z; \begin{pmatrix} 0 \\ \mathrm{Id}_W \end{pmatrix}, (\mathrm{Id}_Z, 0), 0).$$

Let f, g be as in the statement of the octahedral axiom. Writing $v = (c_1, c_2)$ where $c_1 \in \mathrm{Hom}(Z, Z')$ and $c_2 \in \mathrm{Hom}(W, Z')$, we have $v[1]w' = -c_2[1]$. Turning the triangle $(Z', Y', X'; f, g, v[1]w')$ therefore gives a triangle $(W, Z', Y'; c_2, f, g)$ which shows that Z' is in D (since D is closed under extensions in C; note Y' is in D since u'' is an admissible monomorphism) and that c_2 is an admissible monomorphism in D. The triangle $(X, Y, Z'; u, v, w)$ gives a short exact sequence $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z' \to 0$ of D and hence we conclude that $v = \operatorname{coker} u$ in D by the first paragraph of the proof. Thus, we have a pushout square as desired. Together with the dual argument, this establishes 1.1(v) and completes the proof of 2.1(a). Observe for future reference that we have a morphism of triangles

$$(a, c_1, \mathrm{Id}_{Y'})\colon (X, Z, Y'; u'', v'', w'') \to (W, Z', Y'; c_2, f, g). \tag{2.2.1}$$

2.3. **Proof of 2.1(b).** We construct a natural isomorphism

$$\eta\colon \mathrm{Ext}^1_D(-, ?) \cong \mathrm{Hom}^1_C(-, ?)$$

with components $\eta_{Y',X}\colon \mathrm{Ext}^1_D(Y', X) \to \mathrm{Hom}^1_C(Y', X)$ for Y', X in D. Consider an element e of $\mathrm{Ext}^1_D(Y', X)$ represented by the class of an extension

$$0 \to X \xrightarrow{u''} Z \xrightarrow{v''} Y' \to 0 \tag{2.3.1}$$

in D. There is a unique element $w'' \in \mathrm{Hom}^1_C(Y', X)$ such that $(X, Z, Y'; u'', v'', w'')$ is a triangle, and we set $\eta_{Z,X}(e) = w''$. One readily verifies this gives a well-defined map. Naturality in X of the maps $\eta_{Y',X}$ follows from (2.2.1), and naturality in Y' follows by duality. Once one has naturality, it follows that the $\eta_{Y',X}$ are abelian group homomorphisms using 1.4(e) and the definition (see e.g. [6, ChVII]) of Baer sum of extensions. Observe $\eta_{Y',X}$ is injective since $w'' = \eta_{Y',X}(e) = 0$ above implies (2.3.1) is split, by Lemma 1.4(d). Finally, $\eta_{Z,X}$ is surjective since given $w'' \in \mathrm{Hom}^1_C(Y', X)$, the triangulated category axioms assure there is a triangle $(X, Z, Y'; u'', v'', w'')$ for some Z in C and u, v; since D is closed under extensions in C, Z is in D and the triangle gives a short exact sequence (2.3.1) whose class $e \in \mathrm{Ext}^1(Y', X)$ satisfies $\eta_{Y',X}(e) = w$. This completes the proof of 2.1(b).

2.4. **Proof of 2.1(c).** Using the definitions as in [6], the commutativity of the first square in (c) follows readily from 2.1(b) and (2.2.1). The commutativity of the second square follows dually.

Remarks. Using the definition of Yoneda product, one can show the isomorphisms of 2.1(b) can be spliced together to give a well-defined natural map

$$\mathrm{Ext}^n_D(M,N) \to \mathrm{Hom}^n_C(M,N),$$

defined to be zero for $n < 0$, which is compatible with the connecting morphisms of long exact sequences induced by a short exact sequence in D and corresponding triangle in C.

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