# Excess intersections and numerical irreducible decompositions

Daniel J. Bates* David Eklund† Jonathan D. Hauenstein‡
Chris Peterson§

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**Abstract**

A fundamental problem in algebraic geometry is to decompose the solution set of a given polynomial system. A numerical description of this solution set is called a numerical irreducible decomposition and currently all standard algorithms use a sequence of homotopies forming a dimension-by-dimension approach. In this article, we pair a classical result to compute a smooth point on every irreducible component in every dimension using a single homotopy together with the theory of isosingular sets. Examples are presented to compare this approach with current algorithms for computing a numerical irreducible decomposition.
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## 1 Introduction

For a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$, the solution set of $f$ is $\mathcal{V}(f) = \{x \in \mathbb{C}^N \mid f(x) = 0\}$. The solution set $\mathcal{V}(f)$ has a unique decomposition, called the *irreducible decomposition* of $\mathcal{V}(f)$, into a finite number of inclusion maximal irreducible algebraic sets. This geometric decomposition of $\mathcal{V}(f)$ corresponds to the prime decomposition of the radical of the ideal generated by the polynomials of $f$. A third means of describing this decomposition is afforded by numerical methods. A *numerical irreducible decomposition* of $\mathcal{V}(f)$ consists of a set of numerical approximations of points on each irreducible component of $\mathcal{V}(f)$, along with various auxiliary data. There are several techniques for computing numerical irreducible decompositions, each of which requires

*Department of Mathematics, United States Naval Academy, Annapolis, MD, 21402 (dbates@usna.edu). This author was partially supported by Colorado State University, Institut Mittag-Leffler (Djursholm, Sweden), and NSF Grants DMS-1115668 and DMS-1719658.

†DTU Compute, Richard Petersens Plads, Building 324, DK-2800 Kgs. Lyngby, Denmark (daek@math.kth.se, `https://people.kth.se/~daek/`). This author was partially supported by Institut Mittag-Leffler (Djursholm, Sweden).

‡Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556 (hauenstein@nd.edu, `www.nd.edu/~jhauenst`). This author was partially supported by Institut Mittag-Leffler (Djursholm, Sweden), and NSF grant CCF-1812746.

§Department of Mathematics, Colorado State University, Fort Collins, CO 80523 (peterson@math.colostate.edu, `www.math.colostate.edu/~peterson`). This author was partially supported by NSF grants BIGDATA:F:1633830 and CCF-BSF:CIF:Small:1712788.

a sequence of several homotopies, typically one or more for each dimension in which $\mathcal{V}(f)$ might have an irreducible component.

Given a polynomial system $f$, our proposed algorithm computes a numerical irreducible decomposition of $\mathcal{V}(f)$ by using one homotopy to compute points on $\mathcal{V}(f)$ and then applies some post-processing to form the decomposition. First, a finite set of points $S \subset \mathcal{V}(f)$ is computed with a single homotopy such that $S$ is theoretically guaranteed to contain at least one smooth point on every irreducible component of $\mathcal{V}(f)$. This guarantee comes from intersection theory, which we summarize in Appendix A. As is typically the case with numerical computation, one should view the algorithm as a probability one method based on selecting random numbers.

After computing $S$, the proposed algorithm computes a numerical irreducible decomposition by computing the isosingular set [10] associated with each point in $S$ with respect to $f$. In short, for $x \in \mathcal{V}(f)$, the isosingular set of $x$ with respect to $f$ is an irreducible algebraic subset of $\mathcal{V}(f)$ containing $x$ such that every general point in the isosingular set has the same singularity structure, which is described by its deflation sequence, at $x$. The irreducible components of $\mathcal{V}(f)$ are the inclusion maximal elements of the set of isosingular sets for each $s \in S$.

This article is structured as follows. The remainder of this section introduces necessary concepts from algebraic geometry and numerical algebraic geometry with, for example, the books [4, 22] providing more details. Section 2 provides a homotopy to compute at least one smooth point on each irreducible component. Section 3 describes using isosingular theory to complete a numerical irreducible decomposition. After examples are presented in Section 4, Appendix A provides an exposition of the related concepts and results from intersection theory.

## 1.1 Irreducible decomposition and local dimension

A set $V \subset \mathbb{C}^N$ is called an *algebraic set* if there exists a polynomial system $g : \mathbb{C}^N \to \mathbb{C}^n$ such that $V = \mathcal{V}(g)$. An algebraic set $V$ is called *irreducible* if, for any algebraic sets $W_1, W_2 \subset \mathbb{C}^N$ such that $V = W_1 \cup W_2$, then either $V = W_1$ or $V = W_2$. For an irreducible algebraic set $V$, the set of manifold points in $V$ is connected and its dimension as a complex manifold is defined to be the dimension of $V$, denoted $\dim V$.

Every algebraic set $V \subset \mathbb{C}^N$ can be written uniquely as the union of finitely many irreducible algebraic sets, no one of which is contained in the union of the others. That is, there exist irreducible algebraic sets $Z_1, \ldots, Z_m \subset \mathbb{C}^N$, unique up to reordering, such that

$$Z_i \not\subset \bigcup_{j \neq i} Z_j \qquad \text{and} \qquad V = \bigcup_{i=1}^{m} Z_i.$$

The irreducible algebraic sets $Z_1, \ldots, Z_m$ are called the *irreducible components* of $V$ with

$$\dim V = \max_{i=1,\ldots,m} \dim Z_i.$$

Another useful presentation of an algebraic set is to first group by dimension. Let $V$ be an algebraic set of dimension $\ell$ with irreducible components $Z_1, \ldots, Z_m$. For $i = 0, 1, \ldots, \ell$, let

$$V_i = \bigcup_{\dim Z_j = i} Z_j$$

and suppose that $V_{i,1}, \ldots, V_{i,k_i}$ are the $k_i$ irreducible components of $V_i$. Then,

$$V = \bigcup_{i=0}^{\ell} V_i = \bigcup_{i=0}^{\ell} \bigcup_{j=1}^{k_i} V_{i,j}. \tag{1}$$

The set $V_i$ is called the *pure $i$-dimensional component* of $V$. Moreover, given a point $x \in V$, the *local dimension* of $x$ with respect to $f$ is

$$\dim_f(x) = \max\{i \mid x \in V_i\}.$$

**Example 1.1** For $f = \begin{bmatrix} xy \\ xz \end{bmatrix}$, the pure-dimensional components of $V = \mathcal{V}(f) \subset \mathbb{C}^3$ are

$$V_2 = \{(0, y, z) \mid y, z \in \mathbb{C}\} \quad \text{and} \quad V_1 = \{(x, 0, 0) \mid x \in \mathbb{C}\},$$

each of which is irreducible. Thus, $\dim V = 2$ with $\dim_f((0,0,0)) = 2$ and $\dim_f((3,0,0)) = 1$.

## 1.2 Numerical irreducible decomposition and witness sets

A numerical irreducible decomposition provides a representation of algebraic sets similar to (1) using witness sets. This section summarizes these two concepts with more details in [21, 22].

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system, $V \subset \mathcal{V}(f)$ be a pure $i$-dimensional algebraic set of degree $d$, and $\mathcal{L} : \mathbb{C}^N \to \mathbb{C}^i$ be a system of $i$ general linear polynomials. Then, $W = V \cap \mathcal{V}(\mathcal{L})$ is a set of $d$ points, called a *witness point set* for $V$. A *witness set* for $V$ is the triple $\mathcal{W} = \{f, \mathcal{L}, W\}$. A finite set $\widehat{W} \subset \mathbb{C}^N$ is called a *witness point superset* for $V$ if $V \cap \mathcal{V}(\mathcal{L}) \subset \widehat{W} \subset \mathcal{V}(f) \cap \mathcal{V}(\mathcal{L})$.

If $Z_1, \ldots, Z_k$ are the irreducible components of the pure $i$-dimensional algebraic set $V$, then the witness point set $W$ for $V$ is the disjoint union of the finite sets $W_i = Z_i \cap \mathcal{V}(\mathcal{L})$. In particular, $W_i$ is a witness point set for $Z_i$, called an *irreducible witness point set*, and $\{f, \mathcal{L}, W_i\}$ is a witness set for $Z_i$, called an *irreducible witness set.*

Finally, a *numerical irreducible decomposition* of $\mathcal{V}(f)$ has the form

$$\bigcup_{i=0}^{\dim \mathcal{V}(f)} \bigcup_{j=1}^{k_i} \mathcal{W}_{i,j} \tag{2}$$

where $\mathcal{W}_{i,j}$ is an irreducible witness set for a distinct $i$-dimensional irreducible component of $\mathcal{V}(f)$. We note that the union of witness sets in (2) should be considered as a formal union.

**Example 1.2** Continuing with the setup from Ex. 1.1, consider the linear systems

$$\mathcal{L}_1 = 2x + 3y - 4z - 1 \quad \text{and} \quad \mathcal{L}_2 = \begin{bmatrix} \mathcal{L}_2 \\ 3x - 2y + z - 2 \end{bmatrix}.$$

Then, $W_2 = V_2 \cap \mathcal{V}(\mathcal{L}_2) = \{(0, -9/5, -8/5)\}$ and $W_1 = V_1 \cap \mathcal{V}(\mathcal{L}_1) = \{(1/2, 0, 0)\}$ are witness point sets for $V_2$ and $V_1$, respectively. Moreover, $\mathcal{W}_2 = \{f, \mathcal{L}_2, W_2\}$ and $\mathcal{W}_1 = \{f, \mathcal{L}_1, W_1\}$ are witness sets for $V_2$ and $V_1$, respectively, with the formal union $\mathcal{W}_2 \cup \mathcal{W}_1$ being a numerical irreducible decomposition for $\mathcal{V}(f)$.

A common approach for computing a numerical irreducible decomposition for $\mathcal{V}(f)$, given a polynomial system system $f : \mathbb{C}^N \to \mathbb{C}^n$, consists of three steps. The first step is to compute a witness point superset for each pure $i$-dimensional algebraic set $V_i$ of $\mathcal{V}(f)$, say $\widehat{W}_i$. Standard algorithms for this step, listed in chronological order, include the dimension-by-dimension slicing approach [21], the cascade algorithm [16], and the regenerative cascade algorithm [9]. The second step is to compute a witness point set $W_i$ from each witness point superset $\widehat{W}_i$. Standard algorithms for this step are the homotopy membership test [18] and the local dimension test [2]. The third step is to decompose each witness point set $W_i$ into irreducible witness point sets. The standard technique for this step includes the application of monodromy loops [20] with a stopping criterion given by the trace test [19]. We discuss the homotopy membership test, monodromy loops, and the trace test in more detail in the next section.

## 1.3 Membership, monodromy, and traces

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $V \subset \mathcal{V}(f)$ be an irreducible component with witness set $\{f, \mathcal{L}, W\}$. The *multiplicity* of $V$ with respect to $f$ is the multiplicity of any $w \in W$ with respect to $\begin{bmatrix} f \\ \mathcal{L} \end{bmatrix}$. If $V$ has multiplicity one with respect to $f$, then $V$ is said to be *generically reduced* with respect to $f$. Otherwise, $V$ is said to be *generically nonreduced* with respect to $f$. In particular, $V$ is generically reduced with respect to $f$ if and only if $\dim V = \dim \operatorname{null} Jf(x)$ for a general $x \in V$ where $Jf$ is the Jacobian matrix of $f$, i.e., the matrix of first partial derivatives of the polynomials of $f$. In this case, a point $z \in V$ is called a *singular point* of $V$ with respect to $f$ if $\dim \operatorname{null} Jf(z) > \dim V$; otherwise, $z$ is called a *manifold point* or *smooth point*.

Suppose that $V$ is a generically reduced and irreducible component of $\mathcal{V}(f)$ of dimension $d > 0$. To determine whether some given $z \in \mathcal{V}(f)$ lies on the particular component $V$, the homotopy membership test [18] makes use of the witness set $\mathcal{W}$ together with linear slice moving. More specifically, let $\mathcal{L}_z : \mathbb{C}^N \to \mathbb{C}^d$ be a system of general linear polynomials with $z \in \mathcal{V}(\mathcal{L}_z)$. Consider the homotopy $\mathcal{H} : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ which deforms $\mathcal{L}$ to $\mathcal{L}_z$ along $\mathcal{V}(f)$, namely

$$\mathcal{H}(x,t) = \begin{bmatrix} f(x) \\ \mathcal{L}_z(x) \cdot (1-t) + \mathcal{L}(x) \cdot t \end{bmatrix}.$$

Let $E \subset \mathcal{V}(f) \cap \mathcal{V}(\mathcal{L}_z)$ be the set of endpoints of the homotopy paths defined by $\mathcal{H} = 0$ starting at the points in $W = V \cap \mathcal{V}(\mathcal{L})$ at $t = 1$. Then, $z \in V$ if and only if $z \in E$.

Suppose that $z \in \mathcal{V}(f)$ is a smooth point on some generically reduced $d$-dimensional irreducible component so that $\dim_f(z) = \dim \operatorname{null} Jf(z) = d$. Let $\mathcal{L}_z : \mathbb{C}^N \to \mathbb{C}^d$ be a system of $d$ general linear polynomials with $z \in \mathcal{V}(\mathcal{L}_z)$. Suppose that $\Phi(t)$, for $0 \leq t \leq 1$, defines a general continuous, closed path in the space of $d$-tuples of linear polynomials in $N$ variables which starts and ends at $\mathcal{L}_z$. That is, for each $t \in [0,1]$, $\Phi(t) : \mathbb{C}^N \to \mathbb{C}^d$ is a general linear polynomial system with $\Phi(0) = \Phi(1) = \mathcal{L}_z$. The homotopy path starting at $z$ at $t = 1$ for the homotopy $\mathcal{H} : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ defined by

$$\mathcal{H}(x,t) = \begin{bmatrix} f(x) \\ \Phi(t)(x) \end{bmatrix}$$

defines a *monodromy loop*. Let $\widehat{z}$ be the endpoint of the path defined by $\mathcal{H} = 0$ starting at $z$ at $t = 1$. By connectedness of the set of smooth points of an irreducible algebraic set, $z$ and $\widehat{z}$ must lie on the same irreducible component. Therefore, monodromy loops can be used to determine which witness points must lie on the same irreducible component as described in [20].

**Example 1.3** For $f = y - x^2$, consider the parabola $\mathcal{V}(f) \subset \mathbb{C}^2$ with $z = (1,1)$ and linear $\mathcal{L}_z = 2x - 3y + 1$. Consider the loop $\Phi(t)(x,y) = 2x - 3y + e^{2\pi\sqrt{-1}(1-t)}$ so that $\Phi(0) = \Phi(1) = \mathcal{L}_z$. The resulting monodromy loop yields $\widehat{z} = (-1/3, 1/9)$.

As a complement to the necessary condition obtained by monodromy loops, the *trace test* [19] was developed to provide a stopping criterion, that is, to determine if a given set of witness points forms a witness point set. More specifically, suppose that $W \subset \mathcal{V}(f) \cap \mathcal{V}(\mathcal{L})$ is a finite set consisting of smooth points on the union of generically reduced $d$-dimensional irreducible components of $\mathcal{V}(f)$. Let $v \in \mathbb{C}^d$ and $\lambda : \mathbb{C}^N \to \mathbb{C}$ be general. For each $w \in W$, let $w(t)$ be the homotopy path defined by $\mathcal{V}(f) \cap \mathcal{V}(\mathcal{L} + t \cdot v)$, with $w(0) = w$. Then, $W$ is a union of irreducible witness point sets if and only if the following is linear in $t$:

$$\phi_W(t) = \sum_{w \in W} \lambda(w(t)).$$

**Example 1.4** Continuing with the setup from Ex. 1.3 with $\mathcal{L} = 2x - 3y + 1$, the following applies the trace test to $W_1 = \{(1,1)\}$ and $W_2 = \{(1,1), (-1/3, 1/9)\}$ where $v = 1 + \sqrt{-1}$ and $\lambda(x,y) = 5x + 2y$. In particular,

$$\phi_{W_1}(t) = \frac{25 + 6tv + 19\sqrt{4 + 3tv}}{9}$$

is not linear in $t$ while

$$\phi_{W_2}(t) = \frac{50 + 12tv}{9}$$

is linear in $t$. Hence, combining with Ex. 1.3 shows that $W_2$ is an irreducible witness point set.

In practice, linearity of $\phi_W(t)$ is tested by evaluation of the function at three distinct points and even testings its first and second derivatives [5].

## 1.4 Deflation sequences and smooth points

For generically reduced irreducible components $V \subset \mathcal{V}(f)$, we have already described the smooth points with respect to $f$, namely $z \in V$ such that $\dim_f(z) = \dim \operatorname{null} Jf(z) = \dim V$ where $Jf(z)$ is the Jacobian matrix of $f$ evaluated at $z$. The following extends the notion of smooth points to generically nonreduced irreducible components via deflation sequences from [10].

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system, $y \in \mathcal{V}(f)$, and define $\operatorname{dnull}(f,y) = \dim \operatorname{null} Jf(y)$. Consider the deflation operator $\mathcal{D}$ which, to a pair consisting of a polynomial system $f$ and a point $y \in \mathcal{V}(f)$, assigns a polynomial system $\mathcal{F}$ and the point $y$, that is $(\mathcal{F}, y) = \mathcal{D}(f,y)$. The polynomial system $\mathcal{F}$ is the union of $f$ and all $(N - d + 1) \times (N - d + 1)$ minors of $Jf$ where $d = \operatorname{dnull}(f,y)$. By construction, $y \in \mathcal{V}(\mathcal{F})$ and thus we can iterate the deflation operator $\mathcal{D}$ to generate a sequence

$$(\mathcal{F}_k, y) = \mathcal{D}^k(f,y) \text{ for } k = 1, 2, \ldots \tag{3}$$

beginning with $\mathcal{D}^0(f,y) = (f,y)$. The *deflation sequence* of $y$ with respect to $f$ is the sequence of integers $\{d_k(f,y)\}_{k=0}^{\infty}$ such that

$$d_k(f,y) = \operatorname{dnull}(\mathcal{D}^k(f,y)).$$

Every deflation sequence is a monotonically decreasing sequence of nonnegative integers and thus has a limit, called the *isosingular local dimension* of $y$ with respect to $f$, denoted $\operatorname{isodim}_f(y)$. The isosingular local dimension is a lower bound on the local dimension, i.e., $\operatorname{isodim}_f(x) \leq \dim_f(x)$.

**Example 1.5** [10, Ex. 5.3] For $f = x_1^2 - x_2^2 x_3$ defining the Whitney umbrella, consider the points $y_1 = (1,1,1)$, $y_2 = (0,0,1)$, and $y_3 = (0,0,0)$. Clearly, $\dim_f(y_i) = 2$ for $i = 1,2,3$. The deflation sequences are:

$$y_1 : \{2,2,2,2,\ldots\}, \quad y_2 : \{3,1,1,1,\ldots\}, \quad y_3 : \{3,2,0,0,\ldots\}$$

yielding $\operatorname{isodim}_f(y_1) = 2$, $\operatorname{isodim}_f(y_2) = 1$, and $\operatorname{isodim}_f(y_3) = 0$.

One can extend the definition of deflation sequences to nonempty irreducible algebraic sets $V \subset \mathcal{V}(f)$. In particular, there is a nonempty Zariski open set $Z \subset V$ such that every point in $Z$ has the same deflation sequence with respect to $f$. Hence, the deflation sequence of $V$, denoted $\{d_k(f,V)\}_{k=0}^{\infty}$, is defined to be the deflation sequence at any point in $Z$.

**Example 1.6** [10, Ex. 5.3] Continuing with $f$ in Ex. 1.5, consider the following two nonempty positive-dimensional irreducible algebraic subsets of $\mathcal{V}(f)$: $X_1 = \mathcal{V}(f)$ and $X_2 = \mathcal{V}(x_1, x_2)$. The deflation sequence for $X_i$ is the same as for $y_i$ in Ex. 1.5 for $i = 1, 2$.

Let $V \subset \mathcal{V}(f)$ be an irreducible component. A point $y \in V$ is called a *singular point* of $V$ with respect to $f$ if the deflation sequence for $V$ and $y$ are different. Hence, the *smooth points* of $V$ with respect to $f$ consist of the points $y \in V$ which have the same deflation sequence as $V$. These definitions agree with the classical notion of smooth and singular points for generically reduced algebraic sets. In particular, if $\mathrm{Sing}_f(V)$ is the set of singular points of $V$ with respect to $f$, then $\mathrm{Sing}_f(V)$ is an algebraic set with $\dim \mathrm{Sing}_f(V) < \dim V$ [10, Lemma 5.7].

# 2 A smooth point on every irreducible component

The first step of our approach is to utilize a single homotopy to compute a finite set of points which contains at least one smooth point on every irreducible component. In Appendix A, we provide an exposition of some classical results in intersection theory for projective space, such as Corollary A.17. In this section, we present the theory for affine algebraic sets by reducing down to the homogeneous case.

Suppose that $f : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system and let $r$ be the *rank* of $f$ [22, § 13.4]. That is, $r = \dim \overline{f(\mathbb{C}^N)} \leq n$ where $\overline{f(\mathbb{C}^N)} \subset \mathbb{C}^n$ is the Euclidean closure of $f(\mathbb{C}^N)$ in $\mathbb{C}^n$. In particular, $r$ is equal to the rank of the Jacobian of $f$ at a general point in $\mathbb{C}^N$. Since every irreducible component of $\mathcal{V}(f)$ has codimension at most $r$, we aim to construct $h : \mathbb{C}^r \to \mathbb{C}^r$ from $f$ as follows.

First, we employ randomization to reduce from $n$ polynomials to $r$ polynomials. In particular, by Bertini's Theorem [22, Thm. 13.5.1], there is a nonempty Zariski open set $\mathcal{A} \subset \mathbb{C}^{r\times n}$ such that, for all $A \in \mathcal{A}$, if $V \subset \mathcal{V}(f)$ is an irreducible component, then $V$ is also an irreducible component of $\mathcal{V}(A \cdot f)$. One may reorder $f$ so that the degrees of the entries of $f$ are decreasing and select $A \in \mathcal{A}$ to be of the form $[I_r \ A']$ where $I_r$ is the $r\times r$ identity matrix and $A' \in \mathbb{C}^{r\times(n-r)}$. Then, the degree of the $j^{\text{th}}$ polynomial of $f$ and $A \cdot f$ are the same.

Next, we employ intrinsic slicing to reduce from solving $A \cdot f$ on $\mathbb{C}^N$ to solving on $\mathbb{C}^r$. There is a nonempty Zariski open set $\mathcal{B}_A \subset \mathbb{C}^{N\times r} \times \mathbb{C}^N$ such that, for every $(B, b) \in \mathcal{B}_A$, the linear space $\{By + b \mid y \in \mathbb{C}^r\} \subset \mathbb{C}^N$ intersects each irreducible component of $\mathcal{V}(A \cdot f)$ transversely. In particular, fix $(B, b) \in \mathcal{B}_A$ and define

$$h(y) = A \cdot f(By + b) \quad \text{and write} \quad h(y) = \begin{bmatrix} h_1(y) \\ \vdots \\ h_r(y) \end{bmatrix}.$$

The following is derived from Appendix A.

**Theorem 2.1** *Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system of rank $r$ and $h : \mathbb{C}^r \to \mathbb{C}^r$ be constructed from $f$ as above. Suppose that $g : \mathbb{C}^r \to \mathbb{C}^r$ is a general polynomial system such that $d_i = \deg g_i = \deg h_i$ and $p : \mathbb{C}^r \to \mathbb{C}^r$ is defined by $p_i(y) = y_i^{d_i} - 1$. Let $E_h \subset \mathcal{V}(h) \subset \mathbb{C}^r$ be the set of finite endpoints of the homotopy*

$$H(y, t) = (1 - t)h(y) + t(1 - t)g(y) + tp(y) \tag{4}$$

*starting at $t = 1$ with the $d_1 \cdots d_r$ solutions of $p = 0$ and define*

$$E_f = \left\{By + b \in \mathbb{C}^N \;\middle|\; y \in E_h \text{ and } f(By + b) = 0\right\}. \tag{5}$$

*Then, $E_f$ is a finite set containing at least one smooth point on each irreducible component of $\mathcal{V}(f)$.*

**Proof.** Consider the homogenization of the homotopy $H$, which is defined by

$$\mathcal{H}_i([z_0 : z_1 : \cdots : z_r], t) = z_0^{d_i} \cdot H_i\left(\left(\frac{z_1}{z_0}, \ldots, \frac{z_r}{z_0}\right), t\right)$$

and the homogenization of the polynomial system $h$, which is defined by

$$F_i([z_0 : z_1 : \cdots : z_r]) = z_0^{d_i} \cdot h_i\left(\frac{z_1}{z_0}, \ldots, \frac{z_r}{z_0}\right)$$

for $i = 1, \ldots, r$. Hence, $\mathcal{H} = \{\mathcal{H}_1, \ldots, \mathcal{H}_r\}$ is a first order general homotopy for $F = \{F_1, \ldots, F_r\}$ so that Theorem A.16 applies to $\mathcal{H}$ with, say, endpoints $E_F \subset \mathbb{P}^r$ where we construct the proper closed subsets as follows.

For an algebraic set $Y \subset \mathbb{C}^r$, define

$$\overline{Y} = \overline{\{[1 : y] \mid y \in Y\}} \subset \mathbb{P}^r$$

which is an algebraic set. Let $\mathcal{L} : \mathbb{C}^N \to \mathbb{C}^{N-r}$ be a linear system with $\mathcal{V}(\mathcal{L}) = \{By + b \mid y \in \mathbb{C}^r\}$. For an algebraic set $V \subset \mathcal{V}(f)$, define

$$Y_V = \{y \in \mathbb{C}^r \mid By + B \in V \cap \mathcal{V}(\mathcal{L})\}.$$

Suppose that $V \subset \mathcal{V}(f) \subset \mathbb{C}^n$ is an irreducible component. If $\dim V = N - r$, then $Y_V$ consists of finitely many points, say $Y_V = \{y_1, \ldots, y_{\deg V}\}$ where $V \cap \mathcal{V}(\mathcal{L}) = \{By_1 + b, \ldots, By_{\deg V} + b\}$ consists of smooth points of $V$. In particular, each $\{y_i\}$ is an irreducible component of $\mathcal{V}(h)$ so that $\{[1 : y_i]\}$ is an irreducible component of $\mathcal{V}(F)$. Hence, each $[1 : y_i] \in E_F$ so that $V \cap \mathcal{V}(\mathcal{L}) \subset E_f$, i.e., $E_f$ contains $\deg V$ smooth points on $V$.

If $\dim V > N - r$, then $Y_V$ is an irreducible component of $\mathcal{V}(h)$ so that $\overline{Y_V}$ is an irreducible component of $\mathcal{V}(F)$. Define $S_V = \left(\overline{Y_V} \cap \mathcal{V}(z_0)\right) \cup \overline{Y_{\mathrm{Sing}_f(V)}}$ which is a proper closed subset of $\overline{Y_V}$. Hence, $E_F$ contains at least one point in $\overline{Y_V} \setminus S_V$, say $z$. Since $z_0 \neq 0$, we know that

$$y = \left(\frac{z_1}{z_0}, \ldots, \frac{z_r}{z_0}\right) \subset E_h \cap Y_V$$

so that $By + b \in V \setminus \mathrm{Sing}_f(V) \subset \mathcal{V}(f)$. Hence, $By + b \in E_f$ showing that $E_f$ contains at least one smooth point on $V$. $\square$

**Example 2.2** Let $f$, $V_2$, and $V_1$ be as in Ex. 1.1. Since $f$ has rank 2, for ease of presentation, we simply take $A = I_2$, the $2 \times 2$ identity matrix,

$$B = \begin{bmatrix} I_2 \\ \begin{bmatrix} 2 & -3 \end{bmatrix} \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{yielding} \quad h(y) = \begin{bmatrix} y_1 y_2 \\ y_1(2y_1 - 3y_2 - 1) \end{bmatrix}.$$

With this setup, $E_h$ from Theorem 2.1 consists of 4 points: $(1/2, 0)$ and three distinct points of the form $(0, \star)$. Hence, $E_f$ consists of 4 points: $(1/2, 0, 0)$, which is a smooth point on $V_1$, and three distinct points of the form $(0, \star, \star)$, each of which is a smooth point on $V_2$.

Theorem 2.1 provides a theoretical method for generating a finite set of solutions containing at least one smooth point on each irreducible component. When performing this computation in practice, one uses a random number generator to select complex numbers that satisfy the genericity conditions with probability one.

## 3 Forming a numerical irreducible decomposition

Given a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$, Theorem 2.1 provides a method to compute a finite set $E_f \subset \mathcal{V}(f)$ containing at least one smooth point on each irreducible component. This section describes using isosingular sets to form a numerical irreducible decomposition of $\mathcal{V}(f)$ from $E_f$.

### 3.1 Isosingular sets

Deflation sequences introduced in Section 1.4 gives rise to isosingular sets [10] of $f$, which provide a stratification of the singularity structure of $\mathcal{V}(f)$, as follows. For an irreducible algebraic set $V \subset \mathcal{V}(f)$ with deflation sequence $\{d_k(f,V)\}_{k=0}^{\infty}$, one has $\dim V \leq \lim_{k\to\infty} d_k(f,V)$. One aims to identify irreducible algebraic subsets of $\mathcal{V}(f)$ which are as large as possible so that this inequality becomes an equality, which are called *isosingular sets* with respect to $f$ [10, Cor. 5.5]. In fact, every irreducible component of $\mathcal{V}(f)$ is an isosingular set with respect to $f$ [10, Thm. 5.2].

**Example 3.1** [10, Ex. 5.3] With $f$ in Ex. 1.5, the isosingular sets are the umbrella $\mathcal{V}(f)$, the handle $\{(0,0,z) \mid z \in \mathbb{C}\}$, and the pinch point $\{(0,0,0)\}$.

For each $y \in \mathcal{V}(f)$, one can associate to $y$ a unique isosingular set, denoted $\mathrm{Iso}_f(y)$, such that $y$ is a smooth point of $\mathrm{Iso}_f(y)$ [10, Lemma 5.14]. That is, $y \in \mathrm{Iso}_f(y)$ such that $y$ and $\mathrm{Iso}_f(y)$ have the same deflation sequence. In particular, we immediately have $\mathrm{isodim}_f(y) = \dim \mathrm{Iso}_f(y)$. An algorithm for computing a witness set for $\mathrm{Iso}_g(y)$ given $f$ and $y$ is presented in [10] and described in Section 3.2. This algorithm only needs a numerical approximation of $y$ to be accurate enough to compute finitely many terms in the deflation sequence of $y$ with respect to $f$, each of which is a nonnegative integer.

The following lemma provides a means to recognize the irreducible components of $\mathcal{V}(f)$ given $E_f$, a set containing a smooth point on each irreducible component.

**Lemma 3.2** *Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $E_f \subset \mathcal{V}(f)$ be a finite set containing at least one smooth point on each irreducible component of $\mathcal{V}(f)$. Then, the set of irreducible components of $\mathcal{V}(f)$ is the set of inclusion maximal elements of* $\{\mathrm{Iso}_f(y) \mid y \in E_f\}$.

**Proof.** Let $V_1, \ldots, V_k$ be the irreducible components of $\mathcal{V}(f)$.

For each $i = 1, \ldots, k$, let $y_i \in E_f$ be a smooth point of $V_i$. That is, $y_i \in V_i$ such that $y_i$ and $V_i$ have the same deflation sequence. Since $V_i$ is an irreducible component, $V_i$ is an isosingular set with respect to $f$. By uniqueness, we must have $V_i = \mathrm{Iso}_f(y_i)$. Inclusion maximality follows since $V_i$ is an irreducible component.

Conversely, suppose that $y \in E_f$ such that $\mathrm{Iso}_f(y)$ is inclusion maximal. Since $\mathrm{Iso}_f(y)$ is an irreducible algebraic subset of $\mathcal{V}(f)$, there exists $j \in \{1, \ldots, k\}$ with $\mathrm{Iso}_f(y) \subset V_j = \mathrm{Iso}_f(y_j)$. By inclusion maximality, $\mathrm{Iso}_f(y) = V_j$, i.e., $\mathrm{Iso}_f(y) \subset \mathcal{V}(f)$ is an irreducible component. □

### 3.2 Computing isosingular sets

The effectiveness of Lemma 3.2 to compute a numerical irreducible decomposition of $\mathcal{V}(f)$ for a polynomial system $f$ depends upon the ability to compute a witness set for $\mathrm{Iso}_f(y)$ given $y \in \mathcal{V}(f)$. The approach presented in [10] uses two stages. The first stage determines when the deflation sequence $\{d_k(f,y)\}_{k=0}^{\infty}$ has terminated, that is, compute $\ell \geq 0$ such that

$$d_k(f,y) = d_\ell(f,y) \text{ for all } k \geq \ell \quad \text{yielding} \quad \mathrm{isodim}_f(y) = \lim_{k\to\infty} d_k(f,y) = d_\ell(f,y).$$

The second stage uses monodromy loops to compute additional points in the witness point set and the trace test to validate that a witness point set for $\mathrm{Iso}_f(y)$ has been computed.

The deflation operator $\mathcal{D}$ defined in (3) uses determinants to construct the polynomial system $\mathcal{F}$. We note that an equivalent deflation operator, i.e., they yield the same deflation sequence, based on [1] was presented in [10], which is often more amenable to numerical computations. The following can be easily adapted regardless of the deflation operator used.

A deflation sequence has terminates at $\ell \geq 0$ if and only if for $(\mathcal{F}_\ell, y) = \mathcal{D}^\ell(f, y)$, we have $d_\ell(f, y) = \dim \mathrm{null}\, J\mathcal{F}_\ell(y) = \dim_{\mathcal{F}_\ell}(y)$. Suppose that $\mathcal{F}_\ell : \mathbb{C}^M \to \mathbb{C}^m$ and $d = d_\ell(f, y)$. For a general $R \in \mathbb{C}^{(M-d)\times m}$, consider $\mathcal{G}_\ell : \mathbb{C}^M \to \mathbb{C}^{M-d}$ defined by $\mathcal{G}_\ell = R \cdot \mathcal{F}_\ell$. Hence, $y \in \mathcal{V}(\mathcal{G}_\ell)$ is a smooth point on a generically reduced irreducible component $V \subset \mathcal{V}(\mathcal{G}_\ell)$. Therefore, termination at $\ell$ is equivalent to $V \subset \mathcal{V}(\mathcal{F}_\ell)$. This can be checked by sampling a general point $z \in V$ and determining if $\mathcal{F}_\ell(z) = 0$. One approach to sample a general point is to select systems $\mathcal{L}, \mathcal{L}_y : \mathbb{C}^M \to \mathbb{C}^d$ of general linear polynomials with $\mathcal{L}_y(y) = 0$ and consider the homotopy $\mathcal{H} : \mathbb{C}^M \times \mathbb{C} \to \mathbb{C}^M$ defined by

$$\mathcal{H}(x, t) = \begin{bmatrix} \mathcal{G}_\ell(x) \\ (1-t)\mathcal{L}(x) + t\mathcal{L}_y(x) \end{bmatrix}.$$

Starting with $y$ at $t = 1$, the endpoint $z \in V \cap \mathcal{V}(\mathcal{L})$ is a general point on $V$. If $\mathcal{F}_\ell(z) = 0$, then $V \subset \mathcal{V}(\mathcal{F}_\ell)$ with $\mathrm{Iso}_{\mathcal{F}_\ell}(y) = V$. Since this test is often computationally inexpensive, we can reliably perform this test by using various $\mathcal{L}$.

After determining that a deflation sequence has terminated at index $\ell$, one has computed a polynomial system $\mathcal{F}_\ell$ and a smooth point $y \in \mathrm{Iso}_{\mathcal{F}_\ell}(y)$. Hence, one can use random monodromy loops until the trace test determines that a complete witness point set for $\mathrm{Iso}_{\mathcal{F}_\ell}(y)$ has been computed. For efficiency improvements to using monodromy loops in this context, see [11].

### 3.3 Completing the decomposition

By combining Theorem 2.1 and Lemma 3.2, we obtain Algorithm 1 for computing a numerical irreducible decomposition and the following statement of correctness.

**Theorem 3.3** *With probability one, Algorithm 1 computes a numerical irreducible decomposition for $\mathcal{V}(f)$.*

This statement is merely a recognition of the value in pairing together of isosingular theory (Lemma 3.2) and first order general homotopies (Theorem 2.1), along with several standard tools of numerical algebraic geometry.

**Example 3.4** From $f$ as in Ex. 1.1, it was shown in Ex. 2.2 that $E_f$ consists of 4 points. The three points of the form $(0, \star, \star)$ have deflation sequence $\{2, 2, \dots\}$, isosingular local dimension 2, and correspond to the same isosingular set, namely $V_2$ in Ex. 1.1. Thus, using Algorithm 1, one of these points will be used to compute a witness set for $V_2$ and the other 2 points will be discarded using via the membership test. Since the fourth point in $E_f$, namely $(1/2, 0, 0)$ has deflation sequence $\{1, 1, \dots\}$, isosingular local dimension 1, and does not lie on $V_2$, Algorithm 1 identifies this as forming the witness point set for $V_1$ and returns.

## 4 Examples

We describe using Algorithm 1 via `Bertini` [3] on several other examples. See `https://dx.doi.org/10.7274/r0-4rm8-x958` for additional data regarding these examples.

**Algorithm 1** `Numerical irreducible decomposition`

**Input:** Polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$.
**Output:** Numerical irreducible decomposition $\mathcal{W}$ of $\mathcal{V}(f) \subset \mathbb{C}^n$.

Select random numbers to construct a general first order homotopy $H(y, t)$ as in (4) and track all solution paths to compute corresponding finite set $E_f \subset \mathcal{V}(f)$ as in (5).
**for** each point $x \in E_f$ **do**
  Compute the deflation sequence for $x$ yielding $\operatorname{isodim}_f(x)$, the isosingular local dimension of $f$ at $x$.
**end for**
Compute $m = \min\{\operatorname{isodim}_f(x) \mid x \in E_f\}$ and $M = \max\{\operatorname{isodim}_f(x) \mid x \in E_f\}$.
Compute $E_j = \{x \in E_f \mid \operatorname{isodim}_f(x) = j\}$ for $j = m, \dots, M$.
Initialize $\mathcal{W} = \emptyset$.
**for** $j = M, M-1, \dots, m$ **do**
  **for** each $W \in \mathcal{W}$ **do**
    Use component membership test with witness set $W$ to remove from $E_j$ any points which lie on the irreducible component corresponding to $W$.
  **end for**
  **if** $j >$ rank $f$ **then**
    **for** each $x \in E_j$ **do**
      Use monodromy and trace tests to produce a witness set $W$ for $\operatorname{Iso}_f(x)$.
      Append $W$ to $\mathcal{W}$.
      Use component membership test with witness set $W$ to remove from $E_j$ any points which lie on $\operatorname{Iso}_f(x)$.
    **end for**
  **else**
    Use monodromy and trace tests to partition $E_j$ into witness point sets. Append all corresponding witness sets to $\mathcal{W}$.
  **end if**
**end for**

## 4.1 Line with an embedded point

The polynomial system $f(x, y) = \begin{bmatrix} x^2 \\ xy \end{bmatrix}$ algebraically describes the line $\mathcal{V}(x)$ and an embedded point at $(0, 0)$ This system is considered in Ex. A.19 as a demonstration of what can happen when the homotopy is not general enough. When applying Algorithm 1, one tracks 4 paths which end at 2 distinct points of the form $(0, \star)$, which depend on the random choices, and the other 2 end at $(0, 0)$. Thus, $E_f$ consists of 3 distinct points

Since the Jacobian of $f$ is

$$Jf(x, y) = \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix},$$

it is easy to verify that the deflation sequence for the points of the form $(0, \star)$ is $\{1, 1, \dots\}$ while the deflation sequence for $(0, 0)$ is $\{2, 0, 0, \dots\}$.

Starting at dimension 1, Algorithm 1 computes a witness set for the isosingular set associated to one of the points of the form $(0, \star)$, namely $\mathcal{V}(x)$, and uses this witness set to verify that all other points in $E_f$ lie on $\mathcal{V}(x)$. Thus, Algorithm 1 simply returns a witness set for $\mathcal{V}(x) = \mathcal{V}(f)$. However, we note that Algorithm 1 could be easily modified to compute some finer information such as the equivalence of the irreducible components (namely, 2 for this system) and the existence of distinguished components which are not irreducible components (namely, $\{(0,0)\}$).

See Ex. A.9-A.11 for additional examples of lines with a point of interest.

## 4.2 Nested distinguished components

Consider the polynomial system

$$f(x,y,z) = \begin{bmatrix} (xy-z)(x-y)(x+y-z) \\ (xy-z)(xy-z+(x-y)(x+2y-3z)) \\ (xy-z)(xy-z+(x-y)(2x-3y+z)) \end{bmatrix}$$

which is a particular instance of those considered in Example A.20. The irreducible components of $\mathcal{V}(f)$ are the quadric surface $Q = \mathcal{V}(xy - z)$ and the point $P = \{(2/11, 10/11, 12/11)\}$. There are nested distinguished components inside of the quadric surface, namely the conic curve $C = \mathcal{V}(x - y, xy - z) \subset Q$ and the point $O = \{(0,0,0)\} \subset C \subset Q$.

Since each of the three polynomials in $f$ has degree 4, the first order general homotopy tracks $4^3 = 64$ paths. The endpoints of those 64 paths decompose as:

- 1 point at $P$ with deflation sequence $\{0, 0, \ldots\}$,
- 41 points on $Q$, but not on $C$, with deflation sequence $\{2, 2, \ldots\}$,
- 9 points (each reached twice) on $C$, but not on $O$, with deflation sequence $\{3, 1, 1, \ldots\}$, and
- 1 point (reach four times) at $O$ with deflation sequence $\{3, 2, 0, 0, \ldots\}$.

A full accounting of the paths is $64 = 1 \cdot 1 + 41 \cdot 1 + 9 \cdot 2 + 1 \cdot 4$.

## 4.3 "Illustrative example"

The polynomial system

$$f(x,y,z) = \begin{bmatrix} (y-x^2)(x^2+y^2+z^2-1)(x-1/2) \\ (z-x^3)(x^2+y^2+z^2-1)(y-1/2) \\ (y-x^2)(z-x^3)(x^2+y^2+z^2-1)(z-1/2) \end{bmatrix}$$

is called an *illustrative example* in [17, §3]. The irreducible components of $\mathcal{V}(f)$ are a quadric surface $Q = \mathcal{V}(x^2+y^2+^2-1)$, a cubic curve $C = \mathcal{V}(y-x^2, z-x^3)$, three lines $L_1 = \mathcal{V}(2x-1, 8z-1)$, $L_2 = \mathcal{V}(\sqrt{2}x - 1, 2y - 1)$, and $L_3 = \mathcal{V}(\sqrt{2}x + 1, 2y - 1)$, and the point $P = \{(1/2, 1/2, 1/2)\}$.

Algorithm 1 tracks $5 \cdot 6 \cdot 8 = 240$ paths yielding 196 finite endpoints that decompose as:

- 146 points on $Q$ with deflation sequence $\{2, 2 \ldots\}$,
- 24 points on $C$ with deflation sequence $\{1, 1, \ldots\}$,
- 9 points on $L_1$ with deflation sequence $\{1, 1, \ldots\}$, and
- 8 points on each $L_2$ and $L_3$ with deflation sequence $\{1, 1, \ldots\}$, and
- 1 point at $P$ with deflation sequence $\{0, 0, \ldots\}$.

### 4.4 Cyclic-4 system

The solution set of the polynomial system

$$f(x) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 \\ x_1x_2x_3x_4 - 1 \end{bmatrix}$$

consists of two irreducible curves:

$$C_1 = \mathcal{V}(x_1 + x_3, x_2 + x_4, x_3x_4 - 1) \quad \text{and} \quad C_2 = \mathcal{V}(x_1 + x_3, x_2 + x_4, x_3x_4 + 1)$$

together with 8 distinguished points that are also embedded:

$$\begin{array}{llll} P_1 = (1, 1, -1, -1), & P_2 = (1, -1, -1, 1), & P_3 = (-1, 1, 1, -1), & P_4 = (-1, -1, 1, 1), \\ P_5 = (i, i, -i, -i), & P_6 = (i, -i, -i, i), & P_7 = (-i, i, i, -i), & P_8 = (-i, -i, i, i) \end{array}$$

where $i = \sqrt{-1}$. Tracking $4! = 24$ paths yields 24 finite endpoints that decompose as:

- 4 points each on $C_1$ and $C_2$ with deflation sequence $\{1, 1, \ldots\}$ and
- 1 point (each reached twice) to $P_1, \ldots, P_8$ with deflation sequence $\{2, 0, 0, \ldots\}$.

with $24 = 2 \cdot 4 + 8 \cdot 1 \cdot 2$.

### 4.5 Adjacent minors

One place where using the methods behind Algorithm 1 will have an advantage over other numerical irreducible decomposition methods, e.g., [9, 16, 21], is when $f$ consists of $r$ polynomials of degree $d_1, \ldots, d_r$ such that $r = \operatorname{rank} f = \operatorname{codim} \mathcal{V}(f)$ and $\deg \mathcal{V}(f)$ is approximately $d_1 \cdots d_r$. This is due to the fact that one is targeting the bottom dimension components directly with little wasted effort. Conversely, the approaches of [9, 16, 21] check in all possible dimensions with significant wasted effort. In fact, when $\operatorname{rank} f = \operatorname{codim} \mathcal{V}(f)$, the set $E_f$ obtained in the first line of Algorithm 1 consists of the union of witness points sets for the irreducible components of $\mathcal{V}(f)$. Thus, we can have a direct comparison simply on the number of paths to track and time to compute a witness point set for $\mathcal{V}(f)$ using a first order general homotopy in the first line of Algorithm 1 and the three methods in [9, 16, 21].

To demonstrate, for $n \geq 3$, we consider the system $f_n$ consisting of the $3 \times 3$ adjacent minors of a $3 \times n$ matrix $A^{(n)}$ of indeterminants, e.g., see [12]. Thus, $f_n$ consists of $n - 2$ cubic polynomials in $3n$ variables. For example, when $n = 5$,

$$A^{(5)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} \quad \text{and} \quad f_5 = \begin{bmatrix} \det A^{(5)}_{1:3} \\ \det A^{(5)}_{2:4} \\ \det A^{(5)}_{3:5} \end{bmatrix}$$

where $A^{(n)}_{j:j+2}$ is the $3 \times 3$ submatrix of $A^{(n)}$ consisting of columns $j$, $j + 1$, and $j + 2$. As mentioned above, we have a direct comparison using Algorithm 1 and all of [9, 16, 21] for simply computing a witness point set for $\mathcal{V}(f_n)$ in this case. In fact, since we are solving intrinsically on a nontrivial general linear space $\{By + b \mid y \in \mathbb{C}^r\}$ as in Section 2, we can compare taking $g$ to consist of general cubics and $g = 0$ in (4) following the observation in Ex. A.19. The results

| $n$ | Algorithm 1 | | | Regen. cascade [9] | | Cascade [16] | | Dim-by-dim [21] | |
|---|---|---|---|---|---|---|---|---|---|
| | # paths | time (0) | time (general) | # paths | time | # paths | time | # paths | time |
| 5 | 27 | 0.44 | 0.50 | 63 | 1.28 | 81 | 1.09 | 39 | 0.82 |
| 6 | 81 | 0.93 | 1.73 | 198 | 3.29 | 324 | 4.46 | 120 | 2.88 |
| 7 | 243 | 3.65 | 14.20 | 603 | 16.57 | 1,215 | 22.38 | 363 | 13.81 |
| 8 | 729 | 18.90 | 45.11 | 1,818 | 57.09 | 4,374 | 105.80 | 1,092 | 60.33 |
| 9 | 2,187 | 71.99 | 249.96 | 5,463 | 234.25 | 15,309 | 486.11 | 3,279 | 247.41 |
| 10 | 6,561 | 303.34 | 1210.26 | 16,398 | 999.94 | 52,488 | 2093.95 | 9,840 | 1124.28 |

Table 1: Summary for computing a witness point set for the adjacent minor system $f_n$ using various methods with serial computation and time measured in seconds. Algorithm 1 compares timings when $g = 0$ and when $g$ is general.

| $n$ | Algorithm 1 | | | Regen. cascade [9] | | Cascade [16] | | Dim-by-dim [21] | |
|---|---|---|---|---|---|---|---|---|---|
| | # paths | time (0) | time (general) | # paths | time | # paths | time | # paths | time |
| 10 | 6,561 | 11.02 | 24.68 | 16,398 | 28.40 | 52,488 | 43.50 | 9,840 | 31.04 |
| 11 | 19,683 | 24.15 | 74.11 | 49,203 | 63.94 | 177,147 | 202.78 | 29,523 | 140.34 |
| 12 | 59,049 | 63.94 | 299.40 | 147,618 | 202.66 | 590,490 | 1363.36 | 88,572 | 1055.02 |

Table 2: Summary for computing a witness point set for the adjacent minor system $f_n$ using various methods with parallel computation and time measured in seconds. Algorithm 1 compares timings when $g = 0$ and when $g$ is general.

are summarized in Table 1 and Table 2 where the timings reported utilized `Bertini` running in serial and parallel, respectively, on four 2.4 GHz Opteron 6378 processors, for a total of 64 cores, with 64-bit Linux and 128 GB of memory. These show a comparison between the additional evaluation cost of using a first order general approach with general $g$ in Algorithm 1 as well as the cost of checking all possible dimensions independently [21] or via a cascade [9, 16] when codim $\mathcal{V}(f_n) = \text{rank } f_n = n - 2$ and $\deg \mathcal{V}(f_n) = 3^{n-2}$.

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# A Dynamic intersections and limits of homotopy paths

The following describes some concepts from intersection theory and present a few results that relate to numerical homotopy methods with [6] providing for more details. The software `Macaulay2` [8] and the `ReesAlgebra` package can be used to compute distinguished components and other objects related to this topic, which are utilized in the examples appearing below.

## A.1 Notation

All schemes appearing in this appendix are, by convention, algebraic schemes over $\mathbb{C}$. For an $n$-dimensional scheme $X$, we use $A_k(X)$ to denote the Chow group of $k$-cycles on $X$ modulo rational equivalence. We define $A_*(X) = \bigoplus_{k=0}^n A_k(X)$ to be the Chow group. A cycle class $\alpha \in A_k(X)$ is said to be a well-defined $k$-cycle on $X$ in the case where there is only one $k$-cycle on $X$ that represents $\alpha$. For a subscheme $Y \subseteq X$, there is an associated cycle $[Y]$ on $X$ which takes the geometric multiplicities of the irreducible components of $Y$ into account. The corresponding cycle class in $A_*(X)$ will also be denoted by $[Y]$. In the situation where $Y$ is a subscheme of $X$ and $\alpha \in A_*(Y)$, we will sometimes consider $\alpha$ an element of $A_*(X)$ without further comment. Let $R \subseteq X$ be a closed subset and let $\alpha = \sum_{i=1}^q a_i V_i$ be a cycle on $X$ where $a_1, \ldots, a_q \in \mathbb{Z}$ and $V_1, \ldots, V_q \subseteq X$ are irreducible subvarieties. The part of $\alpha$ supported on $R$ is the cycle on $R$ defined by

$$\alpha^R = \sum_{V_i \subseteq R} a_i V_i,$$

where the sum is over all $i$ such that $V_i$ is contained in $R$.

Given a scheme $X$, a vector bundle $\pi : E \to X$ of rank $d$, and an integer $k \geq d$, there is an induced homomorphism $\pi^* : A_{k-d}(X) \to A_k(E)$ known as the *flat pullback* (see [6, § 1.7]). The map $\pi^*$ is given by $\pi^*([Y]) = [\pi^{-1}(Y)]$ for any pure $(k-d)$-dimensional subscheme $Y$ of $X$. It is shown in [6, Thm 3.3 (a)] that $\pi^*$ is an isomorphism. Let $s : X \to E$ be the zero-section. The *Gysin homomorphism* is $s^* : A_k(E) \to A_{k-d}(X)$ and, by definition, is the inverse of $\pi^*$.

A proper morphism of schemes $f : X \to Y$ induces a homomorphism $f_* : A_k(X) \to A_k(Y)$ (see [6, § 1.4]). Let $X$ be a scheme which is proper over $\mathrm{Spec}(\mathbb{C})$ with structure morphism $\eta : X \to \mathrm{Spec}(\mathbb{C})$. Then, the degree of a 0-cycle class $\alpha \in A_0(X)$ is defined by $\deg \alpha = \eta_*(\alpha)$, where $A_0(\mathrm{Spec}(\mathbb{C}))$ is identified with $\mathbb{Z}$.

Let $E$ be a vector bundle on a scheme $X$ and let $\Gamma$ be a finite-dimensional space of global sections of $E$. Then, $\Gamma$ is said to generate $E$ if the induced map $X \times \Gamma \to E$ is surjective.

In this appendix, all varieties are, by definition, reduced and irreducible. An irreducible component of a scheme will be considered as a variety with the induced reduced structure.

Let $d \geq 0$ be an integer. A closed embedding $X \to Y$ of schemes is called a regular embedding of codimension $d$, if for every $x \in X$, there is an affine neighborhood $U$ of $x$ in $Y$ such that the ideal defining $U \cap X$ in the coordinate ring of $U$ is generated by a regular sequence of length $d$.

## A.2 Intersection products and distinguished varieties

The following defines intersection products in a general setting. In a manner similar to Bézout's theorem, intersection products decompose as a sum of contributions from distinguished varieties.

Let $i : X \to Y$ be a codimension $d$ regular embedding of schemes and let $N_X Y$ denote the normal bundle of $X$ in $Y$. Further, for $k \geq d$, let $V$ denote a pure $k$-dimensional scheme and suppose that we are given a morphism $f : V \to Y$. Then, the inverse image scheme $W = f^{-1}(X)$ is the fiber product of $X$ and $V$ over $Y$, yielding the following:

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y \end{array}$$

where $j$ is the embedding of $W$ in $V$ and $g$ is the induced map to $X$. Let $N$ denote the pullback of $N_X Y$ via $g$, that is $N = g^* N_X Y$, and let $\pi : N \to W$ be the projection. There is a substitute for a normal bundle of $W$ in $V$, denoted $C_W V$ and called the *normal cone* of $W$ in $V$ (see [6, App. B.6]). Since $V$ has pure dimension $k$, $C_W V$ is either empty or has pure dimension $k$ by [6, App. B.6.6]. For $C = C_W V$, as is explained in [6, § 6.1], $C$ embeds in $N$ as a subcone which yields a class $[C] \in A_k(N)$. Let $s : W \to N$ denote the zero-section. The intersection product $X \cdot V$ of $V$ by $X$ on $Y$ is an element of $A_{k-d}(W)$ defined as

$$X \cdot V = s^*([C]).$$

The class $[C] \in A_*(N)$ may be represented by a sum $\sum_{i=1}^{r} m_i C_i$ where $C_1, \ldots, C_r$ are the irreducible components of $C$ and $m_1, \ldots, m_r \in \mathbb{Z}$ are the geometric multiplicities of the components of $C$. Put $Z_i = \pi(C_i)$, a closed subvariety of $W$ called the support of $C_i$ (see [6, App. B.5.3]). The varieties $Z_1, \ldots, Z_r$ are called the *distinguished varieties* of the intersection. Thus, a distinguished variety is by definition reduced and irreducible. Note that we might very well have that $Z_i = Z_j$ even though $i \neq j$. Let $N_i$ be the restriction of $N$ to $Z_i$ and let $s_i : Z_i \to N_i$ be the zero-section. Importantly, we have the so-called canonical decomposition of $X \cdot V$,

$$X \cdot V = \sum_{i=1}^{r} m_i \alpha_i,$$

where $\alpha_i = s_i^*([C_i])$. The *equivalence* of a distinguished variety $Z$ is a class in $A_{k-d}(Z)$ denoted by $i(Z)$ and given by the sum of the terms $m_j \alpha_j$ such that $Z_j = Z$, that is,

$$i(Z) = \sum_{Z_j = Z} m_j \alpha_j.$$

Due to the canonical decomposition, the equivalence of $Z$ is also called the contribution of $Z$ to $X \cdot V$. For any closed subset $R \subseteq V$, the part of $X \cdot V$ supported on $R$ is a class in $A_{k-d}(R)$, denoted $(X \cdot V)^R$, and defined by

$$(X \cdot V)^R = \sum_{Z_j \subseteq R} m_j \alpha_j,$$

the sum being over all $j$ such that $Z_j \subseteq R$. Note that

$$(X \cdot V)^R = \sum_{Z \subseteq R} i(Z),$$

where the sum is over all distinguished varieties $Z$ contained in $R$, but without repetition.

The following is Lemma 7.1 (a) in [6].

**Lemma A.1** *Every irreducible component of $W$ is a distinguished variety.*

**Proof.** The projection $\pi : C_W V \to W$ is onto with $\pi(C_i) \subseteq W$ being closed and irreducible for any irreducible component $C_i \subseteq C_W V$. Hence, for every irreducible component $Z$ of $W$, there is an irreducible component $C_i$ of $C_W V$ such that $\pi(C_i) = Z$. □

## A.3 Dynamic intersections

The following is a dynamic interpretation of the construction above developed in [6, 13, 15]. Lazarsfeld [13] extends and corrects Severi's dynamical approach to intersections from [15]. For simplicity, we will follow the treatment given in [6] with special attention to Prop. 11.3, Remark 11.3, and Ex. 11.3.1 of [6].

Let $T$ be a smooth irreducible curve. Fix a point $t_0 \in T$ and consider $T^* = T \setminus \{t_0\}$. For $t \in T$ and a scheme $\mathcal{S}$ over $T$ with $\pi : \mathcal{S} \to T$, the fiber $\pi^{-1}(t)$ is denoted by $\mathcal{S}_t$. We will use the notation $\mathcal{S}^* = \mathcal{S} \setminus \mathcal{S}_{t_0}$.

Let $Y$ be a scheme and let $\mathcal{X} \to Y \times T$ be a regular embedding of codimension $d$, such that the induced embeddings $\mathcal{X}_t \to Y \times \{t\}$ are also regular of codimension $d$, for all $t \in T$. We call $\mathcal{X} \to Y \times T$ a family of regular embeddings of codimension $d$. Let $V \to Y$ be a closed subscheme of pure dimension $k$ where $k \geq d$. In this situation, we will use $\mathcal{V}$ to denote the trivial family over $T$ with fiber $V$, that is $\mathcal{V} = V \times T$. Identify $V$ with $V \times \{t_0\}$, set $X = \mathcal{X}_{t_0}$, and identify $Y$ with $Y \times \{t_0\}$. Then, $\mathcal{X}$ is called a *deformation* of the embedding of $X$ in $Y$.

Let $\mathcal{W} = \mathcal{X} \cap \mathcal{V}$ and set $W = \mathcal{W}_{t_0} = X \cap V$. Remove $W$ from $\mathcal{W}$ by letting $\mathcal{W}^* = \mathcal{W} \setminus W$ and then take the closure, say $\mathcal{W}'$, of $\mathcal{W}^*$ in $\mathcal{W}$.

**Definition A.2** The *limit set* of the deformation $\mathcal{X}$ is $\mathcal{W}'_{t_0}$.

We will consider the limit set as a variety (the induced scheme structure on the limit set will not play any direct role here). Consider the regular embedding $\mathcal{X}^* \to Y \times T^*$ and the embedding $\mathcal{V}^* \subseteq Y \times T^*$. The intersection product $\mathcal{X}^* \cdot \mathcal{V}^*$ is an element of $A_{k+1-d}(\mathcal{W}^*)$. Let $\sum_i a_i \mathcal{D}_i$ be a representative of $\mathcal{X}^* \cdot \mathcal{V}^*$, where $a_i \in \mathbb{Z}$ and $\mathcal{D}_i \subseteq \mathcal{W}^*$ are irreducible subvarieties of dimension $k + 1 - d$, and let $\mathcal{D}'_i$ be the closure of $\mathcal{D}_i$ in $\mathcal{W}'$.

**Definition A.3** The *limit intersection class* is an element of $A_{k-d}(\mathcal{W}'_{t_0})$, defined by

$$\lim_{t \to t_0} (\mathcal{X}_t \cdot \mathcal{V}_t) = \sum_i a_i [(\mathcal{D}'_i)_{t_0}].$$

**Lemma A.4** *The push forward of the limit intersection class from the limit set to $W$ is equal to the intersection product $X \cdot V$.*

For a proof, see [6, Cor. 11.1].

**Remark A.5** Suppose that $Y$ is a smooth variety and that $\mathcal{W}_t = \mathcal{X}_t \cap \mathcal{V}_t$ has pure dimension $k-d$ for all $t \in T^*$, that is the intersection is proper for all $t \neq t_0$. In this case, the limit intersection class is either 0 or a well defined positive $(k-d)$-cycle on the limit set. To see this, note first that since $Y \times T^*$ is smooth, every component of $\mathcal{X}^* \cap \mathcal{V}^*$ has dimension at least $\dim \mathcal{V}^* + \dim \mathcal{X}^* - \dim(Y \times T^*) = k+1-d$. Since $\mathcal{X}_t \cap \mathcal{V}_t$ has pure dimension $k-d$ for all $t \in T^*$, it follows that every component of $\mathcal{X}^* \cap \mathcal{V}^*$ dominates $T$ and has dimension $k+1-d$. Hence, the intersection $\mathcal{X}^* \cap \mathcal{V}^*$ is proper, and consequently $\mathcal{X}^* \cdot \mathcal{V}^*$ is represented by the positive cycle $\sum_{i=1}^q a_i \mathcal{D}_i$, where $\mathcal{D}_1, \ldots, \mathcal{D}_q \subseteq \mathcal{W}^*$ are the irreducible components of $\mathcal{W}^* = \mathcal{X}^* \cap \mathcal{V}^*$ and $a_i > 0$ is the intersection multiplicity of $\mathcal{D}_i$ in $\mathcal{X}^* \cdot \mathcal{V}^*$. The limit intersection class is represented by the $(k-d)$-cycle $\sum_{i=1}^q a_i [(\mathcal{D}'_i)_{t_0}]$ where $\mathcal{D}'_i$ is the closure of $\mathcal{D}_i$ in $\mathcal{W}'$. Note that $(\mathcal{D}'_i)_{t_0}$ is either empty or of pure dimension $k-d$ for all $i$. Thus, the limit set is either empty or of pure dimension $k-d$ since every irreducible component of $\mathcal{W}'$ is the closure of an irreducible component of $\mathcal{W}^*$ where all components of $\mathcal{W}^*$ are of dimension $k+1-d$ and dominate $T$. It follows that the limit intersection class is either 0 or a well defined $(k-d)$-cycle on the limit set.

**Definition A.6** If the limit intersection class is a well defined $(k-d)$-cycle on the limit set, then we call it the *limit cycle*.

As is explained in [6, § 11.2], the normal bundle $N_X(Y \times T)$ is isomorphic to $N_X Y \oplus N_X \mathcal{X}$. The inclusions $X \subseteq \mathcal{X} \subseteq Y \times T$ induce an inclusion $N_X \mathcal{X} \to N_X(Y \times T)$, which in turn induces a map of vector bundles $\rho : N_X \mathcal{X} \to N_X Y$. Now, $N_X \mathcal{X}$ is the trivial bundle and can be identified with $X \times N_{t_0} T$ in a natural way (see [6, App. B.6.1]). Fix a basis of the 1-dimensional vector space $N_{t_0} T$ and let $e : X \to N_X \mathcal{X}$ be the corresponding constant section.

**Definition A.7** The section $\rho \circ e$ of $N_X Y$ is the *characteristic section* of the deformation $\mathcal{X}$.

The characteristic section depends on the choice of basis of $N_{t_0} T$ and is well defined only up to rescaling by $\lambda \in \mathbb{C} \setminus \{0\}$. The following is [6, Prop. 11.3 (ii)].

**Proposition A.8** *Let $X \to Y$ be a regular codimension $d$ embedding of schemes and let $V \subseteq Y$ be a closed subscheme of pure dimension $k$ where $k \geq d$. Assume that $N_X Y$ is generated by a finite-dimensional space of global sections $\Gamma$. Let $R \subseteq X \cap V$ be a closed subset. Then, there is a nonempty open subset $\Gamma(R) \subseteq \Gamma$ which is invariant under multiplication by $\lambda \in \mathbb{C} \setminus \{0\}$ and is such that the following holds. For any deformation $\mathcal{X}$ of $X \to Y$ with characteristic section in $\Gamma(R)$, the limit intersection class is a well defined $(k-d)$-cycle, and*

$$\left( \lim_{t \to t_0} (\mathcal{X}_t \cdot \mathcal{V}_t) \right)^R = (X \cdot V)^R.$$

*That is, the part of the limit cycle supported on $R$ represents $(X \cdot V)^R$ in $A_{k-d}(R)$.*

## A.4 Projective space, distinguished varieties and positivity

Consider hypersurfaces $X_1, \ldots, X_k \subset \mathbb{P}^k$. For the construction of Section A.2, we take

$$X = X_1 \times \cdots \times X_k \quad \text{and} \quad Y = \underbrace{\mathbb{P}^k \times \cdots \times \mathbb{P}^k}_{k \text{ factors}}$$

with $i : X \to Y$ being the natural inclusion and $V = \mathbb{P}^k$ with $f : V \to Y$ the diagonal morphism. Note that $d = k$ in this setting, where (as in Section A.2) $d$ is the codimension of $X$ in $Y$. Let

$W = f^{-1}(X)$. Then, $W \subseteq \mathbb{P}^k$ is the scheme theoretic intersection of $X_1, \ldots, X_k$, that is

$$W = \bigcap_{i=1}^{k} X_i.$$

The fiber product diagram in this instance is thus

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "g"'] & \mathbb{P}^k \arrow[d, "f"] \\ X_1 \times \cdots \times X_k \arrow[r, "i"'] & \mathbb{P}^k \times \cdots \times \mathbb{P}^k \end{tikzcd} \tag{6}$$

where $g$ is the diagonal map and $j$ is the inclusion.

Lemma A.1 provides that the irreducible components of $W$ are distinguished varieties. As we shall see in the examples below, embedded components of $W$ may or may not be distinguished varieties. In the examples, an embedded point or embedded component of a subscheme $W$ of $\mathbb{P}^k$ is considered as a variety with the induced reduced structure.

**Example A.9** Consider the ideal $I = (x^2y, xy^2) \subset \mathbb{C}[x, y, z]$. Let $X_1 = \{x^2y = 0\}$ and $X_2 = \{xy^2 = 0\}$. The curve $W$ in $\mathbb{P}^2$ defined by $I$ is supported on the two lines $L_1 = \{x = 0\}$ and $L_2 = \{y = 0\}$, but $W$ has an embedded point at their intersection, namely $p = [0 : 0 : 1]$. The normal cone $C_V\mathbb{P}^2$ has three components and their respective supports are $L_1$, $L_2$ and $p$. Thus, these three are the distinguished varieties of the intersection of $\mathbb{P}^2$ by $X_1 \times X_2$ on $\mathbb{P}^2 \times \mathbb{P}^2$. This is an example where every embedded component is a distinguished variety.

**Example A.10** Let $W$ be the subscheme of $\mathbb{P}^3$ defined by $I = (x^2, xyz, y^2) \subset \mathbb{C}[x, y, z, w]$. The only distinguished variety in this case is the line defined by $x = y = 0$. Since $W$ has an embedded point at $[0 : 0 : 0 : 1]$, embedded components need not be distinguished varieties.

**Example A.11** The ideal $I = (x^2z, xy^3, y^3z^2) \subset \mathbb{C}[x, y, z, w]$ defines a subscheme $W$ of $\mathbb{P}^3$ that has distinguished varieties that are not irreducible components nor embedded components. The irreducible components are the three lines $L_1 = \{x = y = 0\}$, $L_2 = \{x = z = 0\}$, and $L_3 = \{y = z = 0\}$ with no embedded components. In addition to the irreducible components, there is one additional distinguished variety corresponding to the point $[0 : 0 : 0 : 1]$.

Consider the canonical decomposition of the intersection product

$$X \cdot \mathbb{P}^k = \sum_{j=1}^{r} m_j \alpha_j.$$

For any distinguished variety $Z_j$, $\alpha_j$ is a 0-cycle class on $W$ of positive degree. By [6, Thm. 12.3], $\deg \alpha_j \geq \deg Z_j$. Since $m_j > 0$ for all $j$, we have the following.

**Lemma A.12** *For any distinguished variety $Z$,*

$$\deg i(Z) \geq \deg Z > 0.$$

**Remark A.13** The inequality $\deg \alpha_j \geq \deg Z_j$ can be tightened, e.g., $\deg \alpha_j \geq a^b \deg Z_j$ where $a = \min\{\deg X_1, \ldots, \deg X_k\}$ and $b = \dim Z_j$ as in [6, Ex 12.3.3].

## A.5 Homotopies

The following specializes the situation in Section A.3 to certain families $\mathcal{X}$ over $\mathbb{C}$ related to homotopy methods. Suppose that $X_1, \ldots, X_k \subset \mathbb{P}^k$ are hypersurfaces, let

$$X = X_1 \times \cdots \times X_k, \quad Y = \underbrace{\mathbb{P}^k \times \cdots \times \mathbb{P}^k}_{k \text{ factors}}, \quad \text{and} \quad W = \bigcap_{j=1}^{k} X_j.$$

From the fiber product diagram (6), we obtain the intersection product $X \cdot \mathbb{P}^k$ in $A_0(W)$. The curve $T$ from Section A.3 will be an open subset of $\mathbb{C}$ and $t_0 = 0$. For polynomials $P_1, \ldots, P_k$, define $P = (P_1, \ldots, P_k)$. Let $F = (F_1, \ldots, F_k) \subseteq \mathbb{C}[x_0, \ldots, x_k]$ be homogeneous polynomials such that $F_i$ defines the hypersurface $X_i \subseteq \mathbb{P}^k$ and let $n_i = \deg F_i$. Suppose that $G_1, \ldots, G_k \in \mathbb{C}[x_0, \ldots, x_k]$ are homogeneous polynomials with $\deg G_i = n_i$ and $H_1, \ldots, H_k \in \mathbb{C}[x_0, \ldots, x_k, t]$ such that, in $x_0, \ldots, x_k$, $H_i$ is homogeneous of degree $n_i$. Let

$$\mathcal{X} = \mathcal{X}(F, G, H) = \{(y_1, \ldots, y_k, t) \in Y \times \mathbb{C} : F_i(y_i) + tG_i(y_i) + t^2 H_i(y_i, t) = 0 \text{ for all } i\} \subseteq Y \times \mathbb{C}.$$

Consider the intersection of $\mathcal{X}(F, G, H)$ with $\mathcal{V} = \mathbb{P}^k \times \mathbb{C}$, where $\mathcal{V}$ is embedded in $Y \times \mathbb{C}$ via the diagonal embedding $\mathbb{P}^k \to Y$. This is a family $\mathcal{W} = \mathcal{X} \cap \mathcal{V} \subseteq \mathbb{P}^k \times \mathbb{C}$ over $\mathbb{C}$ of subschemes of $\mathbb{P}^k$ such that the fiber over 0 is $W = \bigcap_i X_i$. Let $n = \Pi_{i=1}^k n_i$ be the Bézout number. If $\mathcal{W}_t$ is finite with cardinality $|\mathcal{W}_t| = n$ for general $t \in \mathbb{C}$, then we call $\mathcal{X}$ a *total degree homotopy*, or simply a *homotopy*.

**Definition A.14** Given $F$, we say that a property holds for a *first order general homotopy* if, for a general $G$, the property holds for any homotopy $\mathcal{X}(F, G, H)$.

Fix $G$ and $H$ such that $\mathcal{W}_t$ is finite and $|\mathcal{W}_t| = n$ for general $t \in \mathbb{C}$. Let $\Theta \subseteq \mathbb{C}$ be the set of $t \in \mathbb{C}$ such that $\mathcal{W}_t$ is finite and $|\mathcal{W}_t| = n$ and consider the restriction of $\mathcal{X}(F, G, H)$ to $T = \Theta \cup \{0\}$. We use the same notation $\mathcal{X}$, $\mathcal{W}$, and $\mathcal{V}$ to denote the restrictions $(Y \times T) \cap \mathcal{X}$, $(Y \times T) \cap \mathcal{W}$, and $(Y \times T) \cap \mathcal{V}$. Note that $\mathcal{X} \to Y \times T$ is a family of regular embeddings of codimension $k$.

Let $\epsilon > 0$ be such that the closed disc in $\mathbb{C}$ of radius $\epsilon$ centered at 0 is contained in $T$. In other words, $\mathcal{W}_t$ is finite and consists of exactly $n$ points if $0 < |t| \leq \epsilon$. Thus, there are $n$ continuous paths

$$\psi_i : [0, \epsilon] \to \mathbb{P}^k$$

such that $\{\psi_1(t), \ldots, \psi_n(t)\} = \mathcal{W}_t$ for $t \in (0, \epsilon]$ and $\{\psi_1(0), \ldots, \psi_n(0)\} \subseteq \mathcal{W}_0$ (see [22]). Hence, all the paths converge to points on $W$ as $t \to 0$. The points $\{\psi_i(0)\}_{i=1}^n$ are called the end points of the homotopy and we will call the associated 0-cycle $\sum_i \psi_i(0)$ on $W$ the *cycle of endpoints* (counted with multiplicity). The multiplicity of an endpoint in this cycle is equal to the number of paths that converge to that point.

**Remark A.15** By Lemma A.4, the limit intersection class $\lim_{t \to 0}(\mathcal{X}_t \cdot \mathcal{V}_t)$ pushed forward to $W$ is $X \cdot \mathbb{P}^k$ which, by Bézout's theorem, has degree $n > 0$. By Remark A.5, the limit intersection class is a well defined 0-cycle on the limit set. The cycle of endpoints is equal to the limit cycle. To see this, we will use the notation of Section A.3. Let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the irreducible components of $\mathcal{W}^*$ (these are all curves and they dominate $T$) and let $\mathcal{D}'_i$ be the closure of $\mathcal{D}_i$ in $\mathcal{W}'$. By Remark A.5, the limit cycle is equal to $\sum_{i=1}^q a_i (\mathcal{D}'_i)_0$ for some positive integers $a_i$. By above, the limit cycle has degree $n$. Also, the projections $\mathcal{D}'_i \to T$ are flat and therefore $n = \sum_{i=1}^q a_i \deg (\mathcal{D}'_i)_t$ for all $t \in T$. Using the fact that $\mathcal{W}_t$ consists of $n$ points of multiplicity 1 for all $t \in T \setminus \{0\}$, we get that $n = \sum_{i=1}^{q} \deg(\mathcal{D}'_i)_t$ for all $t \in T \setminus \{0\}$. It follows that $a_i = 1$ for all $i$ and $n = \sum_{i=1}^{q} \deg(\mathcal{D}'_i)_t$ for all $t \in T$. Note that any endpoint of the homotopy is contained in some $\mathcal{D}'_i$. Let $p \in (\mathcal{D}'_i)_0$. Then, there is an analytic neighborhood $U$ of $p$ in $\mathcal{D}'_i$ such that $U$ does not contain any other point of $(\mathcal{D}'_i)_0$ and the map $U \to T$ is a flat map of complex spaces. In particular, $U$ does not contain any endpoint of the homotopy distinct from $p$. Using flatness and the corollary to [7, Prop. 3.13], it follows that the geometric multiplicity of $p$ in $(\mathcal{D}'_i)_0$ is equal to the number of paths contained in $\mathcal{D}'_i$ that converge to $p$. Hence, the cycle of endpoints is equal to the limit cycle.

Let $R \subseteq W$ be a closed subset. Then, the cycle of endpoints contained in $R$ (counted with multiplicity), which by the previous remark, is equal to the part of the limit cycle supported on $R$.

Next, we will consider the characteristic section of the deformation $\mathcal{X}$. Fix a basis for the normal space $N_0T$. The normal bundle $N_XY$ is isomorphic to $\bigoplus_{i=1}^{k} \mu_i^*(\mathcal{O}_{\mathbb{P}^k}(n_i))$ where $\mu_i : X \to \mathbb{P}^k$ is the $i^{th}$ projection, see [6, App. B.7.4]. A tuple of homogeneous polynomials $P = (P_1, \ldots, P_k)$ with $\deg P_i = n_i$ induces a global section $\sigma(P)$ of $N_XY$ which is the sum of corresponding sections of the line bundles $\mu_i^*(\mathcal{O}_{\mathbb{P}^k}(n_i))$. The space of such sections $\sigma(P)$ generates $N_XY$. Moreover, by [13] and [6, Ex. 11.3.1], the characteristic section of the deformation $\mathcal{X}(F, G, H)$ is equal to $\sigma(G)$ up to multiplication by a scalar which depends on the choice of basis of $N_0T$. Proposition A.8 yields that, given a closed subset $R \subseteq W$, a first order general homotopy is such that the cycle of endpoints contained in $R$ represents the class $(X \cdot \mathbb{P}^k)^R$ in $A_0(R)$.

**Theorem A.16** *Let $X$ and $W$ be as above. Let $Z$ be a distinguished variety and fix a proper closed subset $S \subset Z$. Then, a first order general homotopy has at least one endpoint which lies on $Z$ but not on $S$.*

**Proof.** Note that
$$(X \cdot \mathbb{P}^k)^Z = i(Z) + \sum_{Z' \in D(Z)} i(Z')$$
where $D(Z)$ is the set of all distinguished varieties properly contained in $Z$. Consider the set $D_S(Z) = \{Z' \in D(Z) : Z' \nsubseteq S\}$. Then,
$$(X \cdot \mathbb{P}^k)^Z - (X \cdot \mathbb{P}^k)^S = i(Z) + \sum_{Z' \in D_S(Z)} i(Z').$$
Applying Prop. A.8 twice, once with $R = S$ and once with $R = Z$ yields that a first order general homotopy will be such that the number of paths (counted with multiplicity) converging to $Z$ but not $S$ is $\deg(X \cdot \mathbb{P}^k)^Z - \deg(X \cdot \mathbb{P}^k)^S$. But
$$\deg(X \cdot \mathbb{P}^k)^Z - \deg(X \cdot \mathbb{P}^k)^S = \deg((X \cdot \mathbb{P}^k)^Z - (X \cdot \mathbb{P}^k)^S) \geq \deg i(Z) > 0,$$
by Lemma A.12 □

Lemma A.1 yields the following.

**Corollary A.17** *Let $W_1, \ldots, W_s$ be the irreducible components of $W$. The endpoints of a first order general homotopy include $s$ points $p_1, \ldots, p_s$ such that, for all $i = 1, \ldots, s$, $p_i$ is a smooth point on $W_i$, $p_i \notin W_j$ for $j \neq i$, and $p_i$ is not on any embedded component of $W$.*

**Remark A.18** For nonconstant homogeneous polynomials $F_1, \ldots, F_k \in \mathbb{C}[x_0, \ldots, x_k]$, define $n_i = \deg F_i$. Homotopies of the following form are useful in practice:

$$(1-t)F + t(1-t)A + tB = 0 \tag{7}$$

where $A_1, \ldots, A_k$ are general homogeneous polynomials with $\deg A_i = n_i$ and $B_i = x_i^{n_i} - x_0^{n_i}$. At $t = 1$, the system of equations (7) has exactly the Bézout number $n = \Pi_{i=1}^k n_i$ solutions and, consequently, this is also true for general $t \in \mathbb{C}$. Moreover, the solutions at $t = 1$ are easy to compute. Sine $A_1, \ldots, A_k$ are general and the coefficient of the linear term in $t$ in (7) is $A - F + B$, (7) is a first order general homotopy.

## A.6 Examples

We close this appendix with some examples of homotopies associated to subschemes of affine space $\mathbb{C}^n$. If $W \subseteq V = \mathbb{C}^n$ is a subscheme, we have the normal cone $\pi : C_W V \to V$ and, as in Section A.2, the distinguished varieties $Z_i = \pi(C_i)$ where $C_1, \ldots, C_r$ are the irreducible components of $C_W V$.

**Example A.19** Consider the subscheme $W \subset \mathbb{C}^2$ defined by the ideal $I = (x^2, xy) \subset \mathbb{C}[x, y]$. The solution set of $I$ is the line $\{x = 0\}$, but there is an embedded point at the origin. In fact, the distinguished components are the irreducible component $\{x = 0\}$ and the point $\{(0, 0)\}$.

Suppose that $q_1, q_2 \in \mathbb{C}[x, y]$ are degree 2 polynomials such that $q_1 = q_2 = 0$ has 4 distinct solutions. Consider the homotopy

$$\begin{bmatrix} (1-t)x^2 + tq_1 \\ (1-t)xy + tq_2 \end{bmatrix} = 0$$

which defines 4 solution paths whose endpoints depend upon $q_1$ and $q_2$.

1. If $q_1$ and $q_2$ are general, two paths end at the origin and two paths end at distinct points on the line $x = 0$ away from the origin whose $y$-coordinates depend on $q_1$ and $q_2$.
2. If, for general $\gamma \in \mathbb{C}$, one takes $q_1 = \gamma(x^2 - 1)$ and $q_2 = \gamma(y^2 - 1)$ which is a standard start system for a total degree homotopy (e.g., see [22, Eq. 8.4.2]), then two paths end at the origin and the other two paths diverge to infinity.
3. If, for general $\gamma \in \mathbb{C}$, one takes $q_1 = \gamma(y^2 - 1)$ and $q_2 = \gamma(x^2 - 1)$, then one obtains the general behavior in Item 1.

Consider replacing $x$ and $y$ with general linear forms $\ell_1, \ell_2$ in $x$ and $y$. That is, for general linear forms $\ell_1, \ell_2$, consider the ideal $I = (\ell_1^2, \ell_1\ell_2) \subset \mathbb{C}[x, y]$. The distinguished components are the line $\{l_1 = 0\}$ and the origin. In this case, a standard total degree homotopy yields the generic behavior, namely one endpoint at the origin (of multiplicity 2) and two distinct endpoints on the line $l_1 = 0$ away from the origin.

The following example illustrates a case of nested distinguished components.

**Example A.20** Let $q_0 = xy - z \in \mathbb{C}[x, y, z]$ and $Q \subset \mathbb{C}^3$ be the quadric surface $Q \subset \mathbb{C}^3$ defined by $q_0 = 0$. Let $\ell_0 = x - y$ and $C \subset Q$ be the conice defined by $q_0 = l_0 = 0$. Let $\ell_1$ and $\ell_2$ be general linear forms and $\ell_3 = x + y - z$. Define $q_1 = q_0 + \ell_0\ell_1$ and $q_2 = q_0 + \ell_0\ell_2$. Consider the ideal $I = (q_0\ell_0\ell_3, q_0q_1, q_0q_2)$ which is generated by 3 quartics. The distinguished components are $Q$, $C$, $O = \{(0, 0, 0\}$, and $\{P\}$ where $P \in \mathbb{C}^3 \setminus Q$. Since $O \subset C \subset Q$, the irreducible components are $Q$ and $\{P\}$. Using a first order general homotopy, the endpoints of the 64 paths are distributed as follows:

1. 41 distinct points on the quadric $Q$ not on $C$,
2. 9 points of multiplicity 2 on the conic $C$ not on $O$,
3. 1 point of multiplicity 4 at the origin, and
4. 1 point at $P$

with $64 = 41 + 9 \cdot 2 + 1 \cdot 4 + 1$.