ALGEBRAIC BOUNDARIES OF HILBERT'S SOS CONES

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Abstract. We study the geometry underlying the difference between non-negative polynomials and sums of squares. The hypersurfaces that discriminate these two cones for ternary sextics and quaternary quartics are shown to be Noether-Lefschetz loci of K3 surfaces. The projective duals of these hypersurfaces are defined by rank constraints on Hankel matrices. We compute their degrees using numerical algebraic geometry, thereby verifying results due to Maulik and Pandharipande. The non-SOS extreme rays of the two cones of non-negative forms are parametrized respectively by the Severi variety of plane rational sextics and by the variety of quartic symmetroids.

1. Introduction

A fundamental object in convex algebraic geometry is the cone $\Sigma_{n,2d}$ of homogeneous polynomials of degree $2d$ in $\mathbb{R}[x_1, \ldots, x_n]$ that are sums of squares (SOS). Hilbert [13] showed that the cones $\Sigma_{3,6}$ and $\Sigma_{4,4}$ are strictly contained in the corresponding cones $P_{3,6}$ and $P_{4,4}$ of non-negative polynomials. Blekherman [5] furnished a geometric explanation for this containment. In spite of his recent progress, the geometry of the sets $P_{3,6} \backslash \Sigma_{3,6}$ and $P_{4,4} \backslash \Sigma_{4,4}$ remains mysterious.

We here extend known results on Hilbert's SOS cones by characterizing their *algebraic boundaries*, that is, the hypersurfaces that arise as Zariski closures of their topological boundaries. The algebraic boundary of the cone $P_{n,2d}$ of non-negative polynomials is the *discriminant* [19], and this is also always one component in the algebraic boundary of $\Sigma_{n,2d}$. The discriminant has degree $n(2d-1)^{n-1}$, which equals 75 for $\Sigma_{3,6}$ and 108 for $\Sigma_{4,4}$. What we are interested in are the other components in the algebraic boundary of the SOS cones.

Theorem 1. *The algebraic boundary of $\Sigma_{3,6}$ has a unique non-discriminant component. It has degree 83200 and consists of forms that are sums of three squares of cubics. Similarly, the algebraic boundary of $\Sigma_{4,4}$ has a unique non-discriminant component. It has degree 38475 and consists of forms that are sums of four squares of quadrics. Both hypersurfaces define Noether-Lefschetz divisors in moduli spaces of K3 surfaces.*

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Our characterization of these algebraic boundaries in terms of sums of few squares is a consequence of [5, Corollaries 5.3 and 6.5]. What is new here is the connection to K3 surfaces, which elucidates the hypersurface of ternary sextics that are rank three quadrics in cubic forms, and the hypersurface of quartic forms in 4 variables that are rank four quadrics in quadratic forms. Their degrees are coefficients in the modular forms derived by Maulik and Pandharipande in their paper on Gromov-Witten and Noether-Lefschetz theory [18]. In Section 2 we explain these concepts and present the proof of Theorem 1.

Section 3 is concerned with the cone dual to $\Sigma_{n,2d}$ and with the dual varieties to our Noether-Lefschetz hypersurfaces in Theorem 1. Each of them is a determinantal variety, defined by rank constraints on a 10×10-Hankel matrix, and it is parametrized by a Grassmannian via the global residue map in [6, §1.6]. We note that Hankel matrices are also known as moment matrices or catalecticants.

Section 4 features another appearance of a Gromov-Witten number [16] in convex algebraic geometry. Building on work of Reznick [23], we shall prove:

Theorem 2. *The Zariski closure of the set of extreme rays of $P_{3,6}\backslash\Sigma_{3,6}$ is the Severi variety of rational sextic curves in the projective plane $\mathbb{P}^2$. This Severi variety has dimension* 17 *and degree* 26312976 *in the $\mathbb{P}^{27}$ of all sextic curves.*

We also determine the analogous variety of extreme rays for quartics in $\mathbb{P}^3$:

Theorem 3. *The Zariski closure of the set of extreme rays of $P_{4,4}\backslash\Sigma_{4,4}$ is the variety of* quartic symmetroids *in $\mathbb{P}^3$, that is, the surfaces whose defining polynomial is the determinant of a symmetric 4×4-matrix of linear forms. This variety has dimension* 24 *in the $\mathbb{P}^{34}$ of all quartic surfaces.*

Section 5 offers an experimental study of the objects in this paper using numerical algebraic geometry. We demonstrate that the degrees 83200 and 38475 in Theorem 1 can be found from scratch using the software `Bertini` [4]. This provides computational validation for the cited results by Maulik and Pandharipande [18]. Motivated by Theorem 3, we also show how to compute a symmetric determinantal representation (7) for a given quartic symmetroid.

A question one might ask is: *What's the point of integers such as* 38475*?* One answer is that the exact determination of such degrees signifies an understanding of deep geometric structures that can be applied to a wider range of subsequent problems. A famous example is the number 3264 of plane conics that are tangent to five given conics. The finding of that particular integer in the 19th century led to the development of intersection theory in the 20th century, and ultimately to numerical algebraic geometry in the 21st century. To be more specific, our theorems above furnish novel geometric representations of boundary sums of squares that are strictly positive, and of extremal non-negative polynomials that are not sums of squares. Apart from its intrinsic appeal within algebraic geometry, we expect that our approach, with its focus on explicit degrees, will be useful for applications in optimization and beyond.

2. Noether-Lefschetz Loci of K3 Surfaces

Every smooth quartic surface in $\mathbb{P}^3$ is a K3 surface. In our study of Hilbert's cone $\Sigma_{4,4}$ we care about quartic surfaces containing an elliptic curve of degree 4. As we shall see, these are the quartics that are sums of four squares. K3 surfaces also arise as double covers of $\mathbb{P}^2$ ramified along a smooth sextic curve. In our study of $\Sigma_{3,6}$ we care about K3 surfaces whose associated plane sextic is a sum of three squares. This constraint on K3 surfaces also appeared in the proof by Colliot-Thélène [8] that a general sextic in $\Sigma_{3,6}$ is a sum of four but not three squares of rational functions.

Our point of departure is the Noether-Lefschetz Theorem (cf. [9]) which states that a general quartic surface S in $\mathbb{P}^3$ has Picard number 1. In particular, the classical result by Noether [20] and Lefschetz [17] states that every irreducible curve on S is the intersection of S with another surface in $\mathbb{P}^3$. This has been extended to general polarized K3 surfaces, i.e. K3 surfaces S with an ampel divisor A. For each even $l \geq 2$, the moduli space M_l of K3 surfaces with a polarization A of degree $A^2 = l$ has dimension 19 and is irreducible. For the general surface S in M_l, the Picard group is generated by A. The locus in the moduli space M_l where the Picard number of S increases to 2 has codimension one. We are here interested in one irreducible component of that locus in M_2, and also in M_4. The relevant enumerative geometry was developed only recently, by Maulik and Pandharipande [18], and our result rests on theirs.

Proof of Theorem 1. It was shown in [5] that $\partial\Sigma_{3,6}\backslash\partial P_{3,6}$ consists of ternary sextics F that are sums of three squares over $\mathbb{R}$. Over the complex numbers $\mathbb{C}$, such a sextic F is a rank three quadric in cubic forms, so it can be written as

$$F \;=\; fh - g^2 \qquad \text{where } f, g, h \in \mathbb{C}[x_1, x_2, x_3]_3.$$

Let S be the surface of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the polynomial

$$G \;=\; fs^2 + 2gst + ht^2.$$

If f, g and h are general, then the surface S is smooth. The canonical divisor on $\mathbb{P}^1 \times \mathbb{P}^2$ has bidegree $(-2,-3)$, so, by the adjunction formula, S is a K3-surface. The projection $S \to \mathbb{P}^2$ is two-to-one, ramified along the curve $\{F = 0\} \subset \mathbb{P}^2$. Up to the actions of $SL(2,\mathbb{C})$ and $SL(3,\mathbb{C})$, there is an 18-dimensional family of surfaces of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. These surfaces determine a divisor $D(2,3)$ in the moduli space M_2 of K3-surfaces with a polarization of degree 2. The Picard group of a general point S in this divisor has rank ≤ 2.

Let $A \subset S$ be the preimage of a general line in $\mathbb{P}^2$, and let B be a general fiber of the projection $S \to \mathbb{P}^1$. The classes of the curves A and B are independent in the Picard group of S. Therefore the Picard group has rank at least 2 for every surface $S \in D(2,3)$. The curves A and B determine the intersection matrix

$$\begin{pmatrix} A^2 & A\cdot B \\ A\cdot B & B^2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}. \tag{1}$$

Conversely, any K3-surface with Picard group generated by classes A and B having the intersection matrix (1) has a natural embedding in $\mathbb{P}^1 \times \mathbb{P}^2$ as a divisor of bidegree $(2,3)$: The linear system $|A+B|$ defines an embedding of S into the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ ([26, Proposition 7.15 and Example 7.19]).

A general pencil of plane sextic curves contains a finite number of curves that are ramification loci of K3 double covers with Picard group of rank 2 and intersection matrix with a given discriminant. In our situation, this number is the degree of the hypersurface that forms the Zariski closure of $\Sigma_{3,6} \backslash P_{3,6}$.

We shall derive this number from results of [18]. Let R be a general surface of bidegree $(2,6)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, and let X be the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified along R. The general fiber of the projection $X \to \mathbb{P}^1$ is a curve $\Pi \cong \mathbb{P}^1$ of K3 surfaces with a polarization A of degree $l = A^2 = 2$. Since the surface R has degree two in the first factor, the curve Π defines a conic in the space of ternary sextics. Section 6 in [18] computes the Noether-Lefschetz number $NL^{\Pi}_{1,3}$ of pairs (S, B) where $[S] \in \Pi$, and B is the class of a curve of genus $g(B) = 1$ on S and degree $A \cdot B = 3$. By adjunction, the self-intersection equals $B^2 = 2g(B) - 2 = 0$, and hence the intersection matrix is as above.

There are two curve classes on the K3 surface S, namely B and $3A - B$, that have self-intersection 0 and intersection number $A \cdot B = A \cdot (3A - B) = 3$. Therefore each such surface S appears twice in the count of [18]. Also, the curve Π is a conic in the space of plane sextic curves. So, to the get the count of surfaces S in the Noether-Lefschetz locus for a line in the space of sextics, we altogether must divide the number $NL^{\Pi}_{1,3}$ by 4.

In [18, Corollary 3], the Noether-Lefschetz number $NL^{\Pi}_{g(B),A\cdot B}$ is expressed as the coefficient of the monomial q^{δ} in the expansion of a modular form Θ^{Π}_l of weight $21/2$ as a power series in $q^{1/2l}$, where $l = A^2$ is the degree of the polarization. The exponent of the relevant monomial is $\delta = \Delta_l(g(B), A\cdot B)/2l$, where $\Delta_l(g(B), A\cdot B)$ is the discriminant of intersection matrix $\begin{pmatrix} l & A\cdot B \\ A\cdot B & B^2 \end{pmatrix}$. For the conic Π in the space of sextics, the modular form Θ^{Π}_2 has the expansion

$$\Theta^{\Pi}_2 = -1 + 150q + 1248q^{\frac{5}{4}} + 108600q^2 + 332800q^{\frac{9}{4}} + 5113200q^3 + \dots.$$

In our case, we have $\delta = 9/4$, since $l = 2$ and the intersection matrix (1) has discriminant 9. We conclude that the number

$$\frac{1}{4} NL^{\Pi}_{1,3} = \frac{1}{4} 332800 = 83200$$

equals the degree of the hypersurface of sextics that are sums of three squares.

We now come to the case of quartic surfaces in $\mathbb{P}^3$. It was shown in [5] that $\partial\Sigma_{4,4}\backslash\partial P_{4,4}$ consists of quartic forms F that are sums of four squares over $\mathbb{R}$. Over the complex numbers $\mathbb{C}$, such a quartic F is a rank 4 quadric in quadrics:

$$(2) \qquad F \;=\; fg - hk \;=\; \det\begin{pmatrix} f & h \\ k & g \end{pmatrix} \;\; \text{for some } f,g,h,k \in \mathbb{C}[x_1,x_2,x_3,x_4]_2.$$

The K3 surface S defined by F contains two distinct pencils of elliptic curves on S, one defined by the rows and one by the columns of the 2×2 matrix. Up to the action of $SL(4,\mathbb{C})$, the determinantal quartics (2) form an 18-dimensional family, hence a divisor in the moduli space M_4. A general surface S in this family has Picard rank 2, and its Picard group is generated by the class of a plane section and the class of an elliptic curve in one of the two elliptic pencils.

Conversely, any smooth quartic surface S that contains an elliptic quartic curve is defined by a determinant F as in (2). This form of the equation is therefore characterized by the intersection matrix of S. Let A be the class of the plane section of S in $\mathbb{P}^3$ and let B and E be the classes of the curves in the two elliptic pencils. Then A and $B = 2A - E$ have intersection numbers

$$(3) \qquad \begin{pmatrix} A^2 & A\cdot B \\ A\cdot B & B^2 \end{pmatrix} \;=\; \begin{pmatrix} A^2 & A\cdot E \\ A\cdot E & E^2 \end{pmatrix} \;=\; \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix}.$$

For general S, the classes A and B generate the Picard group and have intersection matrix (3) with discriminant $\Delta_4(1,4) = 16$. Let Π be a general linear pencil of quartic surfaces in $\mathbb{P}^3$. The Noether-Lefschetz number $NL^{\Pi}_{1,4}$ counts pairs (S,B) where $[S]\in\Pi$ and B is a curve class on S of degree 4 and genus $g(B) = 1$. Since there are two classes of such curves on S, we get the number of surfaces in the pencil containing such a curve class, by dividing $NL^{\Pi}_{1,4}$ by 2.

As above, the number $NL^{\Pi}_{g(B),A\cdot B}$ is the coefficient of the monomial q^{δ} in the expansion of a modular form Θ^{Π}_{l} of weight 21/2 as a power series in $q^{1/2l}$, where $l = A^2$ is the degree of the polarization. Here

$$\delta \;=\; \frac{\Delta_4(g(B), A\cdot B)}{8} \;=\; \frac{16}{8} \;=\; 2.$$

The modular form for the general line Π in the space of quartic surfaces equals

$$\Theta^{\Pi}_4 \;=\; -1 + 108q + 320q^{\frac{9}{8}} + 5016q^{\frac{3}{2}} + 76950q^2 + 136512q^{\frac{17}{8}} + \dots$$

This was shown in [18, Theorem 2]. We conclude that the degree of the hypersurface of homogeneous quartics in 4 unknowns that are sums of 4 squares is

$$\frac{1}{2}NL^{\Pi}_{1,4} \;=\; \frac{1}{2}76950 \;=\; 38475.$$

This completes the proof of Theorem 1. $\square$

Remark 4. It was pointed out to us by Giorgio Ottaviani that the smooth ternary sextics that are rank three quadrics in cubic forms are known to coincide with the smooth sextics that have an effective even theta characteristic (cf. [21, Proposition 8.4]). Thus the algebraic boundary of Hilbert's SOS cone for ternary sextics is also related to the theta locus in the moduli space $\overline{\mathcal{M}}_{10}$. □

3. Rank Conditions on Hankel Matrices

We now consider the convex cone $(\Sigma_{n,2d})^\vee$ dual to the cone $\Sigma_{n,2d}$. Its elements are the linear forms ℓ on $\mathbb{R}[x_1,\ldots,x_n]_{2d}$ that are non-negative on squares. Each such linear form ℓ is represented by its associated quadratic form on $\mathbb{R}[x_1,\ldots,x_n]_d$, which is defined by $f \mapsto \ell(f^2)$. The symmetric matrix which expresses this quadratic form with respect to the monomial basis of $\mathbb{R}[x_1,\ldots,x_n]_d$ is denoted H_ℓ, and it is called the *Hankel matrix* of ℓ. It has format $\binom{n+d-1}{d} \times \binom{n+d-1}{d}$, and its rows and columns are indexed by elements of $\{(i_1,i_2,\ldots,i_n) \in \mathbb{N}^n : i_1+i_2+\cdots+i_n = d\}$. We shall examine the two cases of interest.

The Hankel matrix for ternary sextics ($n = d = 3$) is the 10×10-matrix

$$(4) \qquad H_\ell \;=\; \begin{bmatrix}
a_{006} & a_{015} & a_{024} & a_{033} & a_{105} & a_{114} & a_{123} & a_{204} & a_{213} & a_{303}\\
a_{015} & a_{024} & a_{033} & a_{042} & a_{114} & a_{123} & a_{132} & a_{213} & a_{222} & a_{312}\\
a_{024} & a_{033} & a_{042} & a_{051} & a_{123} & a_{132} & a_{141} & a_{222} & a_{231} & a_{321}\\
a_{033} & a_{042} & a_{051} & a_{060} & a_{132} & a_{141} & a_{150} & a_{231} & a_{240} & a_{330}\\
a_{105} & a_{114} & a_{123} & a_{132} & a_{204} & a_{213} & a_{222} & a_{303} & a_{312} & a_{402}\\
a_{114} & a_{123} & a_{132} & a_{141} & a_{213} & a_{222} & a_{231} & a_{312} & a_{321} & a_{411}\\
a_{123} & a_{132} & a_{141} & a_{150} & a_{222} & a_{231} & a_{240} & a_{321} & a_{330} & a_{420}\\
a_{204} & a_{213} & a_{222} & a_{231} & a_{303} & a_{312} & a_{321} & a_{402} & a_{411} & a_{501}\\
a_{213} & a_{222} & a_{231} & a_{240} & a_{312} & a_{321} & a_{330} & a_{411} & a_{420} & a_{510}\\
a_{303} & a_{312} & a_{321} & a_{330} & a_{402} & a_{411} & a_{420} & a_{501} & a_{510} & a_{600}
\end{bmatrix}$$

The Hankel matrix for quaternary quartics ($n = 4, d = 2$) also has size 10×10:

$$(5) \quad H_\ell \;=\; \begin{bmatrix}
a_{0004} & a_{0013} & a_{0022} & a_{0103} & a_{0112} & a_{0202} & a_{1003} & a_{1012} & a_{1102} & a_{2002}\\
a_{0013} & a_{0022} & a_{0031} & a_{0112} & a_{0121} & a_{0211} & a_{1012} & a_{1021} & a_{1111} & a_{2011}\\
a_{0022} & a_{0031} & a_{0040} & a_{0121} & a_{0130} & a_{0220} & a_{1021} & a_{1030} & a_{1120} & a_{2020}\\
a_{0103} & a_{0112} & a_{0121} & a_{0202} & a_{0211} & a_{0301} & a_{1102} & a_{1111} & a_{1201} & a_{2101}\\
a_{0112} & a_{0121} & a_{0130} & a_{0211} & a_{0220} & a_{0310} & a_{1111} & a_{1120} & a_{1210} & a_{2110}\\
a_{0202} & a_{0211} & a_{0220} & a_{0301} & a_{0310} & a_{0400} & a_{1201} & a_{1210} & a_{1300} & a_{2200}\\
a_{1003} & a_{1012} & a_{1021} & a_{1102} & a_{1111} & a_{1201} & a_{2002} & a_{2011} & a_{2101} & a_{3001}\\
a_{1012} & a_{1021} & a_{1030} & a_{1111} & a_{1120} & a_{1210} & a_{2011} & a_{2020} & a_{2110} & a_{3010}\\
a_{1102} & a_{1111} & a_{1120} & a_{1201} & a_{1210} & a_{1300} & a_{2101} & a_{2110} & a_{2200} & a_{3100}\\
a_{2002} & a_{2011} & a_{2020} & a_{2101} & a_{2110} & a_{2200} & a_{3001} & a_{3010} & a_{3100} & a_{4000}
\end{bmatrix}$$

We note that what we call Hankel matrix is known as *moment matrix* in the literature on optimization and functional analysis, and it is known as *(symmetric) catalecticant* in the literature on commutative algebra and algebraic geometry.

The dual cone $(\Sigma_{3,3})^\vee$ is the spectrahedron consisting of all positive semidefinite Hankel matrices (4). The dual cone $(\Sigma_{4,2})^\vee$ is the spectrahedron consisting of all positive semidefinite matrices (5). This convex duality offers a way of representing our Noether-Lefschetz loci via their projective dual varieties.

Theorem 5. *The Hankel matrices (4) having rank* ≤ 7 *constitute a rational projective variety of dimension* 21 *and degree* 2640*. Its dual is the hypersurface of sums of three squares of cubics. Likewise, the Hankel matrices (5) having rank* ≤ 6 *constitute a rational projective variety of dimension* 24 *and degree* 28314*. Its dual is the hypersurface of sums of four squares of quadrics.*

Proof. The fact that these varieties are rational and irreducible of the asserted dimensions can be seen as follows. Consider the Grassmannian $\mathrm{Gr}(3,10)$ which parametrizes three-dimensional linear subspaces F of the 10-dimensional space $\mathbb{R}[x_1,x_2,x_3]_3$ of ternary cubics. This Grassmannian is rational and its dimension equals 21. The *global residue* in $\mathbb{P}^2$, as defined in [6, §1.6], specifies a rational map $F \mapsto \mathrm{Res}_{\langle F\rangle}$ from $\mathrm{Gr}(3,10)$ into $\mathbb{P}((\mathbb{R}[x_1,x_2,x_3]_6)^*) \simeq \mathbb{P}^{27}$. The base locus of this map is the resultant of three ternary cubics, so $\mathrm{Res}_{\langle F\rangle}$ is well-defined whenever the ideal $\langle F\rangle$ is a complete intersection in $\mathbb{R}[x_1,x_2,x_3]$. The value $\mathrm{Res}_{\langle F\rangle}(P)$ of this linear form on a ternary sextic P is the image of P modulo the ideal $\langle F\rangle$, and it can be computed via any Gröbner basis normal form. Our map $F \mapsto \ell$ is birational because it has an explicit inverse: $F = \mathrm{kernel}(H_\ell)$. The inverse simply maps the rank 7 Hankel matrix representing ℓ to its kernel.

The situation is entirely analogous for $n=4, d=2$. Here we consider the 24-dimensional Grassmannian $\mathrm{Gr}(4,10)$ which parametrizes 4-dimensional linear subspaces F $\mathbb{R}[x_1,x_2,x_3,x_4]_2$. The global residue in $\mathbb{P}^3$ specifies a rational map

$$\mathrm{Gr}(4,10) \dashrightarrow \mathbb{P}((\mathbb{R}[x_1,x_2,x_3,x_4]_4)^*) \simeq \mathbb{P}^{34}, \quad F \mapsto \mathrm{Res}_{\langle F\rangle}.$$

This map is birational onto its image, the variety of rank 6 Hankel matrices (5), and the inverse of that map takes a rank 6 Hankel matrix (5) to its kernel.

To determine the degrees of our two Hankel determinantal varieties, we argue as follows. The variety S_r of all symmetric 10×10-matrices of rank $\leq r$ is known to be irreducible and arithmetically Cohen-Macaulay, it has codimension $\binom{11-r}{2}$, and its degree is given by the following formula due to Harris and Tu [10]:

$$\mathrm{degree}(S_r) \;=\; \prod_{j=0}^{9-r}\left(\binom{10+j}{10-r-j}\Big/\binom{2j+1}{j}\right). \tag{6}$$

Thus S_r has codimension 6 and degree 2640 for $r=7$, and it has codimension 10 and degree 28314 for $r=6$. The projective linear subspace of Hankel matrices (4) has dimension 27. Its intersection with S_7 was seen to have dimension 21. Hence the intersection has the expected codimension 6 and is proper. That the intersection is proper ensures that the degree remains 2640. Likewise, the projective linear subspace of Hankel matrices (5) has dimension 34, and its intersection with S_6 has dimension 24. The intersection has the expected codimension 10, and we conclude as before that the degree equals 28314.

It remains to be seen that the two Hankel determinantal varieties are projectively dual to the Noether-Lefschetz hypersurfaces in Theorem 1. This follows from [5, Corollary 5.2] for sextics curves in $\mathbb{P}^2$ and from [5, Corollary 5.7] for quartic surfaces in $\mathbb{P}^3$. These results characterize the relevant extreme rays of $\Sigma^*_{3,6}$ and $\Sigma^*_{4,4}$ respectively. These rays are dual to the hyperplanes that support $\partial\Sigma_{3,6}$ and $\partial\Sigma_{4,4}$ at smooth points representing strictly positive polynomials. By passing to the Zariski closures, we conclude that the algebraic boundaries of $\Sigma_{3,6}\backslash P_{3,6}$ and $\Sigma_{4,4}\backslash P_{4,4}$ are projectively dual to the Hankel determinantal varieties above. For a general introduction to the relationship between projective duality and cone duality in convex algebraic geometry we refer to [25]. □

Remark 6. In the space $\mathbb{P}(\mathrm{Sym}^2 V)$ of quadratic forms on a 10-dimensional vector space V^*, the subvariety S_r of forms of rank $\leq r$ is the dual variety to $S^*_{10-r} \subset \mathbb{P}(\mathrm{Sym}^2 V^*)$. Identifying V with ternary cubics, the space of 10×10 Hankel matrices (4) form a 27-dimensional linear subspace $H \subset \mathbb{P}(\mathrm{Sym}^2 V^*)$. For $r \leq 3$, we have $\dim(S_r) < 27$ and the variety dual to $H_{10-r} = S^*_{10-r} \cap H$ coincides with the image Σ_r of the birational projection of S_r into H^*. That image is the variety of sextics that are quadrics of rank $\leq r$ in cubics. Furthermore, when $r \leq 2$ the projection from S^*_r to Σ_r is a morphism, so the degrees of these two varieties coincide. When $r = 3$, the projection is not a morphism and the degree drops to 83200. A similar analysis works for $V = \mathbb{R}[x_1, x_2, x_3, x_4]_2$ with $r \leq 4$. □

4. Extreme Non-negative Forms

For each of Hilbert's two critical cases, in Section 2 we examined the hypersurface separating $\Sigma_{n,2d}$ and $P_{n,2d}\backslash\Sigma_{n,2d}$. In this section we take an alternative look at this separation, namely, we focus on the extreme rays of the cone $P_{n,2d}$ of non-negative forms that do not lie in the SOS subcone $\Sigma_{n,2d}$. We begin with the following result on zeros of non-negative forms in the two Hilbert cases.

Proposition 7. *Let p be a non-negative form in $P_{3,6}$ or $P_{4,4}$. If p has more than* 10 *zeros, then p has infinitely many zeros and it is a sum of squares.*

Proof. The statement for $P_{3,6}$ was proved by Choi, Lam and Reznick in [7]. They also showed the statement for the cone $P_{4,4}$ but with "11 zeros" instead of "10 zeros". To reduce the number from 11 to 10, we use Kharlamov's theorem in [15] which states that the number of connected components of any quartic surface in real projective 3-space is ≤ 10. See also Rohn's classical work [24]. □

Recall that a face of a closed convex set K in a finite-dimensional real vector space is *exposed* if it is the intersection of K with a supporting hyperplane. The extreme rays of K lie in the closure (and hence in the Zariski closure) of the set of exposed extreme rays [27]. A polynomial $p \in P_{n,2d} \backslash \Sigma_{n,2d}$ that generates an extreme exposed ray of $P_{n,2d}$ will be called an *extreme non-negative form.*

Our first goal is to prove Theorem 2, which characterizes the Zariski closure of the semi-algebraic set of all extreme non-negative forms for $n = d = 3$.

Proof of Theorem 2. Suppose $p \in P_{3,6} \backslash \Sigma_{3,6}$ is an extreme form. By [23, Lemma 7.1], the polynomial p is irreducible. Moreover, we claim that $|\mathcal{V}_{\mathbb{R}}(p)| \geq 10$. It is not hard to show that p is an extreme non-negative form if and only if $\mathcal{V}_{\mathbb{R}}(p)$ is maximal among all forms in $P_{n,2d}$. In other words, if p is an extreme non-negative form and $\mathcal{V}_{\mathbb{R}}(p) \subseteq \mathcal{V}_{\mathbb{R}}(q)$ for some $q \in P_{n,2d}$ then $q = \lambda p$ for some $\lambda \in \mathbb{R}$. Now suppose that $|\mathcal{V}_{\mathbb{R}}(p)| \leq 9$. Then there is a ternary cubic $q \in P_{3,3}$ that vanishes on $\mathcal{V}_{\mathbb{R}}(p)$. We have $q^2 \in P_{3,6}$ and $\mathcal{V}_{\mathbb{R}}(p) \subseteq \mathcal{V}_{\mathbb{R}}(q)$. This contradicts maximality of $\mathcal{V}_{\mathbb{R}}(p)$. By Proposition 7 we conclude that $|\mathcal{V}_{\mathbb{R}}(p)| = 10$.

Let C be the sextic curve in the complex projective plane $\mathbb{P}^2$ defined by $p = 0$. Since C is irreducible, it must have non-negative genus. Each point in $\mathcal{V}_{\mathbb{R}}(p)$ is a singular point of the complex curve C. As this gives C ten singularities, it follows by the genus formula that C can have no more singularities, and furthermore that all of the real zeros of p are ordinary singularities. The genus of C is zero and therefore it is an irreducible rational curve.

Let $\mathcal{S}_{6,0}$ denote the Severi variety of rational sextic curves in $\mathbb{P}^2$. We have shown that $\mathcal{S}_{6,0}$ contains the semi-algebraic set of extreme forms in $P_{3,6} \backslash \Sigma_{3,6}$. This is a subvariety in the $\mathbb{P}^{27}$ of ternary sextics. The Severi variety $\mathcal{S}_{6,0}$ is known to be irreducible, and the general member C has exactly 10 nodes. Moreover, that set of 10 nodes in $\mathbb{P}^2$ uniquely identifies the rational curve C.

Each rational sextic curve in $\mathbb{P}^2$ is the image of a morphism $\mathbb{P}^1 \to \mathbb{P}^2$ defined by three binary forms of degree 6. To choose these, we have $3 \cdot 7 = 21$ degrees of freedom. However, the image in $\mathbb{P}^2$ is invariant under the natural action of the 4-dimensional group $GL(2, \mathbb{C})$ on the parametrization, and hence $\mathcal{S}_{6,0}$ has dimension $21 - 4 = 17$. The degree of $\mathcal{S}_{6,0}$ is the number of rational sextics passing through 17 given points in $\mathbb{P}^2$, which is one of the Gromov-Witten numbers of $\mathbb{P}^2$. For rational curves, these numbers were computed by Kontsevich and Manin [16] using an explicit recursion formula equivalent to the WDVV equations. From their recursion, one gets degree$(\mathcal{S}_{6,0}) = 26312976$.

To complete the proof, it remains to be shown that the semi-algebraic set of extreme forms in $P_{3,6} \backslash \Sigma_{3,6}$ is Zariski dense in the Severi variety $\mathcal{S}_{6,0}$. We deduce this from [23, Theorem 4.1 and Section 5]. There, starting with a specific set Γ of 8 points in $\mathbb{P}^2$, an explicit 1-parameter family of non-negative sextics with 10 zeros, 8 of which come from Γ, was constructed using *Hilbert's Method.* Furthermore, by Theorem 4.1, Hilbert's Method can be applied to any 8 point configuration in the neighborhood of Γ. By a continuity argument, all 8 point configurations sufficiently close to Γ will also have a 1-parameter family of non-negative forms with 10 zeros. All such forms are exposed extreme rays.

This identifies a semi-algebraic set of extreme non-negative forms having dimension $16 + 1 = 17$. We conclude that this set is Zariski dense in $\mathcal{S}_{6,0}$. $\square$

Remark 8. Our analysis implies the following result concerning $\partial P_{3,6} \backslash \Sigma_{3,6}$. All exposed extreme rays are sextics with ten acnodes, and all extreme rays are limits of sextics with ten acnodes. This proves the second part of Reznick's Conjecture 7.9 in [23]. Indeed, in the second paragraph of the above proof we saw that C has ten ordinary singularities. These cannot be cusps since $p \geq 0$. Hence they have to be what is classically called *acnodes*, or *round zeros* in [23].

Our next goal is to derive Theorem 3, the analogue to Theorem 2 for quartic surfaces in $\mathbb{P}^3$. The role of the Severi variety $\mathcal{S}_{6,0}$ is now played by the variety $\mathcal{QS}$ of quartic symmetroids, *i.e.* the surfaces whose defining polynomial equals

$$F(x_1, x_2, x_3, x_4) \quad = \quad \det\big(A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4\big), \tag{7}$$

where A_1, A_2, A_3, A_4 are complex symmetric 4×4-matrices.

Lemma 9. *The variety $\mathcal{QS}$ is irreducible and has codimension* 10 *in* $\mathbb{P}^{34}$.

Proof. Each of the four symmetric matrices A_i has 10 free parameters. The formula (7) expresses the 35 coefficients of F as quartic polynomials in the 40 parameters, and hence defines a rational map $\mathbb{P}^{39} \dashrightarrow \mathbb{P}^{34}$. Our variety $\mathcal{QS}$ is the Zariski closure of the image of this map, and so it is irreducible. To compute its dimension, we form the 35×40 Jacobian matrix of the parametrization. By evaluating at a generic point $(A_1, \ldots, A_4)$, we find that the Jacobian matrix has rank 25. Hence the dimension of the symmetroid variety $\mathcal{QS} \subset \mathbb{P}^{34}$ is 24. For a theoretical argument see [14, page 168, Chapter IX.101]. $\square$

A general complex symmetroid S has 10 nodes, but not every 10-nodal quartic in $\mathbb{P}^3$ is a symmetroid. To identify symmetroids, we employ the following lemma from Jessop's classical treatise [14] on singular quartic surfaces. Let S be a 10-nodal quartic with a node at $p = (0 : 0 : 0 : 1)$. Its defining polynomial equals $F = f x_4^2 + 2 g x_4 + h$ where $f, g, h \in \mathbb{C}[x_1, x_2, x_3]$ are homogeneous of degrees 2,3,4 respectively. The projection of S from p is a double cover of the plane with coordinates x_1, x_2, x_3 ramified along the sextic curve C_p defined by $g^2 - fh$. The curve C_p has nodes exactly at the image of the nodes on S that are distinct from p. Since no three nodes on S are collinear, the curve C_p has 9 nodes in $\mathbb{P}^2$. The following result appears on page 14 in Chapter I.8 of [14].

Lemma 10. *If the sextic ramification curve C_p is the union of two smooth cubics that intersect in* 9 *distinct points, then the quartic surface S is a symmetroid and, moreover, the ramification curve C_q for the projection from any node q on S is the union of two smooth cubic curves.*

Proof of Theorem 3. Let $\mathcal{E}$ denote the semialgebraic set of all non-negative extreme forms F in $P_{4,4} \backslash \Sigma_{4,4}$. Each $F \in \mathcal{E}$ satisfies $|\mathcal{V}_{\mathbb{R}}(F)| = 10$, by Proposition 7 and the same argument as in the first paragraph in the proof of Theorem 2. Thus $\mathcal{E}$ consists of those real quartic surfaces in $\mathbb{P}^3$ that have precisely 10 real points.

We shall prove that $\mathcal{E}$ is a subset of $\mathcal{QS}$. Let $F \in \mathcal{E}$ and $S = \mathcal{V}_{\mathbb{C}}(F)$ the corresponding complex surface. Then S is a real quartic with 10 nodes, and these nodes are real. Our goal is to show that S is a symmetroid over $\mathbb{C}$. If $p \in \mathbb{P}^3_{\mathbb{R}}$ is one of the nodes of F, then the ramification curve C_p is a real sextic curve with 9 real nodes at the image of the nodes distinct from p. Since the nodes on S are the only real points, these nodes are the only real points on C_p.

Through any nine of the nodes of S there is a real quadratic surface. This quadric is unique; otherwise there is a real quadric through all ten nodes and F is not extreme. Let q be a node on S distinct from p and A a real quadratic form vanishing on all nodes on S except q. Consider the pencil of quartic forms

$$F_t \;=\; F + tA^2 \quad \text{for } t \in \mathbb{R}.$$

Suppose $p = (0:0:0:1)$. The polynomial A has the form $ux_4 + v$, where $u, v \in \mathbb{R}[x_1, x_2, x_3]$ have degree 1 and 2. The equation of F_t is then given by

$$F_t \;=\; (f + tu^2)x_4^2 \;+\; 2(g + tuv)x_4 \;+\; h + tv^2.$$

Any surface $S_t = \{F_t = 0\}$ has at least 9 real singular points, namely the nodes of S other than q. Since F is non-negative, F_t is non-negative for $t > 0$ with zeros precisely at the 9 nodes. On the other hand, F has an additional zero at q. Since A^2 is positive at q, the real surface $\{F_t = 0\}$ must have a 2-dimensional component when $t < 0$. Projecting from p we get a pencil of ramification loci $C_p(t)$. In the above notation, this family of sextic curves is defined by the forms

$$G_t \;=\; fh - g^2 + t(hu^2 - 2guv + fv^2) \quad \in \; \mathbb{R}[x_1, x_2, x_3]_6.$$

The curves in this pencil have common nodes at eight real points $p_1, \ldots, p_8$ in the plane $\mathbb{P}^2$, namely the images of the nodes on S other than p and q.

Consider the vector space V of real sextic forms that are singular at $p_1, \ldots, p_8$. Since each p_i imposes 3 linear conditions, we have $\dim V \geq 28 - 3 \cdot 8 = 4$. We claim that $\dim V = 4$. To see this, consider a general curve $C_p(t)$ with $t > 0$. It has only eight real points, so as a complex curve it is irreducible and smooth outside the eight nodes. Hence the geometric genus of $C_p(t)$ is 2. Let X denote the blow-up of the plane in the points $p_1, \ldots, p_8$, and denote by C the strict transform of $C_p(t)$ on X. By Riemann-Roch, $\dim H^0(\mathcal{O}_X(C)\big|_C) = 3$, since $C^2 = 4$. Combined with the cohomology of the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)\big|_C \to 0,$$

we conclude that $\dim V = \dim H^0(X, \mathcal{O}_X(C)) \leq 4$, and hence $\dim V = 4$.

The pencil $\mathbb{R}\{k_1, k_2\}$ of real cubic forms through the eight points $p_1, \ldots, p_8$ determine a 3-dimensional subspace $U = \mathbb{R}\{k_1^2, k_1k_2, k_2^2\}$ of V, while the sextic forms G_t span a 2-dimensional subspace L of V. Since G_t has no real zeros except the nodes when $t > 0$, we see that L is not contained in U. Hence L and U intersect in a 1-dimensional subspace of V, so there exists a unique value $t_0 \in \mathbb{R}$ such that $C_p(t_0) = K_1 \cdot K_2$, where $K_1, K_2 \in \mathbb{C}[x_1, x_2, x_3]_3$.

We now have two possibilities: either K_1 and K_2 are both real, or K_1 and K_2 are complex conjugates. We claim that the latter is the case. Consider the intersection $\{G_t = 0\} \cap \{K_1 \cdot K_2 = 0\}$. This scheme is the union of a scheme of length 32 supported on the 8 nodes and a scheme Z of length $6 \cdot 6 - 4 \cdot 8 = 4$. Its defining ideal $\langle fh - g^2, hu^2 - 2guv + fv^2 \rangle$ contains the square $(gu - 2fv)^2$, and thus each of its points has even length. Hence Z is either one point of length 4 or two points of length 2. Since the general G_t does not contain the ninth intersection point of K_1 and K_2, each component of Z is contained in only one of the K_i. In particular, since $K_i \cap C_p(t)$ contains a scheme of length 2 disjoint from the points $p_1, \ldots, p_8$, this shows that Z has two points, one in each of the K_i. If both Z_i were real then $K_i \cap Z$ would be real, contradicting the fact that G_t has only 8 real points. We conclude that the two cubics K_1, K_2 are complex conjugates and their only real points are the 9 common intersection points.

We now claim that $t_0 = 0$. Indeed, if $t_0 < 0$ then S_{t_0} has 2-dimensional real components and the real points in the ramification locus $C_p(t_0)$ would have dimension 1. If $t_0 > 0$ then S_{t_0} has only 9 real points and $C_p(t_0)$ has only 8 real points. Since $C_p(t_0) = K_1 \cdot K_2$ has 9 real zeros, it follows that $t_0 = 0$. Using Jessop's Lemma 10, we now conclude that $F = F_0$ is a symmetroid.

We have shown that the semi-algebraic set $\mathcal{E}$ is contained in the symmetroid variety $\mathcal{QS}$. It remains to be proved that $\mathcal{E}$ is Zariski dense in $\mathcal{QS}$. To see this, we start with any particular extreme quartic. For instance, take the following extreme quartic due to Choi, Lam and Reznick [7, Proof of Proposition 4.13]:

$$F_b \;=\; \sum_{i,j} x_i^2 x_j^2 \;+\; b \sum_{i,j,k} x_i^2 x_j x_k \;+\; (4b^2 - 4b - 2) x_1 x_2 x_3 x_4 \quad \text{for } 1 < b < 2, \tag{8}$$

where the sums are taken over all distinct pairs and triples of indices. The complex surface defined by F_b has 10 nodes, namely, the points in $\mathcal{V}_{\mathbb{R}}(F_b)$. Our proof above shows that F_b is a symmetroid. Since the Hessian of F_b is positive definite at each of the 10 real points, we can now perturb these freely in a small open neighborhood inside the variety of 10-tuples of real points that are nodes of a symmetroid. Each corresponding quartic is real, non-negative and extreme. This adaptation of "Hilbert's method" constructs a semi-algebraic family of dimension 24 in $\mathcal{E}$. We conclude that $\mathcal{QS}$ is the Zariski closure of $\mathcal{E}$. $\square$

Our proof raises the question whether Lemma 10 can be turned into an algorithm. To be precise, given an extreme quartic, such as (8), what is a practical method for computing a complex symmetric determinantal representation (7)? We shall address this question in the second half of the next section.

5. Numerical Algebraic Geometry

We verified the results of Theorems 1 and 5 using the algorithms implemented in `Bertini` [4]. In what follows we shall explain our methodology and findings. An introduction to numerical algebraic geometry can be found in [29].

The main computational method used in `Bertini` is homotopy continuation. Given a polynomial system F with the same number of variables and equations, *basic homotopy continuation* computes a finite set $\mathcal{S}$ of complex roots of F which contains the set of isolated roots. By "computes $\mathcal{S}$" we mean a numerical approximation of each point in $\mathcal{S}$ together with an algorithm for computing each point in $\mathcal{S}$ to arbitrary accuracy. The basic idea is to consider a parameterized family $\mathcal{F}$ of polynomial systems which contains F. One first computes the isolated roots of a sufficiently general member of $\mathcal{F}$, say G, and then tracks the solution paths starting with the isolated roots of G at $t = 1$ of the homotopy

$$H(x,t) \;=\; F(x)(1-t) + tG(x).$$

The solution paths are tracked numerically using predictor-corrector methods. For enhanced numerical reliability, the adaptive step size and adaptive precision path tracking methods of [3] is used. The endpoints at $t = 0$ of these paths can be computed to arbitrary accuracy using endgames with the set of finite endpoints being the set $\mathcal{S}$. If F has finitely many roots, then $\mathcal{S}$ is the set of all roots of F. If the variety of F is not zero-dimensional, then the set of isolated roots of F is obtained from $\mathcal{S}$ using the local dimension test of [2].

Our computations to numerically verify the degrees in Theorem 1 only used basic homotopy continuation. For the $\Sigma_{3,6}$ case, we computed the intersection of the set of rank three quadrics in cubics with a random line in the space $\mathbb{P}^{27}$ of ternary sextics. In particular, for random $p, q \in \mathbb{C}[x_0, x_1, x_2]_6$, we computed the complex values of s such that there exists $f, g, h \in \mathbb{C}[x_0, x_1, x_2]_3$ with

$$fh - g^2 \;=\; p + sq.$$

We used the two degrees of freedom in the parametrization of a rank three quadric in cubics by taking the coefficient of x_0^3 in g and $x_0^2 x_1$ in f to be zero, and we dehomogenized by taking the coefficient of x_0^3 in f to be one. The resulting system $F = 0$ consists of 26 quadratic and two linear equations in 28 variables. Since the solution set of $F = 0$ is invariant under the action of negating g, we considered F as a member of the family $\mathcal{F}$ of all polynomial systems in 28 variables consisting of two linear and 26 quadratic polynomials which are invariant under this action. It is easy to verify that a general member of $\mathcal{F}$ has 2^{26} roots, which consist of 2^{25} orbits of order 2 under the action of negating g. We took the system G to be a dense linear product polynomial system [30] with random coefficients which respected this action. By tracking one path from each of the 2^{25} orbits, which took about 40 hours using 80 processors, this yielded 166400 points which correspond to 83200 distinct values of s.

The $\Sigma_{4,4}$ case of Theorem 1 was solved similarly, and the number 38475 was verified. We took advantage of the bi-homogeneous structure of the system

$$fg - hk \;=\; p + sq.$$

Numerical algebraic geometry can be used to compute all irreducible components of a complex algebraic variety. Here the methods combine the ability to compute isolated solutions with the use of random hyperplane sections. Each irreducible component V of $F = 0$ is represented by a witness set which is a triple $(F, \mathcal{L}, W)$ where $\mathcal{L}$ is a system of $\dim V$ random linear polynomials and W is the finite set consisting of the points of intersection of V with $\mathcal{L} = 0$.

Briefly, the basic approach to compute a witness set for the irreducible components of $F = 0$ of dimension k is to first compute the isolated solutions W of $F = \mathcal{L}_k = 0$ where $\mathcal{L}_k$ is a system of k random linear polynomials. The set W is then partitioned into sets, each of which corresponds to the intersection of $\mathcal{L}_k = 0$ with an irreducible component of $F = 0$ of dimension k. The cascade [28] and regenerative cascade [12] algorithms use a sequence of homotopies to compute the isolated solutions of $F = \mathcal{L}_k = 0$ for all relevant values of k.

We applied these techniques to verify the results of Theorem 5 concerning our 10×10 Hankel matrices. Our computations combined the regenerative cascade algorithm with the *numerical rank-deficiency method* of [1]. In short, if $A(x)$ is an $n \times N$ matrix with polynomial entries, consider the polynomial system

$$F_r \;=\; A(x) \cdot B \cdot \begin{bmatrix} I_{N-r} \\ \Xi \end{bmatrix}$$

where $B \in \mathbb{C}^{N \times N}$ is random, I_{N-r} is the $(N-r) \times (N-r)$ identity matrix, and Ξ is an $r \times (N-r)$ matrix of unknowns. One computes the irreducible components of $F_r = 0$ whose general fiber under the projection $(x, \Xi) \mapsto x$ is zero-dimensional. The images of these components are the components of

$$\mathcal{S}_r(A) \;=\; \{\, x \;:\; \operatorname{rank} A(x) \leq r \}.$$

The degree of such degeneracy loci is then computed using the method of [11].

The results on degree and codimension in Theorem 5 were thus verified, with the workhorse being the regenerative cascade algorithm. For instance, we ran `Bertini` for 12 hours on 80 processors to find that the variety of Hankel matrices (4) of rank ≤ 7 is indeed irreducible of dimension 21 and degree 2640.

We now shift gears and discuss the problem that arose at the end of Section 4, namely, how to compute a symmetric determinantal representation (7) for a given extremal quartic $F \in \mathcal{E} \subset \partial P_{4,4} \backslash \Sigma_{4,4}$. For a concrete example let us consider the Choi-Lam-Reznick quartic in (8) with $b = 3/2$. We found that $F_{3/2} = \det(M)/\gamma$, where $\gamma = -54874315598400(735\omega + 2201)$, with $\omega = \frac{2}{7}\sqrt{-10}$, and M is the symmetric matrix with entries

$$
\begin{array}{rcl}
m_{11} &=& (-11844\omega + 8100)x_1 + (3024\omega + 13140)x_3, \\
m_{12} &=& (7980\omega + 14820)x_3, \\
m_{13} &=& (19971\omega - 17460)x_1 + (4494\omega + 9600)x_3, \\
m_{14} &=& (-1596\omega - 26790)x_3 + (15561\omega - 6840)x_4, \\
m_{22} &=& (30324\omega - 7220)x_2 + (20216\omega + 21660)x_3, \\
m_{23} &=& (20216\omega + 21660)x_2 + (6384\omega + 27740)x_3, \\
m_{24} &=& (-20216\omega - 21660)x_2 - 39710x_3 + (7581\omega - 21660)x_4, \\
m_{33} &=& (-13230\omega + 31860)x_1 + 39710x_2 + (-28910\omega + 29910)x_3, \\
m_{34} &=& -39710x_2 + (25004\omega - 17100)x_3 + ((5187/2)\omega - 1140)x_4, \\
m_{44} &=& 39710x_2 + (-20216\omega + 37905)x_3 + (-30324\omega + 27075)x_4.
\end{array}
\tag{9}
$$

A naive approach to obtaining such representations is to extend the numerical techniques introduced for quartic curves in [22, §2]: after changing coordinates so that x_1^4 appears with coefficient 1 in F, one assumes that A_1 is the identity matrix, A_2 an unknown diagonal matrix, and A_3 and A_4 arbitrary symmetric 4×4-matrices with unknown entries. The total number of unknowns is 4+10+10 = 24, so it matches the dimension of the symmetroid variety $\mathcal{QS}$. With this, the identity (7) translates into a system of 34 polynomial equations in 24 unknowns. Solving these equations directly using `Bertini` is currently not possible. Since the system is overdetermined, `Bertini` actually uses a random subsystem which has a total degree of $3^6 4^{15}$. The randomization destroys much of the underlying structure and solving this system is currently infeasible.

In what follows, we outline a better algorithm based on the underlying geometry of the problem. The input is a 10-nodal quartic surface $S = \{F = 0\}$. After changing coordinates, so that $p = (0{:}0{:}0{:}1)$ is one of the nodes, the quartic has the form:

$$F \;=\; f x_4^2 + 2 g x_4 + h \qquad \text{where } f, g, h \in \mathbb{R}[x_1, x_2, x_3].$$

The projection from p defines a double cover $\pi : S \to \mathbb{P}^2$ and the ramification locus is the sextic curve whose defining polynomial is $fh - g^2$ and splits into a product of two complex conjugate cubic forms K_1, K_2. The intersection of S with $\{K_1 = 0\}$, regarded as a cubic cone in $\mathbb{P}^3$, is supported on the branch locus of the double cover and therefore equals two times a curve C of degree 6. The curve C has a triple point at the vertex p, its arithmetic genus is 3, and it is arithmetically Cohen-Macaulay. By the Hilbert-Burch Theorem, the ideal of C is generated by the 3×3-minors $g_1, \ldots, g_4$ of a 3×4 matrix whose entries are linear forms in $\mathbb{C}[x_1, x_2, x_3, x_4]_1$:

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \end{bmatrix} \tag{10}$$

The rows of this matrix give three linear syzygies between the four cubics g_i. Furthermore, F itself is in the ideal generated by these cubics, and so there is a linear relation $F = l_1 g_1 + \cdots + l_4 g_4$. Hence the quartic F is equal, up to multiplication by a non-zero scalar in $\mathbb{C}$, to the determinant of the matrix

$$L \quad = \quad \begin{bmatrix} l_1 & -l_2 & l_3 & -l_4 \\ l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \end{bmatrix}.$$

To find a *symmetric* matrix M with the same property, we solve the linear system $PL = (PL)^T$ for some matrix $P \in \mathrm{GL}(4, \mathbb{C})$ and define $M = PL$.

A numerical version of the above algorithm is almost exactly as explained above except that a basis for the ideal I_C of the genus 3 curve C is found by computing a large sample of points in the intersection $\{F = K_1 = 0\}$, and then computing a basis $g_1, \ldots, g_4$ for the 4-dimensional space of cubic forms vanishing on this set. Next, a basis for the 3-dimensional set of linear syzygies between these cubics is computed. This yields the matrix in (10) whose 3×3 minors are the four cubics g_i. For the quartic (8) with $b = 3/2$, we used `Bertini` to compute 100 random points in this intersection and then used standard numerical linear algebra algorithms. In total, it took 30 seconds to compute a symmetric determinantal representation for $F_{3/2}$. To four digits, with $i = \sqrt{-1}$, the output we found is the symmetric matrix M with entries

$$\begin{array}{lcl}
m_{11} & = & (15.5378 + 5.6547i)x_1 - (20.4008 - 5.8116i)x_2 - (23.1956 + 16.9236i)x_3 + (12.4987 + 26.8206i)x_4, \\
m_{12} & = & (18.3458 - 5.8125i)x_1 - (14.0867 - 25.1505i)x_2 - (35.0029 - 5.2948i)x_3 + (36.1417 + 15.7167i)x_4, \\
m_{13} & = & (11.6232 + 5.6624i)x_1 - (15.6076 - 5.9393i)x_2 - (17.3057 + 12.3685i)x_3 + (11.0079 + 22.8305i)x_4, \\
m_{14} & = & (25.7222 + 1.2098i)x_1 - (27.4233 - 22.3864i)x_2 - (45.8046 + 14.1068i)x_3 + (35.3836 + 37.8454i)x_4, \\
m_{22} & = & (12.6315 - 18.4638i)x_1 + (9.6932 + 37.6953i)x_2 - (26.0269 - 34.9909i)x_3 + (49.9098 - 16.2993i)x_4, \\
m_{23} & = & (14.6285 - 3.0705i)x_1 - (9.5983 - 20.6203i)x_2 - (25.8489 - 4.2265i)x_3 + (31.1616 + 13.0794i)x_4, \\
m_{24} & = & (24.1544 - 17.3589i)x_1 - (5.2755 - 47.6528i)x_2 - (52.3363 - 27.5281i)x_3 + (68.7313 + 6.4353i)x_4, \\
m_{33} & = & (\ 8.5030 + 5.3275i)x_1 - (11.9127 - 5.6822i)x_2 - (12.9473 + 8.9555i)x_3 + (9.6646 + 19.4288i)x_4, \\
m_{34} & = & (19.6130 + 2.9165i)x_1 - (20.0754 - 19.3371i)x_2 - (34.2042 + 9.9911i)x_3 + (30.7454 + 32.0581i)x_4, \\
m_{44} & = & (37.6831 - 10.7034i)x_1 - (27.3051 - 52.2852i)x_2 - (80.4558 - 2.6947i)x_3 + (79.5452 + 43.7001i)x_4.
\end{array} \tag{11}$$

The symbolic solution (9) and the numerical solution (11) are in the same equivalence class of symmetric matrix representations. In fact, we close with the result that the output of the algorithm is essentially unique, independant of the choice of node p and cubic form K_i:

Proposition 11. *For any* 10*-nodal symmetroid* $F \in \mathcal{QS}$*, the representation (7) is unique up to the natural action of* $\mathrm{GL}(4, \mathbb{C})$ *via* $A_i \mapsto U A_i U^T$ *for* $i = 1, 2, 3, 4$.

Proof. Let $M = \sum x_i A_i$ be a symmetric matrix such that $F = \det(M)$. Any three of the four rows of M determine a curve C by taking 3×3 minors. This gives a 4-dimensional linear system L_M of curves of arithmetic genus 3 and degree 6 on S. The doubling of any curve in L_M is the complete intersection of S and a cubic surface defined by a 3×3-symmetric submatrix of M. Conversely, the linear system determines the matrix M up to a change of basis.

Each curve in L_M passes through all the nodes of S, and these are the common zeros of the curves in L_M. If $\tilde{S}$ is the smooth K3 surface obtained by resolving the nodes, then by Riemann-Roch, L_M defines a *complete* linear system on $\tilde{S}$. Since $\mathrm{Pic}(\tilde{S})$ is torsion-free, we see that L_M is uniquely determined as the linear system of degree 6 curves on S passing through all nodes and whose doubling form a complete intersection. Therefore the equivalence class of the symmetric matrix representation is also unique. □

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References

[1] D. Bates, J. Hauenstein, C. Peterson, and A. Sommese: Numerical decomposition of the rank-deficiency set of a matrix of multivariate polynomials, *Approximate Commutative Algebra*, ed. L. Robbiano and J. Abbott, a volume of *Texts and Monographs in Symbolic Computation*, Springer, 55–77, 2009.

[2] D. Bates, J. Hauenstein, C. Peterson, and A. Sommese: A numerical local dimension test for points on the solution set of a system of polynomial equations, *SIAM J. Num. Anal.* **47** (2009) 3608–3623.

[3] D. Bates, J. Hauenstein, A. Sommese, and C. Wampler: Stepsize control for adaptive multiprecision path tracking, *Contemp. Math.* **496** (2009) 21–31.

[4] D. Bates, J. Hauenstein, A. Sommese, and C. Wampler: `Bertini`: Software for Numerical Algebraic Geometry. Available at `http://www.nd.edu/~sommese/bertini`.

[5] G. Blekherman: Non-negative polynomials and sums of squares, `arXiv:1010.3465`.

[6] E. Cattani and A. Dickenstein: Introduction to residues and resultants, in *Solving Polynomial Equations: Foundations, Algorithms, and Applications* (eds. A. Dickenstein and I.Z. Emiris), Algorithms and Computation in Mathematics **14**, Springer, 2005.

[7] M.D. Choi, T.Y. Lam and B. Reznick: Real zeros of positive semidefinite forms. I. *Mathematische Zeitschrift* **171** (1980) 1–26.

[8] J.-L. Colliot-Thélène: The Noether-Lefschetz theorem and sums of 4 squares in the rational function field $\mathbb{R}(x, y)$, *Compositio Mathematica* **86** (1993) 235–243.

[9] P. Griffiths and J. Harris: On the Noether-Lefschetz theorem and some remarks on codimension two cycles, *Mathematische Annalen* **271** (1985) 31–51.

[10] J. Harris and L. Tu: On symmetric and skew-symmetric determinantal varieties, *Topology* **23** (1984) 71–84.

[11] J. Hauenstein and A. Sommese: Witness sets of projections, *Appl. Math. Comput.* **217** (2010) 3349–3354.

[12] J. Hauenstein, A. Sommese, and C. Wampler: Regenerative cascade homotopies for solving polynomial systems, to appear in *Appl. Math. Comput.*

[13] D. Hilbert: Über die Darstellung definiter Formen als Summe von Formenquadraten, *Mathematische Annalen* **32** (1888) 342–350.

[14] C.M. Jessop: *Quartic Surfaces with Singular Points*, Cambridge University Press, 1916.

[15] V.M. Kharlamov: The maximal number of components of a fourth degree surface in $\mathbb{RP}^3$, *Functional Analysis and its Applications* **6** (1972) 345–346.

[16] M. Kontsevich and Y. Manin: Gromov-Witten classes, quantum cohomology, and enumerative geometry. Comm. Math. Phys. **164** (1994), no. 3, 525–562.

[17] S. Lefschetz: On certain numerical invariants of algebraic varieties with application to Abelian varieties, *Transactions Amer. Math. Soc.* **22** (1921) 327–482.

[18] D. Maulik and R. Pandharipande: Gromov-Witten theory and Noether-Lefschetz theory, `arXiv:0705.1653`.

[19] J. Nie: Discriminants and non-negative polynomials, `arXiv:1002.2230`.

[20] M. Noether: Zur Grundlegung der Theorie der algebraischen Raumkurven, *J. Reine Angew. Mathematik* **92** (1882) 271–318.

[21] G. Ottaviani: Symplectic bundles on the plane, secant varieties and Lüroth quartics revisited, in *Vector Bundles and Low-codimensional Subvarieties: State of the Art and Recent Developments*, 315–352, Quad. Mat., 21, Dept. Math., Seconda Univ. Napoli, Caserta, 2007.

[22] D. Plaumann, B. Sturmfels and C. Vinzant: Computing linear matrix representations of Helton-Vinnikov curves, to appear in *Mathematical Methods in Systems, Optimization and Control*, (volume dedicated to Bill Helton, edited by H. Dym, M. de Oliveira, M. Putinar), Operator Theory: Advances and Applications, Birkhäuser, Basel, 2011, `arXiv:1011.6057`.

[23] B. Reznick: On Hilbert's construction of positive polynomials, `arXiv:0707.2156`.

[24] K. Rohn: Die Maximalzahl und Anordnung der Ovale bei der ebenen Kurve 6. Ordnung und bei der Fläche 4. Ordnung, *Mathematische Annalen* **73** (1913) 177–228.

[25] P. Rostalski and B. Sturmfels: Dualities in convex algebraic geometry, *Rendiconti di Mathematica, Serie VII* **30** (2010) 285-327.

[26] B. Saint-Donat: Projective models of K3 surfaces, *Amer. J. Math.* **96** (1974) 602–639.

[27] R. Schneider: *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge University Press, 1993.

[28] A. Sommese and J. Verschelde: Numerical homotopies to compute generic points on positive dimensional algebraic sets, *J. Complexity* **16** (2000) 572–602.

[29] A. Sommese and C. Wampler: *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*, World Scientific, Singapore, (2005).

[30] J. Verschelde and R. Cools: Symbolic homotopy construction, *Appl. Algebra Engrg. Comm. Comput.* **4** (1993) 169–183.

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