polyTop: Software for computing topology of smooth real surfaces

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Abstract. A common computational problem is to compute topological information about a real surface defined by a system of polynomial equations. Our software, called `polyTop`, leverages numerical algebraic geometry computations from `Bertini` and `Bertini_real` with topological computations in `javaPlex` to compute the Euler characteristic, genus, Betti numbers, and generators of the fundamental group of a smooth real surface. Several examples are used to demonstrate this new software.

Keywords: Numerical algebraic geometry, topology, cell decomposition, graphs, Euler characteristic, Betti numbers, fundamental group

1 Introduction

Let $X \subset \mathbb{R}^N$ be a smooth, closed, and orientable surface defined by the vanishing of a system of polynomial equations. Common topological quantities of interest regarding X include the Euler characteristic, genus, Betti numbers, and generators of the fundamental group [11,12]. This paper presents an approach to compute these quantities that combines numerical algebraic geometry with computational topology, and is implemented in the new software `polyTop`.[3]

The input to `polyTop` is a cell decomposition of X, which is computed from the polynomial system f as follows. First, `Bertini` [2,3] is used to compute a numerical irreducible decomposition of the solution set of $f = 0$ over the complex numbers. From this numerical irreducible decomposition, the software `Bertini_real` [6,7] computes a cell decomposition of the real surface X.

Using the cell decomposition as input, `polyTop` computes a topologically equivalent simplicial complex that immediately yields the Euler characteristic, genus, and Betti numbers. Interfacing with the computational topology software `javaPlex` [15] yields confirmation of the Betti numbers and generators of the fundamental group.

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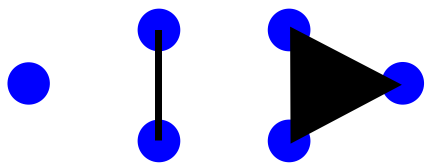
There are several alternatives to our numerical approach. One could compute topological data from numerical sampling X, e.g., [4,9,10,13]. Another approach is to utilize symbolic computations to perform similar computations for a Riemann surface arising from a complex curve [16]. For example, similar topological computations are implemented in the software package `algcurves` in `Maple`.

The remainder is organized as follows. In Section 2, a method to move from a cell decomposition computed by `Bertini_real` to a topologically equivalent simplicial complex is presented. Section 3 explains the use of MATLAB and `javaPlex` in order to compute the Euler characteristic, genus, Betti numbers, and generators of the fundamental group. We demonstrate the software with various examples in Section 4 and conclude in Section 5.

2 Cell decomposition and simplicial complex

For a smooth, closed, and orientable surface $X \subset \mathbb{R}^N$, we compute a simplicial complex $S(X)$ that is topologically equivalent to X. In our case, the simplicial complex $S(X)$ is a set composed of 0-, 1-, and 2-simplices, called vertices, edges, and faces, respectively and visually represented in Figure 1. The key aspect is that such a topologically equivalent simplicial complex $S(X)$ for X can be constructed from a cell decomposition of X computed by `Bertini_real`.

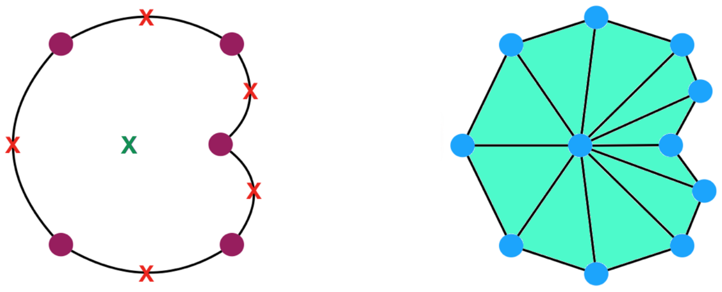


(a) (b) (c)

Fig. 1. A visual representation of (a) a 0-simplex or vertex, (b) a 1-simplex or edge, and (c) a 2-simplex or face.

Following [1,5], the real surface X can be decomposed into a finite union of cells that mirror a simplicial complex. Each 2-cell, called a face, of X is a subset of X that has a generic interior point and a boundary consisting of 1-cells. Each 1-cell, called an edge, of X is a subset of X that has a generic interior point and a 0-cell (vertex) at each end. Figure 2(a) provides an illustration of a 2-cell.

The software `polyTop` constructs a topologically equivalent simplicial complex $S(X)$ of X by looping over each cell of the cell decomposition and constructing a corresponding simplicial complex as follows. The vertices of the simplicial complex consist of the generic interior point of the 2-cell, each generic interior point of the 1-cells, and the vertices at the end of each 1-cell. The edges of the simplicial complex consist of “interior” edges connecting the generic interior point of the 2-cell with each vertex on the boundary of the 2-cell and “boundary” edges connecting the generic interior point of each 1-cell with its vertices. The faces consist of the generic interior point of the 2-cell and two vertices connected by a "boundary" edge. This construction is illustrated in Figure 2(b). Naturally connecting the simplicial complexes along neighboring cells of the decomposition yields a simplicial complex $S(X)$ that is topologically equivalent to X.



(a) (b)

Fig. 2. A visual representation of a 2-cell and corresponding simplicial complex.

Let V, E, and F denote the number of vertices, edges, and faces, respectively, of the simplicial complex $S(X)$. The Euler characteristic χ of X is computed from $S(X)$ via

$$\chi = V - E + F. \tag{1}$$

Suppose that X is connected. Then, the Euler characteristic χ and genus g are related via

$$\chi = 2 - 2g \tag{2}$$

and the Betti numbers of X are $\beta_0 = \beta_2 = 1$ and $\beta_1 = 2g$. A basis for the fundamental group $\pi_1(X)$, which consists of $2g$ loops, is computed by `javaPlex` from the simplicial complex $S(X)$.

In order to test for connectivity, we take the transitive closure A^+ of the adjacency matrix A of edges of $S(X)$. Since $S(X)$ has V vertices, then A is an $V \times V$ symmetric matrix where $A_{ij} = A_{ji}$ is 1 if there is an edge in $S(X)$ between vertex i and vertex j and 0 otherwise. The transitive closure A^+ of A describes which vertices are connected, i.e., A^+_{ij} is 1 if there is a path connecting vertex i and vertex j and 0 otherwise. The transitive closure can be computed using Boolean matrix multiplication and addition via the following:

$$A^+ = A + A^2 + A^3 + \cdots + A^V. \tag{3}$$

In particular, X is connected if and only if every entry of A^+ is 1. If X is not connected, A^+ can be used to decompose X into connected components.

3 Software

The software polyTop is written in MATLAB to utilize the preexisting MATLAB interfaces of both Bertini_real and javaPlex.

Given a polynomial system f, one first uses Bertini to compute a numerical irreducible decomposition which is then used by Bertini_real to compute a cell decomposition. The data for the cell decomposition is loaded into MATLAB utilizing via gather_br_samples from Bertini_real which creates a file called BRinfo#.mat which can be used by all of the other MATLAB functions in the Bertini_real interface. For example, one can plot the surface using this file via the command bertini_real_plotter within MATLAB.

After using the command load_javaplex to load the javaPlex library and separately loading the cell decomposition data in MATLAB, polyTop can be executed. The first task of polyTop is to organize the cell decomposition data to create a topologically equivalent simplicial complex using the method described in Section 2.

Next, a stream is created in javaPlex that organizes the simplicial complex data for use in topological computations within javaPlex. Vertices are added using the command `stream.addVertex(i,0)`. Edges between vertices a and b are added via the command `stream.addElement([a, b])` while faces consisting of vertices a, b, and c are added via `stream.addElement([a, b, c])`.

Finally, a call to javaPlex performs homology computations on the simplex stream. The homology is computed with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

4 Examples

The following summarizes several computations using polyTop. The input is computed via a numerical irreducible decomposition using Bertini followed by a cell decomposition using Bertini_real. A topologically equivalent simplicial complex is then constructed yielding the Euler characteristic, genus, and Betti numbers. The software javaPlex is then used for confirming the Betti numbers and generators of the fundamental group. The following timings are based on using an 2.4 GHz Intel Core i5 processor: the sphere and torus examples ran in under 0.1 seconds while the tanglecube and Crixxi examples completed in under 15 seconds.

4.1 Sphere

The unit sphere $X \subset \mathbb{R}^3$, defined by $x^2 + y^2 + z^2 = 1$, is a simply connected real surface. That is, the fundamental group of X is trivial, the Euler characteristic is $\chi = 2$, genus is $g = 0$, and the Betti numbers are $\beta_0 = \beta_2 = 1$ with $\beta_1 = 0$. A topologically equivalent simplicial complex derived from a cell decomposition computed by Bertini_real is shown in Figure 3 consisting of $V = 6$ vertices, $E = 12$ edges, and $F = 8$ faces in agreement with (1).

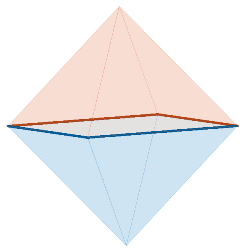


Fig. 3. Topologically equivalent simplicial complex for the unit sphere.

4.2 Torus

For an illustrative real surface with a nontrivial fundamental group, we consider the torus $X \subset \mathbb{R}^3$ defined by

$$(x^2 + y^2 + z^2 + 15/4)^2 - 16(x^2 + y^2) = 0.$$

Figure 4(a) shows a topologically equivalent simplicial complex derived from a cell decomposition computed by `Bertini_real`. In particular, there is a single hole which is commensurate with the fact that the genus is $g = 1$, Euler characteristic is $\chi = 0$, and the Betti numbers are $\beta_0 = \beta_2 = 1$ with $\beta_1 = 2$. This simplicial complex consists of $V = 32$ vertices, $E = 96$ edges, and $F = 64$ faces in agreement with (1).

Interfacing with `javaPlex` using this simplicial complex yields representatives for the two generators of the fundamental group. The output is

$$\begin{aligned}&\text{Dimension : } 1\\&[0.0, \text{infinity}) : [1, 14] + [2, 9] + [2, 14] + [1, 9]\\&[0.0, \text{infinity}) : [3, 18] + [3, 17] + [1, 17] + [1, 18]\end{aligned}$$

In this notation, an edge connecting vertices v and w is represented by $[v, w]$ and a loop is a sum of edges. Hence, this shows that each of the two generating loops consists of 4 edges which we can equivalently write as

$$1 \to 14 \to 2 \to 9 \to 1 \quad \text{and} \quad 3 \to 18 \to 1 \to 17 \to 3$$

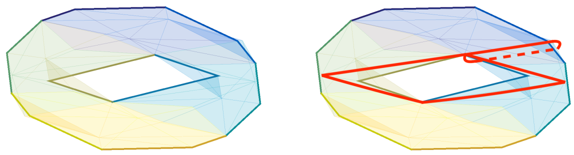
and are visually represented in Figure 4(b).

4.3 Tanglecube

The tanglecube is a degree four surface in $\mathbb{R}^3$ of genus $g = 5$ defined by

$$x^4 - 5x^2 + y^4 - 5y^2 + z^4 - 5z^2 + 11.8 = 0.$$

This surface, shown in Figure 5(a), was used, for example, in [14] to demonstrate creating a meshing of the volume inside of the tanglecube surface using

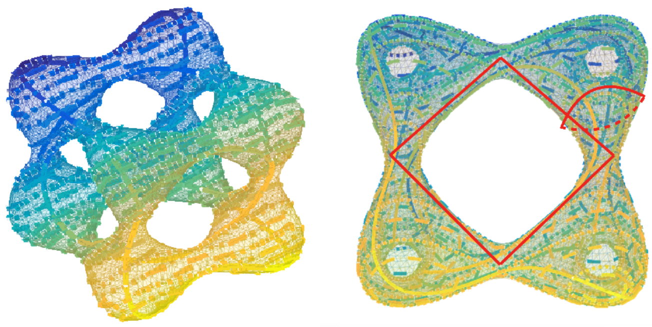


(a) (b)

Fig. 4. (a) Simplicial complex for a torus with (b) visualizing two generating loops of the fundamental group.

approximately 50,000 vertices and 200,000 tetrahedra. Using a cell decomposition computed by `Bertini_real`, a topologically equivalent simplicial complex of just the tanglecube surface consists of $V = 296$ vertices, $E = 912$ edges, and $F = 608$ faces. This confirms that the surface has genus $g = 5$ with Euler characteristic $\chi = -8$ via (1) and (2).

Passing the simplicial complex to `javaPlex` confirms that the Betti numbers are $\beta_0 = \beta_2 = 1$ with $\beta_1 = 10$ and computes ten loops that generate the fundamental group. Figure 5(b) shows two representatives of these ten loops.



(a) (b)

Fig. 5. (a) Simplicial complex for the tanglecube with (b) visualizing two representative loops of the ten generating loops of the fundamental group.

4.4 Crixxi

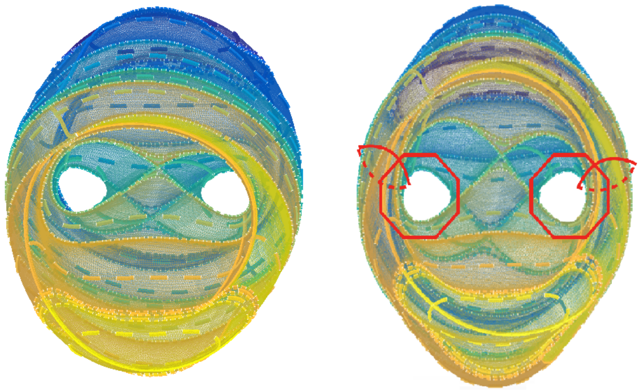
The Crixxi surface defined by

$$\left(\frac{1}{25}x^2 + \frac{1}{25}y^2 - 1\right)^3 + \left(\frac{1}{25}y^2 + \frac{1}{25}z^2 - 1\right)^2 = 0$$

is singular. By perturbing the right-hand side, say by replacing 0 with 1/10, the real surface becomes smooth and orientable [8, pg. 110] as shown in Figure 6(a).

This visualization suggests that the genus is $g = 2$ so that the Euler characteristic is $\chi = -2$ and Betti numbers are $\beta_0 = \beta_2 = 1$ with $\beta_1 = 4$. This is confirmed by (1) and (2) after computing a topologically equivalent simplicial complex from a cell decomposition computed by `Bertini_real` having $V = 346$ vertices, $E = 1044$ edges, and $F = 696$ faces.

Passing the simplicial complex to `javaPlex` yields a confirmation of the Betti numbers above and computes four loops that generate the fundamental group. Two of the loops consist of eight edges while the other two loops consist of 32 and 40 edges. A visualization of these four loops is shown in Figure 6(b).



(a) (b)

Fig. 6. (a) Simplicial complex for the perturbed Crixxi surface with (b) visualizing four generating loops of the fundamental group.

5 Conclusion

Computing topological information about a real surface defined by a system of polynomial equations is a recurrent problem within computational algebraic geometry. Using numerical algebraic geometry computations from `Bertini` and `Bertini_real` and topological computations in `javaPlex`, `polyTop` computes the Euler characteristic, genus, Betti numbers, and generators of the fundamental group of a real surface.

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