Robust Numerical Algebraic Geometry

Emma R. Cobian, Jonathan D. Hauenstein, and Charles W. Wampler

University of Notre Dame

Abstract

The field of numerical algebraic geometry consists of algorithms for numerically solving systems of polynomial equations. When the system is exact, such as having polynomials with rational coefficients, the solution set is well-defined. However, for a member of a parameterized family of polynomial systems where the parameter values may be measured with imprecision or arise from prior numerical computations, uncertainty may emerge in the structure of the solution set, including the number of isolated solutions, the existence of higher dimensional solution components, and the number of irreducible components along with their multiplicities. The loci where these structures change form a stratification of exceptional algebraic sets in the space of parameters. We describe methodologies for making the interpretation of numerical results more robust by searching for nearby parameter values on an exceptional set. We demonstrate these techniques on several illustrative examples and then treat several more substantial problems arising from the kinematics of mechanisms and robots.

1 Introduction

Numerical algebraic geometry concerns algorithms for numerically solving systems of polynomial equations, primarily based on homotopy methods, often also referred to as polynomial continuation. Reference texts for numerical algebraic geometry are [6, 45], and software packages that implement its algorithms are available in [5, 10, 26, 49]. Built on a foundation of methods for finding all isolated solutions, the field has grown to include algorithms for computing the irreducible decomposition of algebraic sets along with operations such as membership testing, intersection, and projection. The basic construction of the field is a *witness set*, say $\mathcal{W}$, in which a pure D-dimensional algebraic set, say $X \subset \mathbb{C}^n$, is represented by a structure having three members:

$$\mathcal{W} = \{f, L, W\} \tag{1}$$

where $f : \mathbb{C}^n \to \mathbb{C}^k$ is a polynomial system such that X is a D-dimensional component of the algebraic set $V(f) = \{x \in \mathbb{C}^n \mid f(x) = 0\}$, $L : \mathbb{C}^n \to \mathbb{C}^D$ is a *slicing system* of D generic linear polynomials, and $W = X \cap V(L)$ is a *witness point set* for X. Given a witness set, one can sample the set it represents by moving its slicing system in a homotopy. Given witness sets for two components, one can compute their intersection, obtaining witness sets for the components of the intersection [21]. A witness set can be decomposed into its irreducible components using monodromy [42] to group together points on the same irreducible component and using a trace test [9, 11, 27, 43] to verify when this process is complete. Algorithms built on these and related techniques compute a numerical irreducible decomposition of $V(f)$, producing a collection of witness sets, one for each irreducible component. In a similar fashion, one may construct pseudo-witness sets for projections of algebraic sets [18], which is the geometric equivalent of symbolic elimination. Thus, using algorithms for intersection, union, projection, and membership testing, one can represent and manipulate constructible algebraic sets.

Applications of algebraic geometry, such as kinematics, computer vision, and chemical equilibrium, often involve parameterized families of polynomial systems of the form $f(x;p) : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^k$, where x is an array of variables and p is an array of parameters. For example, in kinematic analysis, x may be variables describing the relative displacement at joints between parts while p may describe the length of links or the axis of a rotational joint. In kinematic synthesis, where one seeks to find a mechanism to produce a desired motion, these roles may be interchanged. The long history of research in kinematics and its applications to mechanisms and robotics is extensive; [38] provides a useful overview. In multi-view computer vision, the variables of a scene reconstruction describe the location of objects and cameras in three-dimensional space while the parameters are the coordinates of features observed in the camera images [1, 22, 29]. In chemical equilibrium, concentrations are variables and reaction rates are parameters [33, Chap. 9] and [13, 37]. In short, in a single instance of the family, variables are the unknowns while parameters are given, and the parameter space defines a family of problems having the same polynomial structure. In the simplest case, the parameters are merely the coefficients of polynomials with a fixed set of monomials, while in many applications the coefficients are often polynomial functions of the parameters. In general, one can consider a parameter space that is an irreducible algebraic set or, with minor additional conditions, even a complex analytic set [35]. However, for simplicity, we will assume here that the parameter space is the complex Euclidean space $\mathbb{C}^m$.

In this article, we consider the algorithms of numerical algebraic geometry which compute using floating point arithmetic, so function evaluations and solution points are consequently inexact. Furthermore, in applications, parameters may be values measured with imprecision or they may be numerical values produced in prior stages of computation. In light of these uncertainties, the results reported by the numerical algorithms may require interpretation. For example, a coordinate of a solution point computed as near zero may indicate that there is an exact solution nearby with that coordinate exactly zero, but this is not assured. Moreover, if the polynomial system is given with uncertain parameters or with coefficients presented as floating point values, there is some uncertainty about which problem is being posed. The interpretation of the solution set, such as classifying how many endpoints of a homotopy have converged to finite solution points versus how many have diverged to infinity, is subject to judgements about round-off errors from inexactly evaluating inexactly specified polynomials. Similar judgements must be made when categorizing singular versus nonsingular solutions, sorting solutions versus nonsolutions when using randomization, concluding that a witness set is complete via a trace test, or counting multiplicities according to how many homotopy paths converge to (nearly) the same point. Our aim in this article is to lay out methodologies for making these interpretations more robust yielding *robust numerical algebraic geometry*.

In a parameterized family, the irreducible decomposition has the same structure for generic parameters; that is, the sets of parameters where the structure changes, collectively called the *discriminant variety*, lie on proper algebraic subsets of the parameter space. By structure, we mean features over the complex numbers described by integers, such as the number of irreducible components, their dimensions and degrees, and the multiplicity of the witness points. Within any irreducible algebraic subset of parameter space, there may exist algebraic subsets of lower dimension where these integer features change again, creating a stratification of algebraic sets of successively greater specialization. In an early discussion of this phenomenon, Kahan [24] called such sets *pejorative manifolds*, but we prefer the more neutral term *exceptional sets*. Typically, an analyst would like to be able to detect when the problem they have posed is close to an exceptional set. Moreover, since such sets have zero measure in their containing parameter space, the fact that a posed problem falls quite close to an exceptional set may indicate that this is not a coincidence, and in fact, the exceptional case is the true item of interest. Additionally, constraining the problem to lie on an exceptional set can convert an ill-conditioned problem into a well-conditioned one [24].

Factoring a single multivariate polynomial presents a special case of the phenomena we presently address. Wu and Zeng [52] note that factorization of such a polynomial is ill-posed when coefficient perturbations are considered since the factors change discontinuously as the coefficients approach a factorization submanifold. Their answer to this problem is to define a metric for the distance between polynomials and a partial ordering on the factorization structures. This partial ordering corresponds to algebraic set inclusion in the stratification of factorization structures, and in the parlance of Wu and Zeng, each such set is said to be more singular than any set that contains it. (Roughly, a polynomial with more factors is more singular than one with fewer, and for the same degree, a factor appearing with multiplicity is more singular that a product of distinct factors.) Wu and Zeng regularize the numerical factorization problem by requiring the user to provide an uncertainty ball around the given polynomial. Among all the possible factorization manifolds that intersect the uncertainty ball, they define the numerical factorization to be the nearest polynomial on the exceptional set of highest singularity. As the radius of the uncertainty ball grows, the numerical factorization may change to one of higher singularity. So, while the problem remains ill-posed at critical radii, it has become well-posed everywhere else. In the case of a single polynomial of moderate degree, the number of possible specializations is small enough that one can enumerate them all. This gives the potential of finding the unique numerical factorization for most uncertainty radii, and a finite list of alternatives for any range of radii, functionalities provided by Wu and Zeng's software package.

Since parameterized families arise often in applications, there have been several methods proposed for studying the solutions of parameterized systems. For example, [39] describes a parametric geometric resolution for solving parameterized square systems, i.e., when $n = k$. Symbolic approaches for computing discriminant varieties for paramertized systems are considered in [25, 36]. Since the enumeration of all possible structures requires recursively computing discriminant varieties of discriminant varieties, this is a a formidable task that we do not attempt here. Nonetheless, an approach to compute such an enumeration for the special case of exceptional dimension was described in [46] via fiber products.

In the process of computing a numerical irreducible decomposition, there typically will be only a few places where the uncertainty in judging how to classify points warrants exploration of the alternatives. The same can be said for more limited objectives, such as computing only the isolated solutions of a system by homotopy, where one may question the number of solution paths deemed to have landed at infinity or observe a cluster of path endpoints that might indicate a single solution point of higher multiplicity. If the uncertainty is due solely to floating point round-off and we have access to either a symbolic description or a refinable numerical description of the parameter values, then the correct judgement can be made with high confidence by increasing the precision of the computation. Such results might then be certified either by symbolic computations or by verifiable numerics, such as Smale's alpha theory [40] (see also [8, Ch. 8]) or by techniques from interval analysis [31]. Our present concern is for cases where there is inherent uncertainty in the parameters due to empirical measurements of the parameters or, perhaps, the parameters arise from prior computations in finite precision. Therefore, the novel topic that we consider here is the following:

> Assuming that a questionable structural element has been identified, determine the nearest point in parameter space where the special structure occurs. (2)

Our contributions are the two-stage robust framework described in Section 3 which first uses fiber products as in Proposition 3.1 and then a local optimization approach, and subsequent corollaries. Remark 3.3 describes stacking up fiber product systems to form polynomial systems whose solution sets correspond to a target structure. This robust framework is then specialized to consider solutions at infinity (Theorem 4.1), solutions lying along a common linear space (Theorem 5.1), factorization of components (Theorem 6.1), and solutions with prescribed multiplicity and local Hilbert function (Theorem 7.1). In particular, our factorization of components uses a novel methodology built from the second derivative trace test [9] and applies to components of any dimension while [52] only considers hypersurfaces.

The rest of the paper is structured as follows. After describing some robustness scenarios in Section 2, Section 3 provides a framework for robustness in numerical algebraic geometry. This framework is then applied to a variety of structures: fewer finite solutions in Section 4, existence of higher-dimensional components in Section 5, components that further decompose into irreducible components in Section 6, and solution sets of higher multiplicity in Section 7. We formulate the algebraic conditions implied by each type of structure and use local optimization techniques to find a nearby set of parameters satisfying them. After treating each type of specialized structure and illustrating on a small example, in Section 8, we show the effectiveness of the approach on three more substantial problems coming from the kinematics of mechanisms and robots. A short conclusion is provided in Section 9. Files for the examples, which are all computed using `Bertini` [5], are available at https://doi.org/10.7274/25328878.

2 Robustness scenarios

Before presenting our approach to robustifying numerical algebraic geometry, we discuss scenarios where proximity to an exceptional set leads to ill-conditioning.

To abbreviate the discussion, we use the phrase "with probability one" as shorthand for the more precise and stronger condition that exceptions are a proper algebraic subset of the relevant parameter space, where the parameter space in question should be clear from context. Similarly, a point in parameter space is "generic" if it lies in the dense Zariski-open set that is the complement of the set of exceptions. For example, the reference to "a system of D generic linear polynomials," $L : \mathbb{C}^n \to \mathbb{C}^D$, just after (1) means a system of the form $Ax + b$ where matrix $[A\ \ b]$ has been chosen from the dense Zariski-open subset of $\mathbb{C}^{D\times(n+1)}$ where $V(L)$ intersects X transversely. In numerical work, we make the assumption that a random number generator suffices for choosing generic points.

Our discussion uses the concept of a *fiber product* [46]. For algebraic sets A and B with algebraic maps $\pi_A : A \to Y$ and $\pi_B : B \to Y$, the fiber product of A with B over Y is

$$A \times_Y B = \{(a, b) \in A \times B \mid \pi_A(a) = \pi_B(b)\}. \tag{3}$$

One may similarly form fiber products between three or more algebraic sets. In this article, the maps involved in forming fiber products will all be natural projections of the form $(x, p) \mapsto p$. Moreover, for polynomial systems $f, g : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^k$, if A and B in $\mathbb{C}^n \times \mathbb{C}^m$ are irreducible components of $V(f(x, p))$ and $V(g(x, p))$, respectively, then the fiber product of A with B over $\mathbb{C}^m$ is an algebraic set in $V(f(x_1, p_1), g(x_2, p_2), p_1 - p_2)$, a so-called reduction to the diagonal, which is isomorphic to an algebraic set in $V(f(x_1, p), g(x_2, p)) \subset \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^m$. In this situation, we refer to $\{f(x_1, p), g(x_2, p)\}$ as a fiber product system. We note the convention used throughout is that $f(x, p)$ means that both x and p are considered as variables, which is in contrast to $f(x; p)$ where x are variables and p are parameters.

2.1 Multiplicity example

As a simple first example, consider solving $V(f)$ for the single polynomial $f = x^2 + 2\sqrt{2}x + 2$, which has the factorization $(x + \sqrt{2})^2$. Hence, $V(f)$ is the single point $x = -\sqrt{2}$ which has multiplicity 2 with respect to f. If instead of f, we are given an eight-digit version of it, say

$$\tilde{f}_8 = x^2 + 2.8284271x + 2,$$

the Matlab `roots` command, operating in double precision, returns the two roots

$$x_8 = -1.414213550000000 \pm 0.000187073241389\mathrm{i}$$

where $\mathrm{i} = \sqrt{-1}$. Using these roots as initial guesses for Newton's method, the solutions of the sixteen-digit version of f,

$$\tilde{f}_{16} = x^2 + 2.828427124746190x + 2,$$

are computed in double precision as

$$x_{16} = -1.414213553589213 \pm 0.000000183520060\mathrm{i}.$$

One may work the problem in increasingly higher precision by considering a sequence of approximations a_ℓ of $2\sqrt{2}$ rounded off to ℓ digits. For every ℓ, $\tilde{f}_\ell = x^2 + a_\ell x + 2$ has a pair of roots in the vicinity of $-\sqrt{2}$. A numerical package with adjustable precision can refine $\tilde{f}_\ell$ and a solution x_ℓ until $|x_\ell + \sqrt{2}|$ is smaller than any positive error tolerance one might set.

Whether a computer program using this refinement process reports two isolated roots or one double root depends on settings for its precision and tolerance. If the same program is only given $\tilde{f}_8$ or $\tilde{f}_{16}$, the roots stay distinct no matter what precision is used for Newton's method.

We can stabilize this situation by asking if there exists a nearby polynomial with a double root. That is, we ask if there is a polynomial near to $\tilde{f}_8$ of the form $\hat{f}(x;p) = x^2 + px + 2$ with a root that also satisfies the derivative $\hat{f}'(x;p) = 2x + p$. Solving $V(\hat{f}(x,p), \hat{f}'(x,p))$ in $\mathbb{C}^2$ with Newton's method and an initial guess taking $x = x_8$ and $p = a_8$, returns

$$(\hat{x}, \hat{p}) = (-1.414213562373095, 2.828427124746190),$$

where the imaginary parts have converged to zero within machine epsilon. Moreover, the Jacobian matrix for this structured system is clearly nonsingular:

$$J\begin{bmatrix} \hat{f} \\ \hat{f}' \end{bmatrix} = \begin{bmatrix} 2x+p & x \\ 2 & 1 \end{bmatrix} \approx \begin{bmatrix} 0 & -1.414213562373095 \\ 2 & 1 \end{bmatrix}.$$

The good conditioning of this double-root problem not only leads to a quickly convergent, accurate answer, but also it could be used to certify the answer via alpha theory [40] or interval analysis [31]. The acceptance of the double root as the "correct" answer depends on whether $\hat{p}$ is within the tolerance of the given data. After all, if the user really wants the roots of $\tilde{f}_8$ as given, then x_8 is the better answer. However, if the coefficient on x is acknowledged to be only known to eight digits, then the double root $\hat{x}$ is the preferred answer.

Suppose that we reformulate such that the parameter space has two entries, say

$$\tilde{f}(x;p) = x^2 + p_1 x + p_2,$$

then we may search for the system with a double root nearest to $\tilde{f}_8$ as

$$\min \|p - (2.8284271, 2)\| \text{ subject to } (\tilde{f}(x,p), \tilde{f}'(x,p)) = 0.$$

Using the Euclidean norm, the global optimum is attained at the nonsingular point

$$(x, p_1, p_2) = (-1.414213558248730,\ 2.828427116497461,\ 1.999999988334534),$$

which again shows that there is a polynomial $\tilde{f}$ near the given polynomial $\tilde{f}_8$ such that $\tilde{f}$ has a double root. This illustrates the flavor of the approach put forward in [52], although they also treat multivariate polynomials, which have a richer set of factorization structures than just multiplicity.

One contribution of the present paper is to treat exceptional factorization and exceptional multiplicity for systems of polynomials while [52] only considers the hypersurface case.

2.2 Divergent solutions

In numerical algebraic geometry, one of the most common objectives is to find the isolated solutions of a "square" polynomial system, say $f(x;p) : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$, at a given parameter point, say $p = p^*$. (Here, *square* means the number of equations equals the number of variables.) A standard result of the field states that the number of isolated solutions is constant for all p in a nonempty open Zariski set in $\mathbb{C}^m$. In other words, the exceptions are a proper algebraic subset of $\mathbb{C}^m$, say P^*. A key technique in the field uses this fact to build homotopies for finding all isolated points. In particular, if we have all isolated solutions, say $S_1 \subset \mathbb{C}^n$, of start system $f(x; p_1)$ for a generic set of parameters, p_1, then the homotopy $f(x; \phi(t)) = 0$ for a general enough continuous path $\phi(t) : [0, 1] \to \mathbb{C}^m$ with $\phi(1) = p_1$ and $\phi(0) = p_0$ defines $\#S_1$ continuous paths whose limits as $t \to 0$ include all isolated solutions of a target system $f(x; p_0)$, e.g., see [35] or [45, Theorem 7.1.1]. The conditions for a "general enough" path are very mild; in fact, $\phi(t) = tp_1+(1-t)p_0$ suffices with probability one when p_1 is chosen randomly, independent of p_0. Many *ab initio* homotopies, which solve a system from scratch, fit into this mold. For example, polyhedral homotopies are formed by considering the family of systems having the same monomials as the target system, so that the parameter space consists of the coefficients of the monomials [23, 28, 50]. After solving $f(x; p_1)$ for generic p_1 by such a technique, one may proceed to solve any target system in the family by parameter homotopy. If the target parameters are special, i.e., if $p_0 \in P^*$, then $f(x; p_0)$ has fewer isolated solutions than $f(x; p_1)$ meaning that some solution paths of the homotopy either diverge to infinity or some of the endpoints lie on a positive-dimensional solution component. Diverging to infinity can be handled by homogenizing f and working on a projective space [32]. Thus, paths that originally diverged to infinity are transformed to converge to a point with homogeneous coordinate equal to zero.

When one executes a parameter homotopy using floating point arithmetic, the computed homogeneous coordinate of a divergent path is typically not exactly zero but rather some complex number near zero. This also occurs when p_0 is slightly perturbed off of the special set P^*. Usually, one does not know the conditions that define the algebraic set P^*, but even so, with a solution near infinity in hand, one may wonder how far p_0 is from P^*. Suppose that due to round-off or measurement error, p_0 is uncertain. Then, it could be of high interest to know whether the closest point of P^* is within the uncertainty ball centered on p_0.

It often happens that more than one endpoint of a homotopy falls near infinity. In such cases, it is of interest to find a nearby point in parameter space where all those points land on infinity simultaneously. Let us assume that $f(x; p)$ has been homogenized so that x has homogeneous coordinates $[x_0, x_1, \ldots, x_n] \in \mathbb{P}^n$, with $x_0 = 0$ being the hyperplane at infinity. To consider $j > 1$ points simultaneously, we must introduce a double subscript notation, where point $x_i \in \mathbb{P}^n$ has coordinates $x_i = [x_{i0}, \ldots, x_{in}]$. Point x_i is a solution at infinity if it satisfies the augmented system $F(x_i; p) = \{f(x_i; p), x_{i0}\} = 0$, so sending $j > 1$ points to infinity simultaneously requires

$$\{F(x_1; p), \ldots, F(x_j; p)\} = 0, \tag{4}$$

which is the j^{th} fiber product system for the projection $(x, p) \mapsto p$ [46]. We note that the isolated solutions to $f(x; p)$ are not necessarily independent in the sense that imposing (4) for j points may result in forcing more than j endpoints to lie at infinity.

2.3 Emergent solutions

When a parameterized system has more equations than unknowns, $f(x; p) : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^k$ with $k > n$, there may exist exceptional sets where the number of isolated solutions increases.

A familiar example is a linear system $Ax = b$ where full-rank matrix A has more rows than columns. For most choices of b in Euclidean space, the system is incompatible and has no solutions. However, when b lies in the column space of A, there will be a unique solution. In the more general nonlinear case, one method for finding all isolated solutions is to replace f with a "square" randomization $R_n f : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ wherein each of the n polynomials of system $R_n f$ is a random linear combination of the polynomials in f. Theorem 13.5.1 (item (2)) of [45] implies that, with probability one, the isolated points in $V(R_n f)$ include all the isolated points in $V(f)$. After solving $R_n f$ by homotopy, one sorts solutions vs. nonsolutions by evaluating f at each solution of $R_n f$. If $V(R_n f)$ contains nonsolutions for generic parameters $p \in \mathbb{C}^m$, then there may exist an exceptional set $P^* \subset \mathbb{C}^m$ where one or more of these satisfy f to become solutions. Bertini's Theorem [45, Thm A.8.7] tells us that the nonsolutions will be nonsingular with probability one. However, if they emerge as singular solutions of $V(f)$ as $p \to P^*$, then they will be ill-conditioned near P^*. Even if the nonsolutions remain well-conditioned as solutions of $R_n f$, meaning that the Jacobian matrix with respect to the variables $J(R_n f) = R_n \cdot Jf$ is far from singular, the problem of solving f for parameters in the vicinity of P^* is ill-conditioned from the standpoint that the number of solutions changes discontinuously as we approach P^*. The numerical difficulty arises in deciding whether or not $f(x;p) = 0$ when the floating point evaluation of f is near, but not exactly, zero. For nonsingular emergent solutions, sensitivity analysis, e.g., singular value decomposition, of the full Jacobian matrix with respect to both the variables and the parameters can estimate the distance in parameter space from the given parameters to P^*, while alpha theory or interval analysis can provide provable bounds. In the case of singular emergent solutions, multiplicity conditions will have to be imposed as well (see below). In any case, the simultaneous emergence of j solutions requires them to satisfy the j^{th} fiber product system $\{f(x_1;p), \ldots, f(x_j;p)\}$.

2.4 Sets of exceptional dimension

Polynomial systems often have solution sets of positive dimension. This happens by force if there are fewer equations than unknowns, but it can happen more generally as well. Moreover, a polynomial system can have irreducible solution components at several different dimensions. For $x \in V(f)$, the *local dimension* at x, denoted $\dim_x V(f)$, is the highest dimension of all the irreducible solution components containing x. For a parameterized system with the natural projection $\pi(x,p) = p$, the fiber above $p^* \in \mathbb{C}^m$ is $V(f(x;p^*))$ and the fiber dimension at point $(x^*, p^*) \in V(f(x,p))$ is $\dim_{x^*} V(f(x;p^*))$. Define $\mathcal{D}_h$ as the closure of the set $\{(x,p) \in \mathbb{C}^N \times \mathbb{C}^m \mid \dim_x V(f(x;p)) = h\}$, which is an algebraic set. A parameterized polynomial system has a set of exceptional dimension wherever $\mathcal{D}_H$ intersects $\mathcal{D}_h$ for $H > h$, that is, exceptions occur at parameter values $p^* \in \mathbb{C}^m$ where the fiber dimension increases. The sets $\pi(\mathcal{D}_h)$ form a stratification of parameter space with each containment progressing to higher and higher fiber dimension. Since the structure of the solution set changes every time there is a change in dimension, each such change is another example of ill-conditioning. As presented in [46] and discussed below in Section 5, fiber products provide a way of finding sets of exceptional dimension.

In numerical algebraic geometry, sets of exceptional dimension can be understood as a case of emergent solutions. Holding p constant, a witness point set for a pure D-dimensional component of $V(f(x;p))$ is found by intersecting with a codimension D generic affine linear space, $L_D(x)$. For $D > n-k$, this is accomplished by first computing the isolated solutions of the randomized system $\{R_{n-D}f(x;p), L_D(x)\}$, where $R_\ell f$ denotes ℓ generic linear combinations of the polynomials in f and $L_D(x)$ is a system of D generic affine linear equations. When p^* is on a set of exceptional dimension, nonsolutions emerge as solutions to f as $p \to p^*$. For a degree d irreducible component to emerge, d new witness points must emerge simultaneously, which leads to a fiber product formulation of the same general form as (4), with F now defined as $F(x;p) = \{R_{n-D}f(x;p), L_D(x)\}$. While the d witness points of a degree d component must satisfy the d^{th} fiber product, it may happen that imposing the j^{th} fiber product for $j < d$ suffices. In particular, a different bound based on counting dimensions often comes into play first [46].

2.5 Exceptional decomposition

Once one finds a witness set $\mathcal{W} = \{f, L_D, W\}$ for a pure D-dimensional component X of $V(f)$, it is often of interest to decompose X into its irreducible components, which are the closure of the connected components of X after removing its singularities, $X \setminus X_{\text{sing}}$. For a single polynomial, irreducible components correspond exactly with factors, so irreducible decomposition is the generalization of factorization to systems of polynomials. Ill-conditioning occurs near a point in parameter space where a component decomposes into more irreducible components than general points in the neighborhood. Again, we get a stratification of parameter space where components decompose more and more finely.

Every pure-dimensional algebraic set satisfies a linear trace condition, and irreducible components correspond with the smallest subsets of a witness point set W that satisfy the trace test [9, 11, 27, 43]. Ill-conditioning occurs when the trace for a proper subset $W_1 \subsetneq W$ evaluates to approximately zero. We may then ask if there is a parameter point nearby where the trace is exactly zero, indicating that X decomposes, with W_1 representing a component having lower degree. (The number of points in a witness set is equal to the degree of the algebraic set it represents.) As presented in Section 6, since the trace test involves all the points W_1 simultaneously, a new kind of fiber product system ensues.

2.6 Exceptional multiplicity

Our opening example of a single polynomial with a double root generalizes to systems of polynomials. As we approach a subset of parameter space where witness points merge, the components they represent coincide, forming a component of higher multiplicity. When we speak of the multiplicity of an irreducible component, we mean the multiplicity of its witness points cut out by a generic slice. However, when randomization is used to find witness points of $V(f)$, $f(x) : \mathbb{C}^n \to \mathbb{C}^k$ at dimensions $D > n-k$, the multiplicity of a witness point as a solution of $\{R_{n-D}f(x), L_D(x)\}$ may be greater than its multiplicity as a solution of $\{f, L_D(x)\}$, with equality only guaranteed for either multiplicity 1 (i.e., for nonsingular points) or when the multiplicity with respect to the randomized system is 2 [45, Theorem 13.5.1]. Section 7 discusses in more detail how multiplicity is defined in terms of Macaulay matrices and local Hilbert functions. For the moment, it suffices to say that the Macaulay matrix evaluated at a generic point of an irreducible algebraic set reveals the set's local Hilbert function and multiplicity, and provides an algebraic condition for it. As such, we again get a stratification of parameter space associated with changing the local Hilbert function and increasing the multiplicity.

Since every generic point of an irreducible algebraic set has the same multiplicity, the conditions necessary to postulate the multiplicity of a component may be asserted for several witness points simultaneously. As in the previous cases, asserting an algebraic condition for several points simultaneously is a form of fiber product.

2.7 Summary

Each case discussed above—divergent solutions, emergent solutions, exceptional dimension, exceptional decomposition, and exceptional multiplicity—can lead to a kind of ill-conditioning wherein small changes in parameters result in a discrete change in an integer property of the solution set. Given a parameterized polynomial system $f(x;p)$ along with parameters near such a discontinuity, one may consider variations in the parameters and pose the question of finding the nearest point in parameter space where the exceptional condition occurs. In each case, imposing the exceptional condition on several points simultaneously results in a fiber product system. In particular, when the exceptional condition applies to an irreducible component of degree $d > 1$, it automatically applies to a set of d witness points, and consequently, fiber products are key to robustifying numerical irreducible decomposition.

3 Robustness framework

The following provides a two-stage framework for robust numerical algebraic geometry to address topic (2). The first stage is formulating a polynomial system which, as described in Section 2, results from a *fiber product* [46]. The key part of this construction is to create a polynomial system where the identified structural element of interest corresponds with a union of irreducible components of the fiber product system. The second stage is to solve a local optimization problem to recover a point in parameter space having the special structure.

The following is derived directly from [46] which describes the promotion of exceptional sets with respect to dimension to be irreducible components of a fiber product system.

Proposition 3.1. *Let $g(x;p) : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^k$ be a polynomial system, $\pi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ be the projection map $(x,p) \mapsto p$, $X = V(g(x,p)) \subset \mathbb{C}^n \times \mathbb{C}^m$, and $Y = \overline{\pi(X)} \subset \mathbb{C}^m$. Suppose that $Z \subset \mathbb{C}^n \times \mathbb{C}^m$ is a nonempty irreducible algebraic set contained in X. For $a > 0$, let*

$$Z_Y^a = \{(x_1, \ldots, x_a, p) \mid (x_i, p) \in Z\} \quad \text{and} \quad \mathcal{G}_a(x_1, \ldots, x_a, p) = \begin{bmatrix} g(x_1, p) \\ \vdots \\ g(x_a, p) \end{bmatrix}.$$

Suppose that there exists $h \geq 0$ and a Zariski open dense set $U \subset Z$ such that

1. *for every $(u,q) \in U$, $\dim(Z \cap \pi^{-1}(q)) = h$ and*
2. *for some $(w,s) \in U$, there exists an open set $V \subset X$ in the complex topology that contains (w,s) such that, for every $(u',q') \in V \setminus Z$, $\dim(X \cap \pi^{-1}(q')) < h$.*

Then, there exists $a_Z \in \mathbb{Z}$ with $1 \leq a_Z \leq \dim Y \leq m$ such that, for every $a \geq a_Z$, there is an irreducible component $\mathcal{Z}_a$ of $V(\mathcal{G}_a)$ contained in Z_Y^a with $\overline{\pi_a(\mathcal{Z}_a)} = \overline{\pi(Z)}$ where $\pi_a(x_1, \ldots, x_a, p) = p$.

Proof. This is a combination of Lemma 2.8 and Corollary 2.15 of [46] where $\mathcal{Z}_a$ is called the *main component* of Z_Y^a. In fact, a simple dimension counting argument from Items 1 and 2 shows that

$$\dim \mathcal{Z}_a = a \cdot h + \dim \overline{\pi(Z)}$$

and the maximum dimension of an irreducible component of $V(\mathcal{G}_a)$ which could contain $\mathcal{Z}_a$ is

$$a \cdot (h-1) + \dim Y.$$

Hence, when $a \geq \dim Y$, $\dim \mathcal{Z}_a \geq a \cdot (h-1) + \dim Y$, which shows that $\mathcal{Z}_a$ must be an irreducible component of $V(\mathcal{G}_a)$. □

The number a is called the *order* of the fiber product $\mathcal{G}_a$. As the order increases, Proposition 3.1 shows that exceptional sets are promoted to irreducibility in the parlance of [46].

We need a fiber product framework which is flexible enough to apply to all the cases highlighted in Section 2. The approach we take is to allow for auxiliary variables. After constructing the fiber product, one can then simplify the system using appropriate slicing and randomization to construct a well-constrained polynomial system. This approach is demonstrated in the following.

Example 3.2. *Consider finding the values of $p \in \mathbb{C}^2$ such that $f(x;p) = p_1x + p_2$ has a one-dimensional solution set. Of course, $\tilde{p} = (0,0)$ is the only such point.*

By the standard approach of fiber products in [46], Proposition 3.1 applies to $Z = \mathbb{C} \times \{\tilde{p}\}$, $X = V(f(x,p))$, and $Y = \mathbb{C}^2$. Since X is irreducible, one needs $a = 2 = \dim Y$ with

$$\mathcal{G}_2(x_1, x_2, p) = \begin{bmatrix} f(x_1, p) \\ f(x_2, p) \end{bmatrix} = \begin{bmatrix} p_1x_1 + p_2 \\ p_1x_2 + p_2 \end{bmatrix}.$$

Since $Z_Y^2 = \mathbb{C} \times \mathbb{C} \times \{\tilde{p}\}$ is irreducible, Proposition 3.1 yields that Z_Y^2 is an irreducible component of $V(\mathcal{G}_2)$. The other irreducible component of $V(\mathcal{G}_2)$ is $\{(x, x, p) \mid f(x;p) = 0\}$.

For parameter values where the solution set is one-dimensional, any general line will intersect $V(f)$ in an isolated witness point. This condition can be formulated by introducing auxiliary variables $u \in \mathbb{C}^2$ as the coefficients of a linear polynomial in x, so that $f(x;p)$ is replaced with $F(x,u;p)$, where

$$F(x,u;p) = \begin{bmatrix} p_1x + p_2 \\ u_1x + u_2 \end{bmatrix}.$$

In particular, for general $p \in \mathbb{C}^2$, $V(F(x,u;p))$ defines a curve while $V(F(x,u;\tilde{p}))$ is a surface in which the closure of the projection onto $u \in \mathbb{C}^2$ is all of $\mathbb{C}^2$. Hence, one can impose 2 general linear conditions on each $u \in \mathbb{C}^2$, i.e., simply instantiate each instance of u

to be a general point in $\mathbb{C}^2$*. To that end, applying Proposition 3.1 to* $Z = V(F(x,u;\tilde{p})) \times \{\tilde{p}\}$*,* $X = V(f(x,p))$*, and* $Y = \mathbb{C}^2$ *with* $a = 2$ *yields*

$$\mathcal{F}_2(x_1,u_1,x_2,u_2,p) = \begin{bmatrix} F(x_1,u_1,p) \\ F(x_2,u_2,p) \end{bmatrix} = \begin{bmatrix} p_1x_1 + p_2 \\ u_{11}x_1 + u_{12} \\ p_1x_2 + p_2 \\ u_{21}x_2 + u_{22} \end{bmatrix}.$$

After instantiating each u_j *with general* $c_j \in \mathbb{C}^2$*, this becomes the following system in* x_1, x_2, p*:*

$$\begin{bmatrix} p_1x_1 + p_2 \\ c_{11}x_1 + c_{12} \\ p_1x_2 + p_2 \\ c_{21}x_2 + c_{22} \end{bmatrix}.$$

This is a well-constrained system whose solution set is a single point, namely $(x_1, x_2, p) = (-c_{12}/c_{11}, -c_{22}/c_{21}, \tilde{p})$*. One can test that* $V(f(x;\tilde{p})) = \mathbb{C}$ *as requested.*

The particular form of the auxiliary variables and corresponding conditions are dependent on the exceptional condition of interest. Specific formulations for various situations are presented in Sections 4–7.

Remark 3.3. *Proposition 3.1 is formulated based a single exceptional condition associated with a corresponding polynomial system. However, one can also study cases where multiple exceptional conditions are imposed simultaneously. In such cases, Propostion 3.1 can be used sequentially to construct a fiber product system associated with the first exceptional condition, append a fiber product system with the second exceptional condition, and so on. One may view this as a compound fiber product system and the number of fiber products needed for subsequent conditions will be based on the relationship to the previously imposed conditions. An example of this is presented in Section 8.3.*

Remark 3.4. *Proposition 3.1 yields that* m *fiber products, which is the number of parameters, always suffice, but fewer may also work. In practice, the minimal number* a_Z *is often not known* a priori. *If the corresponding irreducible component has multiplicity one with respect to the corresponding fiber product system, Lemma 3 of [18] provides a local linear algebra approach to compute the dimension of the image from such a smooth point on the component, i.e., a point that projects to* $\tilde{p}$*. Moreover, from an approximation of a point that projects to* $\tilde{p}$*, one can use a numerical rank revealing method such as the singular value decomposition to determine the dimension. When all else fails, one could use a guess-and-check method starting at* $a = 1$ *to determine if the fiber product system had an irreducible component with the desired properties. If not, then increment* a *and try again.*

Once one has a properly constructed fiber product system $\mathcal{F}(z,p) = \mathcal{F}_a(z_1,\ldots,z_a,p)$ for an appropriate value of a with each z_j consisting of the original variables x_j and potentially auxiliary variables y_j, the second stage of the robustness framework is to recover a parameter value p^* that has the identified exceptional condition. In particular, we assume that we are given $\hat{p}$ that is close to the set of exceptional parameters, and we wish to find an exceptional point p^* nearby $\hat{p}$. Here, one must choose a notion of "nearby," such as the standard Euclidean distance or an alternative based on knowledge about uncertainty in $\hat{p}$. Over the complex numbers, one may use isotropic coordinates [51] so that the square of the Euclidean distance corresponds with a bilinear polynomial. To keep notation simple, we write this as the local optimization problem

$$p^* = \arg\min \|p - \hat{p}\| \text{ such that } \mathcal{F}(z,p) = 0. \tag{5}$$

In order to obtain local convergence conditions, e.g., see [7, Chap. 2], for the constrained optimization problem (5), one may need to replace $\mathcal{F}$ will a well-constrained randomization. Although such a randomization may introduce new solutions as summarized in Section 2.3, there is still an open neighborhood around $\tilde{p}$ where any $\hat{p}$ in this neighborhood will still result in the corresponding optimal p^* being exceptional. Of course, the size of such neighborhood depends upon the system $\mathcal{F}$, the point $\tilde{p}$, and the optimization method used.

Although there are many local optimization methods and distance metrics, all examples below use the square of the standard Euclidean distance with a gradient descent homotopy [14]. In such cases, $\mathcal{F}$ is constructed to be a well-constrained system and we aim to compute a nearby critical point of (5) using a homogenized version of Lagrange multipliers:

$$\mathcal{G}(z,p,\lambda) = \left[\begin{array}{c} \mathcal{F}(z,p) \\ \lambda_0 \nabla(\|p-\hat{p}\|_2^2) + \sum_{j=1}^M \lambda_j \nabla(\mathcal{F}_j) \end{array} \right]$$

where $\nabla(q)$ is the gradient of q and $\lambda \in \mathbb{P}^M$. If $\hat{z}$ such that $\mathcal{F}(\hat{z},\hat{p}) \approx 0$, then the gradient descent homotopy is simply

$$\mathcal{H}(z,p,\lambda,t) = \left[\begin{array}{c} \mathcal{F}(z,p) - t\mathcal{F}(\hat{z},\hat{p}) \\ \lambda_0 \nabla(\|p-\hat{p}\|_2^2) + \sum_{j=1}^M \lambda_j \nabla(\mathcal{F}_j) \end{array} \right] \tag{6}$$

where the starting point at $t = 1$ is $(\hat{z}, \hat{p}, [1, 0 \ldots, 0])$. Note that such a gradient descent homotopy is local in that it may not work in all cases, particularly when the perturbation is "too large" or a "nearby" component did not actually exist with the given formulation. Here, "too large" and "nearby" are taken with respect to the convergence basin of the gradient descent homotopy. In such cases, one may need to consider alternative formulations, e.g., isotropic coordinates, as well as consider alternative local optimization methods.

4 Projective space and solutions at infinity

Let us begin by applying this robust framework to computing parameter values for which the number of finite solutions of a system decreases.

4.1 Solutions at infinity

For a parameterized polynomial system $f(x;p)$, one can consider solutions at infinity by considering a homogenization (or multihomogenization) of f with variables in projective space (or product of projective spaces). Thus, solutions at infinity correspond with a homogenizing variable being equal to 0. For simplicity, suppose that we have replaced f with a homogenized version which generically has no solutions at infinity. In an application of Proposition 3.1, suppose that we want to reduce the number of finite solutions by forcing solutions to be inside of the hyperplane at infinity defined by $x_0 = 0$. This yields the following.

Theorem 4.1. *Let $f(x;p)$ be a polynomial system which is homogeneous in $x = (x_0, \ldots, x_n)$ such that, for general $p \in \mathbb{C}^m$, $V(f(x;p), x_0) \subset \mathbb{P}^n$ is empty. Let Z^H be a nonempty irreducible component of*

$$\{(x,p) \in \mathbb{P}^n \times \mathbb{C}^m \mid f(x;p) = 0, x_0 = 0\} \tag{7}$$

such that, for general $(y,q) \in Z^H$, $y \in \mathbb{P}^n$ is the unique solution to $f(x;q) = 0$ and $x_0 = 0$. Then, Proposition 3.1 holds when applied to

$$F(x,p) = \begin{bmatrix} f(x,p) \\ x_0 \end{bmatrix}$$

where $x \in \mathbb{C}^{n+1}$ and $p \in \mathbb{C}^m$, and Z is the affine cone over Z^H. In particular, this component system has no auxiliary variables.

Proof. By considering the affine cone over $\mathbb{P}^n$, one has $h = 1$. From the assumption on Z^H, there is a Zariski open dense set $U \subset Z$ such that Item 1 of Proposition 3.1 holds. Moreover, for a general $(w,s) \in U \subset \mathbb{C}^{n+1} \times \mathbb{C}^m$, $([w], s) \in Z^H \subset \mathbb{P}^n \times \mathbb{C}^m$ is not contained in any other irreducible component of (7). Hence, Item 2 of Proposition 3.1 holds when fixing such $(w,s) \in U$ and taking the open set V containing (w,s) to be small enough to not intersect the affine cone over the union of the other irreducible components of (7). □

Remark 4.2. *The condition that $f(x;p) = 0$ generically has no solutions at infinity helps to simplify the proof. Symbolically, this can be accomplished by saturating with respect to x_0. Since such a computation could be challenging, numerically this corresponds with ignoring every irreducible component of $V(f(x;p))$ which maps dominantly onto the parameter space consisting of solutions inside of $x_0 = 0$.*

For perturbed parameter values $\hat{p}$, one is looking for solutions to $f(x;\hat{p}) = 0$ for which a homogenizing coordinate is close to 0. Remark 3.3 applies when aiming to have more than one solution at infinity. If there are s such points, then the order of the fiber product required to send all of them to infinity simultaneously is at most s but could be strictly smaller than s due to relations amongst the solutions. Moreover, in the multiprojective setting where variables are partitioned into several blocks associated to a cross product of projective spaces, Remark 3.3 would also apply to having solutions in different hyperplanes at infinity corresponding to the various blocks, e.g., see Section 8.3. In all cases, after fiber product construction, one can use general affine patches and appropriate slicing and randomization to construct a well-constrained polynomial system.

4.2 Illustrative example

Consider the parameterized family of polynomial systems

$$f(x;p) = \begin{bmatrix} x_1^2 + p_1 x_1 + p_2 \\ (x_1 + p_3)x_2 + 2x_1 - 3 \end{bmatrix}. \tag{8}$$

For generic $p \in \mathbb{C}^3$, $f(x;p) = 0$ has two finite solutions. However, for the exact parameter values $\tilde{p} = (-2.3716, 0.98608803, -0.5377) \in \mathbb{Q}^3$, f has only one finite solution. The reason for this reduction is that, for these exact parameter values, one of the two roots of the first polynomial happens to be $x_1 = -\tilde{p}_3$, at which value the second polynomial evaluates to $0x_2 - 2\tilde{p}_3 - 3 \neq 0$. To demonstrate the robustness framework, we consider starting with a perturbation of $\tilde{p}$, say $\hat{p} = (-2.3728, 0.9607, -0.5349)$ to 4 decimal places. Solving $f(x;\hat{p}) = 0$ yields two finite solutions in $\mathbb{C}^2$ where one solution has large magnitude. Therefore, we aim to recover p^* near $\hat{p}$ with one finite solution by pushing the large magnitude solution to infinity.

The first step is to create a homogenization of f in (8). Using a single homogenizing coordinate, say x_0, this yields

$$f(x;p) = \begin{bmatrix} x_1^2 + p_1 x_0 x_1 + p_2 x_0^2 \\ x_1 x_2 + 2x_0 x_1 + p_3 x_0 x_2 - 3x_0^2 \end{bmatrix}. \tag{9}$$

Viewing $x \in \mathbb{P}^2$, $[0,0,1]$ is always a solution to $f(x;p) = 0$ and is thus ignored as in Remark 4.2. For $\hat{p}$, numerical approximations of the solutions on a randomly selected affine patch are shown in Table 1. The first solution listed corresponds to $[0,0,1]$, so it is not considered further. The second solution listed has x_0 near 0, and therefore we aim to adjust the parameters so that it also lies on the hyperplane at infinity defined by $x_0 = 0$.

Since there is a single solution to push to infinity, Theorem 4.1 along with the addition of a general affine patch yields

$$\mathcal{F} = \begin{bmatrix} x_1^2 + p_1 x_0 x_1 + p_2 x_0^2 \\ x_1 x_2 + 2x_0 x_1 + p_3 x_0 x_2 - 3x_0^2 \\ x_0 \\ c_0 x_0 + c_1 x_1 + c_2 x_2 - 1 \end{bmatrix} \tag{10}$$

for randomly selected $c \in \mathbb{C}^3$. For illustration purposes, one can easily verify that the closure of the image of the projection onto $p \in \mathbb{C}^3$ of the component of $V(\mathcal{F})$ near solution 2 of Table 1 is $V(p_3^2 - p_1 p_3 + p_2)$ which contains $\tilde{p}$. Such a defining equation can be determined using symbolic computation, e.g., via Grobner bases, or exactness recovery methods from numerical values, e.g., [2].

With the first stage of the robustness framework complete, we now move to the second stage involving optimization. The critical point system constructed using homogenized Lagrange multipliers yields

$$\mathcal{G} = \begin{bmatrix} x_1^2 + p_1 x_0 x_1 + p_2 x_0^2 \\ x_1 x_2 + 2x_0 x_1 + p_3 x_0 x_2 - 3x_0^2 \\ x_0 \\ c_0 x_0 + c_1 x_1 + c_2 x_2 - 1 \\ \lambda_1(p_1 x_1 + 2p_2 x_0) + \lambda_2(2x_1 + p_3 x_2 - 6x_0) + \lambda_3 + \lambda_4 c_0 \\ \lambda_1(2x_1 + p_1 x_0) + \lambda_2(x_2 + 2x_0) + \lambda_4 c_1 \\ \lambda_2(x_1 + p_3 x_0) + \lambda_4 c_2 \\ \lambda_0(p_1 - \hat{p}_1) + \lambda_1 x_0 x_1 \\ \lambda_0(p_2 - \hat{p}_2) + \lambda_1 x_0^2 \\ \lambda_0(p_3 - \hat{p}_3) + \lambda_2 x_0 x_2 \end{bmatrix}. \tag{11}$$

Taking the second solution in Table 1 as $\hat{x}$, a gradient descent homotopy (6) recovers a nearby parameter value having the desired structure of only one finite solution, which is provided in Table 2 to 8 decimal places. The exceptional set is two-dimensional, so we do not expect to recover $\tilde{p}$ exactly, just a point p^* nearby consistent with the size of the perturbation.

Since the solution at infinity is singular, Remark 3.4 does not apply for computing dimensions using linear algebra. However, if we instead use a 2-homogeneous formulation, the solution at infinite becomes nonsingular and Remark 3.4 applies. In particular, using two homogenizing coordinates, say x_0 and x_3, and considering solutions at infinity corresponding with $x_3 = 0$, this yields

$$f(x;p) = \begin{bmatrix} x_1^2 + p_1 x_1 x_0 + p_2 x_0^2 \\ x_1 x_2 + 2x_1 x_3 + p_3 x_0 x_2 - 3x_0 x_3 \end{bmatrix} \quad \text{and} \quad \mathcal{F} = \begin{bmatrix} f(x;p) \\ x_3 \\ c_0 x_0 + c_1 x_1 - 1 \\ c_2 x_2 + c_3 x_3 - 1 \end{bmatrix}. \tag{12}$$

for a randomly selected $c \in \mathbb{C}^4$ after adding a general affine patch for each set of variables. A corresponding gradient descent homotopy returns the same results as in Table 2.

5 Witness points and randomization

The next structure to consider for applying this robust framework to is computing parameter values which have solution components of various dimensions and degrees.

5.1 Witness points

As described in Section 2, pure-dimensional solution components can be described by witness sets. A key decision in numerical algebraic geometry, such as part of a dimension-by-dimension algorithm for computing a numerical irreducible decomposition, e.g., [19, 41, 44], is to determine if a floating-point approximation of solution x to a randomized system $Rf(x;p)$ corresponds to an actual solution to the original system $f(x;p)$. For a system whose parameters are given exactly (e.g., rational numbers or radicals of rational numbers), one can refine the approximation of x (typically using Newton's method) and evaluate $f(x;p)$ to higher and higher precision to lower the uncertainty in the decision. If the point is a nonsolution, meaning $Rf(x;p) = 0$ but $f(x;p) \neq 0$, that fact can be certified [20].

In cases where p is only known approximately as $\hat{p}$, and we have an approximate solution $\hat{x}$ to $Rf(x;\hat{p})$, using higher precision to refine $\hat{x}$ as a solution of $Rf(x;\hat{p})$ and then to evaluate $f(\hat{x},\hat{p})$ may not result in a residual that converges to 0. Instead, we must ask whether there exists a set of parameters p^* near to $\hat{p}$ where $f(x;p^*)$ has a root x^* near to $\hat{x}$.

Suppose that we want to $V(f(x;p))$ to have a solution component of dimension D and degree d. Let $\mathrm{Gr}_k(n)$ be the Grassmannian of k-dimensional linear spaces in $\mathbb{C}^n$. For $\ell \in \mathrm{Gr}_k(n)$, let L_ℓ be a system of k linear polynomials such that $V(L_\ell) = \ell$. This yields the following, which is itself a fiber product to impose the degree requirement.

Theorem 5.1. *Suppose that $f(x;p)$ is a polynomial system and Z is a nonempty irreducible component of the set of $(x_1,\ldots,x_d,\ell,p) \in (\mathbb{C}^n)^d \times \mathrm{Gr}_{n-D}(n) \times \mathbb{C}^m$ such that $\{x_1,\ldots,x_d\}$ are d*

distinct points in $V(f(x;p))\cap\ell$ in which, for general $(y_1,\ldots,y_d,b,q)\in Z$, y_j is nonsingular with respect to $\{f(x;q),L_b(x)\}$. Then, Proposition 3.1 holds when applied to

$$F(x_1,\ldots,x_d,\ell,p)=\begin{bmatrix} f(x_1,p)\\ L_\ell(x_1)\\ \vdots\\ f(x_d,p)\\ L_\ell(x_d)\end{bmatrix}.$$

Proof. One has $h=\dim \mathrm{Gr}_{n-D}(n)=D(n-D)$. From the assumption on Z, there is a Zariski open dense set $U\subset Z$ such that Item 1 of Proposition 3.1 holds. Select $(w_1,\ldots,w_d,b,s)\in U$ which is not contained in any other irreducible component of the set of $(x_1,\ldots,x_d,\ell,p)$ in $(\mathbb{C}^n)^d\times \mathrm{Gr}_{n-D}(n)\times\mathbb{C}^m$ such that $\{x_1,\ldots,x_d\}$ are d distinct points in $V(f(x;q))\cap\ell$. Hence, Item 1 of Proposition 3.1 holds when V is an open set containing $(w_1,\ldots,w_d,b,s)$ that is small enough to not intersect the union of the other irreducible components. □

Rather than only considering a single dimension, Remark 3.3 applies to append the conditions on the parameters to ensure that components in various dimensions exist simultaneously starting with the largest dimension. The value of D and d in the definition of F changes accordingly at each invocation. To enforce the factorization into two components requires additional conditions. This is the subject of Section 6 below. Additionally, if one aims to impose that some irreducible components have multiplicity greater than 1, then F must be modified to enforce such extra conditions. This is discussed below in Section 7.

After fiber product construction, one can use appropriate slicing and randomization to construct a well-constrained polynomial system. For example, the Grassmannians can be instantiated as randomly selected elements of $\mathrm{Gr}_{n-D}(n)$.

5.2 Illustrative example

Consider the parameterized family of polynomial systems

$$f(x;p)=\begin{bmatrix} x_1x_2-2x_1+p_1x_2+p_2\\ x_1^2-2x_1+p_1x_1+p_2\end{bmatrix}. \tag{13}$$

For generic $p\in\mathbb{C}^2$, $f(x;p)=0$ consists of two isolated solutions. However, for $\tilde{p}=(1,-2)$, the irreducible decomposition consists of the line $V(x_1+1)$ and the point $(2,2)$ as shown in Fig. 1. Perturbing the parameters produced $\hat{p}=(0.9876,-2.2542)$ to 4 decimal places with the corresponding two isolated solutions also shown in Fig. 1.

When considering witness points on a one-dimensional component of $V(f(x;p))$, the system under consideration has the form

$$f_R(x;p)=\begin{bmatrix} Rf(x;p)\\ L(x)\end{bmatrix}=\begin{bmatrix} x_1x_2-2x_1+p_1x_2+p_2+\Box(x_1^2-2x_1+p_1x_1+p_2)\\ \Box x_1+\Box x_2+\Box\end{bmatrix} \tag{14}$$

where each $\Box$ represents an independent random complex number with $R=[1\ \Box]$. Solving $f_R(x;\hat{p})$, there is one solution for which $f(x;\hat{p})$ is far from vanishing (called a nonsolution)

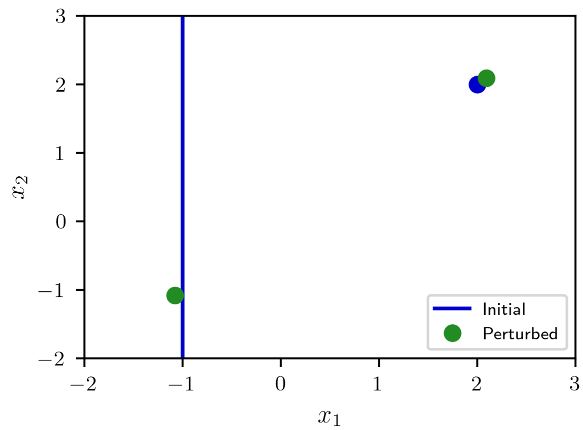


Figure 1: Solution sets for the initial and perturbed parameters

and one solution, call it $\hat{x}$, where the first coordinate is in the vicinity of -1 and $f(\hat{x};\hat{p})$ is close to vanishing. Thus, we aim to recover p^* near $\hat{p}$ for which this later point is an actual witness point for a one-dimensional line. With this, the fiber product system from Theorem 5.1 after instantiating the Grassmannian randomly is

$$\mathcal{F} = \begin{bmatrix} x_1x_2 - 2x_1 + p_1x_2 + p_2 \\ x_1^2 - 2x_1 + p_1x_1 + p_2 \\ c_1x_1 + c_2x_2 + c_3 \end{bmatrix} \tag{15}$$

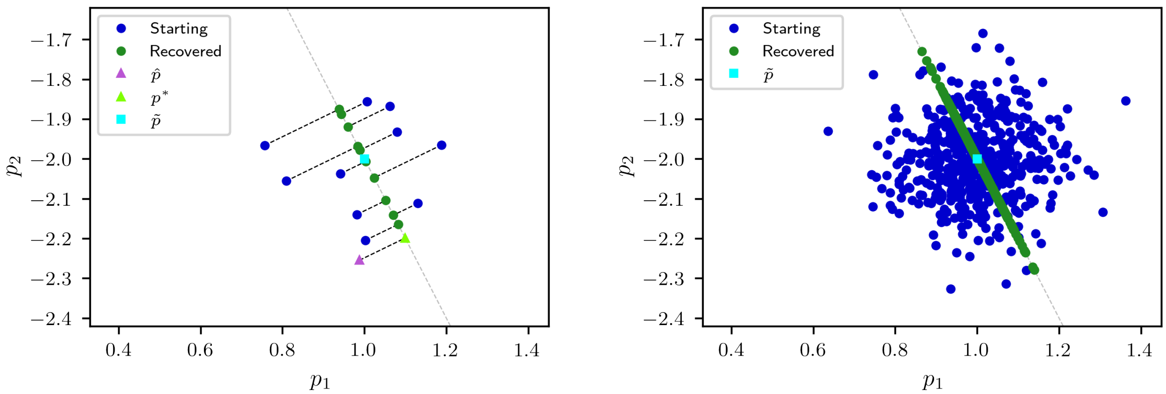
for randomly selected $c \in \mathbb{C}^3$. The critical point system is

$$\mathcal{G} = \begin{bmatrix} x_1x_2 - 2x_1 + p_1x_2 + p_2 \\ x_1^2 - 2x_1 + p_1x_1 + p_2 \\ c_1x_1 + c_2x_2 + c_3 \\ \lambda_1(x_2 - 2) + \lambda_2(2x_1 + p_1 - 2) + \lambda_3c_1 \\ \lambda_1(x_1 + p_1) + \lambda_3c_2 \\ \lambda_0(p_1 - \hat{p}_1) + \lambda_1x_2 + \lambda_2x_1 \\ \lambda_0(p_2 - \hat{p}_2) + \lambda_1 + \lambda_2 \end{bmatrix}. \tag{16}$$

With $(\hat{x}, \hat{p})$, tracking a single path with a gradient descent homotopy (6) recovers a nearby parameter p^* listed in Table 3 to four decimal places. Recomputing a numerical irreducible decomposition for $f(x;p^*)$ yields a line and an isolated point as requested.

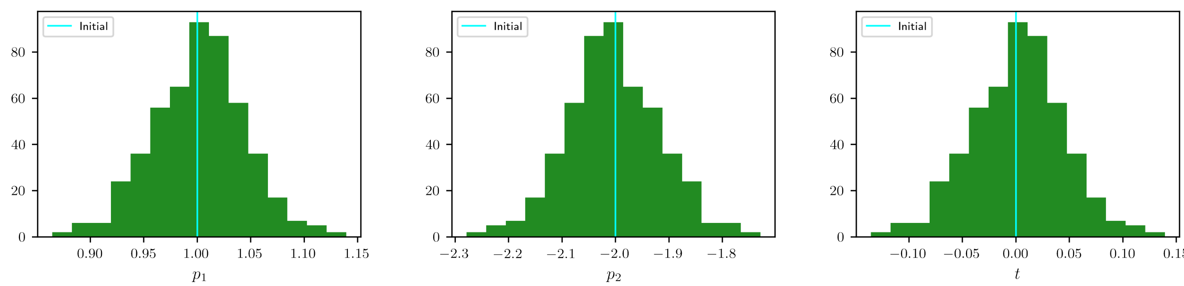
To show that this procedure is robust to arbitrary perturbations, we repeated it on a sample of 500 points from a bivariate Gaussian distribution centered at the initial parameter values $\tilde{p} = (1, -2)$ with covariance matrix $\Sigma = 0.1^2 I_2$ where each sample represents parameter values with error. In Fig. 2, the aforementioned $\tilde{p}$ is shown as a square, and $\hat{p}$ and p^* are triangles, while the additional sampled values and recovered parameters are shown as circles. For this simple problem, it is easy to recover the defining equation for the corresponding exceptional set, namely they all lie along the line $V(2p_1 + p_2)$.

To visualize marginal histograms of the recovered parameter values from the 500 samples, Fig. 3 shows the p_1 and p_2 coordinates along with an intrinsic coordinate parameterizing the line with 0 corresponding to $\tilde{p} = (1, -2)$.



(a) (b)

Figure 2: (a) Illustration of recovering parameters for various perturbations including the example summarized in Table 3; (b) Illustration using 500 samples



(a) (b) (c)

Figure 3: Histograms for (a) p_1, (b) p_2, and (c) intrinsic parameterizing coordinate for recovered parameter values from 500 samples

For a standard multivariate Gaussian, all marginals are Gaussian. So, if we orthogonally project a multivariate Gaussian centered at $\tilde{p}$ onto a linear space passing through $\tilde{p}$, this will yield a Gaussian distribution in the linear space. In the case presented here, the perturbations from the initial parameters are both generated as zero-mean with standard deviation 0.1, so the recovered parameters along the line should be centered on the initial parameters with that same standard deviation. Figure 3(c) is consistent with that expectation. Moreover, orthogonally projecting the distribution of the perturbed parameters onto the line perpendicular to $V(2p_1+p_2)$ will also be distributed as Gaussian with standard deviation 0.1. If one were given just the perturbed parameters and their accuracy, described as a statistical distribution, one could calculate a confidence in the null hypothesis that the given parameters are drawn from a distribution centered on an initial value for which $V(f(x;\tilde{p}))$ has one component that is a line. In this case of a single remaining degree of freedom in parameter space, a Z-score for $||\hat{p}-p^*||$ would be informative.

We will not delve into statistical analyses for more general cases. Nevertheless, we remark that if the exceptional set in parameter space is codimension s and the incoming parameters are perturbed from the exceptional set with a normal distribution $\mathcal{N}(0,\sigma^2 I)$, then the squared distance $\sigma^{-2}||\hat{p}-p^*||^2$ is a chi-squared distribution with s degrees of freedom. (This assumes that the exceptional set is locally smooth and σ is small enough that a local linearization of the exceptional set is accurate on the scale of σ.) If the perturbations have a more general normal distribution, say $\mathcal{N}(0,\Sigma)$, then it would be appropriate to change the norm used in (5) to $(p-\hat{p})^T\Sigma^{-1}(p-\hat{p})$ so that we are searching for a maximum likelihood estimate. The same norm would then enter into a chi-square confidence estimate.

6 Traces and numerical irreducible decomposition

With Section 5 considering witness points of a pure-dimensional component, the next structure to consider applying this robust framework to is computing parameter values which have solution components that decompose into several irreducible components.

6.1 Reducibility

In a numerical irreducible decomposition, the collection of witness points is partitioned into subsets corresponding with the irreducible components. One approach for performing this decomposition is via the trace test [9, 11, 27, 43] and a key decision is to determine when the linear trace vanishes. For exact systems, this can be determined robustly by computing the linear trace to higher precision, but becomes an uncertain task for systems with error as perturbations tend to destroy reducibility.

In the present context, the second-derivative trace test from [9] is appropriate, as it can be employed locally. Suppose that $\{f, L_D, W\}$ is a witness set for a pure D-dimensional component X of $V(f)$. Since discussions about multiplicity are provided in Section 7.1, suppose that each irreducible component of X has multiplicity 1 with respect to f. Moreover, by replacing f with a randomization, we can assume that $f:\mathbb{C}^n\to\mathbb{C}^{n-D}$. Let $W_r\subset W$ consist of r points. Then, there is a pure D-dimensional component $X'\subset X$ with $X'\cap V(L_D)=W_r$ if and only if, for a general $L'_D:\mathbb{C}^n\to\mathbb{C}^D$,

$$\sum_{j=1}^{r}\ddot{w}_j=0 \tag{17}$$

where $\{w_1,\ldots,w_r\}=X'\cap V(L'_D)$, and $\dot{w}_j$ and $\ddot{w}_j$ satisfy

$$\begin{bmatrix} Jf(w_j)\\ JL'_D(w_j)\end{bmatrix}\cdot\dot{w}_j=\begin{bmatrix}0\\1\end{bmatrix},\quad \begin{bmatrix} Jf(w_j)\\ JL'_D(w_j)\end{bmatrix}\cdot\ddot{w}_j=-\begin{bmatrix}\dot{w}_j^T\cdot\text{Hessian}(f_1)(w_j)\cdot\dot{w}_j\\ \vdots\\ \dot{w}_j^T\cdot\text{Hessian}(f_{n-D})(w_j)\cdot\dot{w}_j\\ 0\end{bmatrix}. \tag{18}$$

Since all the coordinates in (17) may not be independent, one can replace (17) with a randomized version, namely $\alpha\cdot\sum_{j=1}^{r}\ddot{w}_j=0$ for a general $\alpha\in\mathbb{C}^n$. Nonetheless, one can use fiber products to recover reducibility of a component of degree r.

Theorem 6.1. *Suppose that $f(x;p)$ is a polynomial system and Z is a nonempty irreducible component of the set of*

$$(w_1, \dot{w}_1, \ddot{w}_1, \ldots, w_r, \dot{w}_r, \ddot{w}_r, \ell, p) \in (\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n)^r \times \mathrm{Gr}_{n-D}(n) \times \mathbb{C}^m$$

such that $\{w_1, \ldots, w_r\}$ are r distinct points in $V(f(x;p)) \cap \ell$, (18) holds for each $(w_j, \dot{w}_j, \ddot{w}_j)$ for $f(x;p)$ and $L_\ell(x)$, and (17) holds. Moreover, suppose that, for general

$$(y_1, \dot{y}_1, \ddot{y}_1, \ldots, y_r, \dot{y}_r, \ddot{y}_r, b, q) \in Z,$$

y_j is nonsingular with respect to $\{f(x;q), L_b(x)\}$. Then, Proposition 3.1 holds when applied to

$$F(x_1, \dot{x}_1, \ddot{x}_1, \ldots, x_r, \dot{x}_r, \ddot{x}_r, \ell, p) = \begin{bmatrix} S(x_1, \dot{x}_1, \ddot{x}_1, \ell, p) \\ \vdots \\ S(x_r, \dot{x}_r, \ddot{x}_r, \ell, p) \\ \ddot{x}_1 + \cdots + \ddot{x}_r \end{bmatrix}$$

where

$$S(x, \dot{x}, \ddot{x}, \ell, p) = \begin{bmatrix} \begin{bmatrix} f(x,p) \\ L_\ell(x) \end{bmatrix} & \text{\% solution} \\ \begin{bmatrix} Jf(x,p) \\ JL_\ell(x) \end{bmatrix} \cdot \dot{x} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{\% } 1^{\text{st}} \text{ derivative} \\ \begin{bmatrix} Jf(x,p) \\ JL_\ell(x) \end{bmatrix} \cdot \ddot{x} + \begin{bmatrix} \dot{x}^T \cdot \mathrm{Hessian}(f_1)(x,p) \cdot \dot{x} \\ \vdots \\ \dot{x}^T \cdot \mathrm{Hessian}(f_{n-D})(x,p) \cdot \dot{x} \\ 0 \end{bmatrix} & \text{\% } 2^{\text{nd}} \text{ derivative} \end{bmatrix}.$$

Proof. One can follow a similar proof as in Theorem 5.1. □

For perturbed parameter values $\hat{p}$, one is looking for collections of points for which (17) is close to 0. The situation in Theorem 6.1 covers the case when X has degree d and one is considering decomposing X into a degree r and degree $d-r$ component (where $1 \le r \le d-r \le d$). If one is considering factorization into more than two components or the factorization of components in different dimensions, then Remark 3.3 applies; that is, one would form a compound fiber product system that stacks up several systems of the form given in Theorem 6.1. Moreover, in the multiplicity 1 case considered here, Remark 3.4 applies.

6.2 Illustrative example

Consider the parameterized family of polynomial systems

$$f(x;p) = p_1 + p_2x_1 + p_3x_1^2 + p_4x_1^3 + p_5x_1x_2 + p_6x_1^2x_2 + p_7x_1^3x_2 + p_8x_2^2 + p_9x_1^2x_2^2 + p_{10}x_1x_2^3 \quad (19)$$

from [52]. For generic $p \in \mathbb{C}^{10}$, $f(x;p) = 0$ defines a quartic plane curve. The problem described in [52, Ex. 2] considers the parameters $\tilde{p} = (-30, 20, 18, -12, 12, -8, 0, -5, 3, 2)$ with perturbation $\hat{p} = (-30, 20, 18, -12, 12.000007, -8, 0.0000003, -5, 3, 2)$ so that

$$f(x;\tilde{p}) = (3x_1^2+2x_1x_2-5)(x_2^2-4x_1+6) \quad \text{and} \quad f(x;\hat{p}) = f(x;\tilde{p})+0.0000003x_1^3x_2+0.000007x_1x_2$$

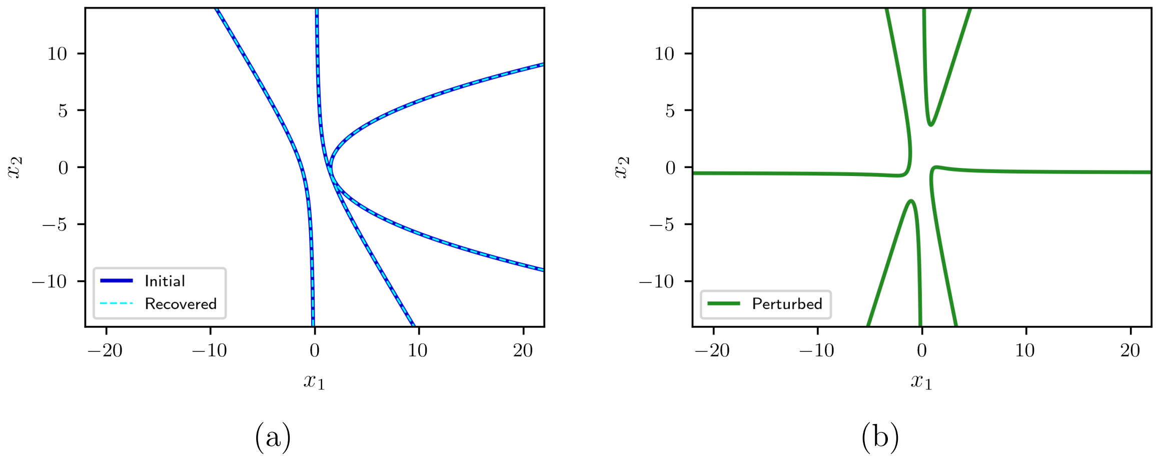


Figure 4: Solution sets corresponding to the (a) initial and recovered parameters, and (b) perturbed parameters for system (19).

with the corresponding quartic plane curves illustrated in Fig. 4.

For illustration, Table 4 considers the result of intersecting $f(x;\hat{p}) = 0$ with the linear space defined by $2x_1 - 3x_2 = 1$. Clearly, we see that both $\ddot{w}_1 + \ddot{w}_2$ and $\ddot{w}_3 + \ddot{w}_4$ are close to zero indicating that we should consider computing parameters p^* near $\hat{p}$ for which $f(x;p^*)$ factors into two quadratics via the second derivative trace test, i.e., apply Theorem 6.1 with $r = 2$.

Table 5 uses Remark 3.4 to determine the dimension of parameter space as the fiber product index a is incremented. In this case, one sees a stabilization of the dimension of the parameter space to 6 when $a = 4$, confirmed as stable when $a = 5$. That is, we expect to recover parameters p^* contained in a 6-dimensional parameter space. After applying appropriate random instantiation of the Grassmannians and randomization (17) as in Section 6.1, the resulting gradient descent homotopy (6) with homogenized Lagrange multipliers is a square system consisting of the same number of variables and equations which is also listed in Table 5. Finally, Table 5 also records the numerical irreducible decomposition of $f(x;p^*)$ for the corresponding recovered p^*. This column also indicates that $\mathcal{F}_4$ suffices for the decomposition to stabilize, with the corresponding recovered parameters (to 7 decimal places) provided in Table 6. For comparison, written using double precision, the recovered factorization from [52] and p^* is provided in Table 7. The key difference is that [52] enforced $p_7 = 0$ so that the third factor maintained the same monomial structure as the exact system while the second derivative trace test only enforced factorability. Hence, with the additional constraint, the recovered factorization in [52] is further away $(3.76 \cdot 10^{-6})$ from p^* than $\hat{p}$ $(3.15 \cdot 10^{-6})$. Of course, one could impose the additional parameter space condition, namely, $p_7 = 0$, with the second derivative trace test approach and recover the same factorization as [52].

7 Multiplicity and local Hilbert function

The final structure we consider for applying this robust framework to is computing parameter values which have solutions with specified multiplicity and local Hilbert function.

7.1 Macaulay matrix

For a univariate polynomial $u(x)$, a number x^* is said to have multiplicity $\mu \geq 0$ if and only if $u(x^*) = u'(x^*) = u''(x^*) = \cdots = u^{(\mu-1)}(x^*) = 0$ and $u^{(\mu)}(x^*) \neq 0$. For a multivariate polynomial system, derivatives are replaced with partial derivatives leading to different ways of having a solution with multiplicity μ. One approach for computing multiplicity in multivariate systems is via Macaulay matrices first introduced in [30] and used in various methods such as [3, 12, 15, 16, 47, 53] to name a few.

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, define

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad \text{and} \quad \partial_\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

For $x^* \in \mathbb{C}^n$, consider the linear functional $\partial_\alpha[x^*]$ from polynomials in x to $\mathbb{C}$ defined by

$$\partial_\alpha[x^*](g) = (\partial_\alpha g)(x^*)$$

which is simply the coefficient of $(x-x^*)^\alpha$ in an expansion of $g(x)$ about x^*. For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^k$ and $d \in \mathbb{Z}_{\geq 0}$, the d^{th} Macaulay matrix of f at x^* is

$$M_d(f, x^*) = \left[\partial_\alpha[x^*]\left((x-x^*)^\beta f_j\right) \text{ such that } |\alpha| \leq d, |\beta| \leq \max\{0, d-1\}, j = 1, \ldots, k\right]$$

where the rows are indexed by (β, j) while the columns are indexed by α. Define $\beta \leq \alpha$ if $\beta_a \leq \alpha_a$ for all $a = 1, \ldots, n$. By the Leibniz rule,

$$\partial_\alpha[x^*]\left((x-x^*)^\beta f_j\right) = \begin{cases} \partial_{\alpha-\beta}[x^*](f_j) & \text{if } \beta \leq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For example, $M_0(f, x^*) = f(x^*)$ and $M_1(f, x^*) = [f(x^*) \;\; Jf(x^*)]$. Moreover, there are matrices $A_d(f, x^*)$ and $B_d(f, x^*)$ such that

$$M_{d+1}(f, x^*) = \begin{bmatrix} M_d(f, x^*) & A_d(f, x^*) \\ 0 & B_d(f, x^*) \end{bmatrix}. \tag{20}$$

The local Hilbert function of f at x^* is

$$h_{f,x^*}(d) = \dim \text{null } M_d(f, x^*) - \dim \text{null } M_{d-1}(f, x^*)$$

where one takes $\dim \text{null } M_{-1}(f, x^*) = 0$. In particular, $x^* \in V(f)$ if and only if $h_{f,x^*}(0) = 1$. Moreover, if $x^* \in V(f)$, then x^* is isolated in $V(f)$ if and only if there exists $d^* \geq 0$ such that $h_{f,x^*}(d) = 0$ for all $d > d^*$ with multiplicity $\mu = \dim \text{null } M_{d^*}(f, x^*) = \sum_{d=0}^{d^*} h_{f,x^*}(d)$.

Let n_d be the number of columns in $M_d(f, x^*)$ and $S(d) = \sum_{j=0}^{d} h_{f,x^*}(j)$. Then, $\Lambda_d = \text{null } M_d(f, x^*) \in \text{Gr}_{S(d)}(n_d)$. By a natural inclusion of Λ_d in $\mathbb{C}^{n_{d+1}}$ by padding with zeros, (20) shows that $\Lambda_{d+1} \in \text{Gr}_{S(d+1)}(n_{d+1})$ arises as the linear span of Λ_d and another a linear space, say, $\Omega_d \in \text{Gr}_{h_{f,x^*}(d+1)}(n_{d+1})$. By abuse of notation, we will write as $\Lambda_{d+1} = \Lambda_d \oplus \Omega_{d+1}$. In particular, $\Lambda_{d^*} = \Omega_0 \oplus \cdots \oplus \Omega_{d^*} \in \text{Gr}_\mu(n_{d^*})$ where $\Omega_j \in \text{Gr}_{h_{f,x^*}(j)}(n_j)$.

The following shows how to impose a local Hilbert function condition.

Theorem 7.1. *Let $d^* \in \mathbb{Z}_{\geq 0}$ and $h = (h_0, h_1, \ldots, h_{d^*}, 0, \ldots)$ with $h_0 = 1$ and $h_j \in \mathbb{Z}_{\geq 1}$ for $j = 1, \ldots, d^*$. With the setup described above, $x^* \in \mathbb{C}^n$ such that $h_{f,x^*}(j) = h_j$ for $j \geq 0$ if and only if there are unique $\Omega_j \in \mathrm{Gr}_{h_{f,x^*}(j)}(n_j)$ for $j \geq 0$ such that*

$$\text{null } M_d(f, x^*) = \Omega_0 \oplus \cdots \oplus \Omega_d \tag{21}$$

for all $d \geq 0$.

Suppose that $F(x, \Omega_0, \ldots, \Omega_{d^*}, p)$ is the polynomial system consisting of $f(x,p)$ along with (21) for $j = 1, \ldots, d^*$. Assume that Z is a nonempty irreducible component of the set $(y, \Omega_0, \ldots, \Omega_{d^*}, q)$ such that $f(y;q) = 0$ and $\Omega_0 \oplus \cdots \oplus \Omega_j \subset \text{null } M_j(f(x;q), y)$ for $j = 0, \ldots, d^*$. Moreover, suppose that, for general $(y, \Omega_0, \ldots, \Omega_{d^*}, q) \in Z$, y is an isolated point in $V(f(x;q))$ with multiplicity $\mu = \sum_{j=0}^{d^*} h_{f(x;q),y}(j)$ and $\Omega_0 \oplus \cdots \oplus \Omega_j = \text{null } M_j(f(x;q), y)$ for $j = 0, \ldots, d^*$. Then, Proposition 3.1 holds when applied to F.

Proof. The first part of the result follows immediately from (20) and the definition of the local Hilbert function. The second part follows analogously as in the other proofs. □

In order to enforce various local Hilbert functions separately at several points, Theorem 7.1 can be applied individually for each point and then all such systems can be collected together. To enforce multiplicity of a component, one applies a local Hilbert function condition at each of the witness points separately with respect to the polynomial system and slicing system together. Then, one takes fiber products resulting from the component systems in Theorem 7.1 stacked together via Proposition 3.1 and Remark 3.3. Additionally, if one aims to enforce a Hilbert function of a zero-scheme, this approach can naturally be generalized following [15].

For perturbed parameter values $\hat{p}$, one is looking for solutions to $f(x;\hat{p}) = 0$ for which the corresponding Macaulay matrices are nearly rank deficient. This can be determined using numerical rank revealing methods such as the singular value decomposition to determine appropriate null space conditions to apply.

As in Section 4.1, one can perform computations on general affine patches of the Grassmannians. In particular, adapting [4, Thm. 2], for a general unitary $R_j \in \mathbb{C}^{(n_j - n_{j-1}) \times (n_j - n_{j-1})}$, one can represent a general open affine patch of $\mathrm{Gr}_{h_j}(n_j)$ as

$$\begin{bmatrix} \Xi_j \\ R_j \begin{bmatrix} I_{h_j} \\ \Psi_j \end{bmatrix} \end{bmatrix}$$

where $\Xi_j \in \mathbb{C}^{n_{j-1} \times h_j}$, $\Psi_j \in \mathbb{C}^{(n_j - n_{j-1} - h_j) \times h_j}$ and I_{h_j} is the $h_j \times h_j$ identity matrix. In particular, the total number of elements in Ξ_j and Ψ_j is

$$n_{j-1}h_j + (n_j - n_{j-1} - h_j)h_j = (n_j - h_j)h_j = \dim \mathrm{Gr}_{h_j}(n_j).$$

7.2 Illustrative example

Similar to Section 5.2, computing a numerical irreducible decomposition of

$$f(x;p) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_1^3 - 2p_1x_1^2 - 2x_1^2 + p_1^2x_1 + 4p_1x_1 - p_1^2 - p_2 \\ x_1^2x_2 - 2x_1^2 - 2p_1x_1x_2 + 4p_1x_1 + p_1^2x_2 - p_1^2 - p_2 \end{bmatrix}, \tag{22}$$

with $\tilde{p} = (1,1)$, has a line and an isolated point in the solution space. However, for this system, the line has multiplicity two. When these parameters are perturbed, say $\hat{p} = (1.2346, 1.0089)$ to 4 decimal places, the point and line structure breaks into three isolated points. In this case, we want to recover nearby parameters giving the special structure of a one-dimensional line with multiplicity two and an isolated point. As in Section 5.2, we randomize to a single equation and add a slice. After solving

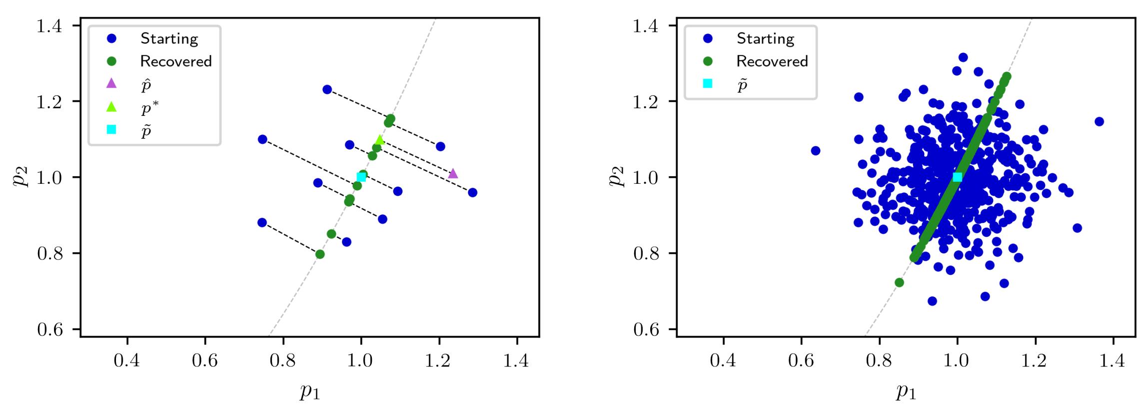
$$f_R(x;p) = \begin{bmatrix} Rf(x;p) \\ L(x) \end{bmatrix} = \begin{bmatrix} f_1 + \Box f_2 \\ \Box x_1 + \Box x_2 + \Box \end{bmatrix} = 0, \tag{23}$$

we choose one of the two solutions near the one-dimensional line. To recover the multiplicity at this component, we add the condition outlined in Theorem 7.1. In particular, as mentioned in Section 2.6, since the randomized system having multiplicity 2 implies the original system has multiplicity 2, we simply work with the randomized system with (21) corresponding to $f_R(x;p) = 0$ from (23) together with

$$Jf_R(x;p) \cdot R_1 \cdot \begin{bmatrix} 1 \\ \psi \end{bmatrix} = 0 \tag{24}$$

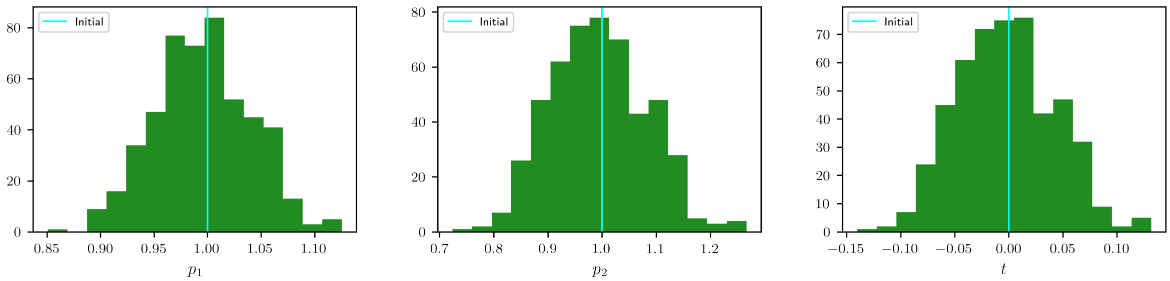
where $R_1 \in \mathbb{C}^{2\times 2}$ is a general unitary matrix. Hence, $\mathcal{F}$ consists of the polynomials in (23) and (24). Using a gradient descent homotopy (6) with homogenized Lagrange multipliers yields the recovered parameters listed in Table 8 and pictorially represented in Fig. 5.

Similar to Section 5.2, we repeated this process with 500 samples from a bivariate Gaussian distribution centered at the initial parameter values $\tilde{p} = (1,1)$ with covariance matrix $\Sigma = 0.1^2 I_2$. The results of this experiment are summarized in Fig. 5. For this simple problem, it is easy to recover the defining equation for the corresponding exceptional set, namely they all lie along the parabola $V(p_1^2 - p_2)$. Histograms of the marginal distributions for p_1, p_2, and along an intrinsic parameterization of the tangent line to the parabola at $\tilde{p}$ are shown in Fig. 6.



(a) (b)

Figure 5: (a) Illustration of recovering parameters for various perturbations including the example summarized in Table 8; (b) Illustration using 500 samples



(a) (b) (c)

Figure 6: Histograms for (a) p_1, (b) p_2, and (c) intrinsic parameterizing coordinate along tangent line for recovered parameter values from 500 samples

8 Kinematic examples

The examples in Sections 4–7 were designed for illustrative purposes. The following considers three examples derived from the field of kinematics.

8.1 Decomposable 4-bar coupler curve

Consider the 4-bar linkage given by the parameterized family of polynomial systems

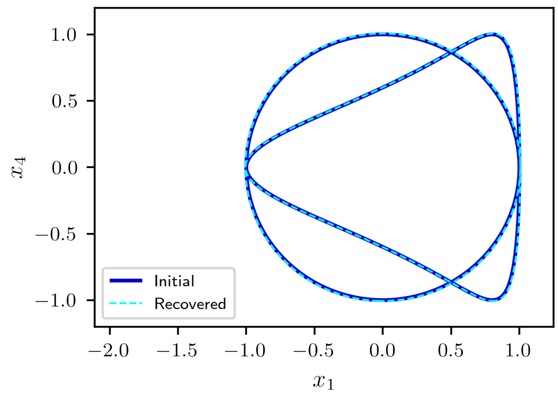
$$f(x;p) = \begin{bmatrix} x_1^2 + x_2^2 - p_1^2 \\ (x_3 - p_2)^2 + x_4^2 - p_3^2 \\ (x_1 - x_3)^2 + (x_2 - x_4)^2 - p_4^2 \end{bmatrix}. \tag{25}$$

For generic $p \in \mathbb{C}^4$, $V(f(x;p))$ is an irreducible sextic ($d = 6$) curve. It is known that $p_1 = p_3$ and $p_2 = p_4$ yields a parallelogram linkage and the solution set factors into a quadratic and quartic curve. Considering initial parameter values $\tilde{p} = (1, 2, 1, 2)$, we perturbed the parameters yielding $\hat{p} = (1.0025, 2.0101, 1.0098, 2.0014)$ rounded to four decimals. For illustration, Fig. 7 shows the solution set projected into (x_1, x_4) space.

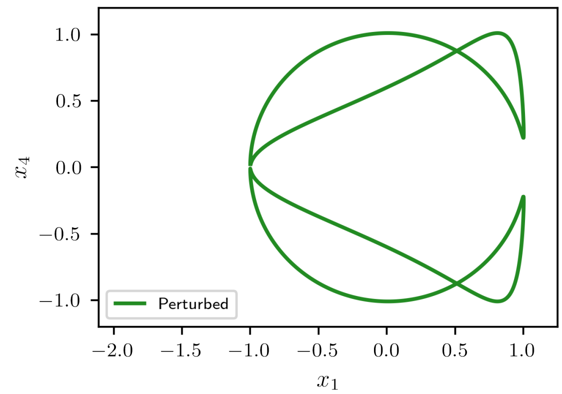
For the perturbed parameter values $\hat{p}$, the sextic curve does not factor. Nonetheless, following Section 6.2, linear traces are close to zero for collections of $r = 2$ and $d - r = 4$ points. Thus, we aim to apply Theorem 6.1 to recover parameters p^* near $\hat{p}$ such that the solution set $V(f(x;p^*))$ factors into a quadratic curve and a quartic curve using the second derivative trace test. Table 9 summarizes the results of applying Remark 3.4 to determine the number of component systems needed. One sees that the dimension stabilizes with two systems. The corresponding system sizes are also reported in Table 9, where the systems are square via homogenized Lagrange multipliers. Using a gradient descent homotopy (6) with two component systems, the recovered parameters p^* are reported in Table 10 to 4 decimals and one clearly sees the parallelogram linkage structure is recovered. The resulting decomposable solution set $V(f(x;p^*))$ is illustrated in Fig. 7(a).

8.2 Stewart-Gough platform

A Stewart-Gough platform consists of two bodies, a base and an end-plate, connected by six legs as illustrated in Fig. 8. For $j = 1, \ldots, 6$, the j^{th} leg imposes a square distance d_j between



(a)



(b)

Figure 7: Projections of the 4-bar coupler curve in (x_1, x_4) space corresponding to (a) initial and recovered parameters, and (b) perturbed parameters

point $a_j \in \mathbb{R}^3$ of the base and point $b_j \in \mathbb{R}^3$ of the end-plate. Letting "$*$" denote quaternion multiplication and letting v' denote the quaternion conjugate of v, the leg constraints may be written as follows, for $j = 1, \ldots, 6$,

$$\begin{aligned} f_j(e, g; a, b, d) = (a_j * a_j' + b_j * b_j' - d_j)e * e' - e * b_j * e' * a_j' * a_j * e * b_j' * e' \\ + g * b_j' * e' + e * b_j * g' - g * e' * a_j' - a_j * e * g' + g * g' = 0 \end{aligned} \quad (26)$$

where e, g are quaternions in a Study coordinate representation of the position and orientation of the end-plate. Hence, e, g must satisfy the Study quadric

$$Q(e, g) = g_0 e_0 + g_1 e_1 + g_2 e_2 + g_3 e_3 = 0. \quad (27)$$

In this example, we set $e_0 = 1$ to dehomogenize the system. For generic parameters $p = (a, b, d)$, this platform can be assembled in 40 rigid configurations over the complex numbers. That is, the solution set of the parameterized polynomial system resulting from the 6 leg constraints in (26) and Study quadratic in (27) consists of 40 isolated points. However, for $\tilde{p}$ reported in Table 12 in Appendix A derived from [17, Ex. 2.2], this platform moves in a circular motion as illustrated in Fig. 8. In particular, this circular motion corresponds to the solution set containing a quadratic curve. To apply the robustness framework, we consider a slight perturbation yielding $\hat{p}$ in Table 12 for which the platform becomes rigid.

Following Section 5.2, we aim to find p^* near $\hat{p}$ for which the solution set contains a quadratic curve. First, we consider a randomization of the 6 leg constraints down to 5 conditions, the Study quadric, and a linear slice, namely

$$f_R(e, g; p) = \begin{bmatrix} f_1 + \square f_6 \\ f_2 + \square f_6 \\ f_3 + \square f_6 \\ f_4 + \square f_6 \\ f_5 + \square f_6 \\ Q \\ \square e_1 + \square e_2 + \square e_3 + \square g_0 + \square g_1 + \square g_2 + \square g_3 + \square \end{bmatrix} \quad (28)$$

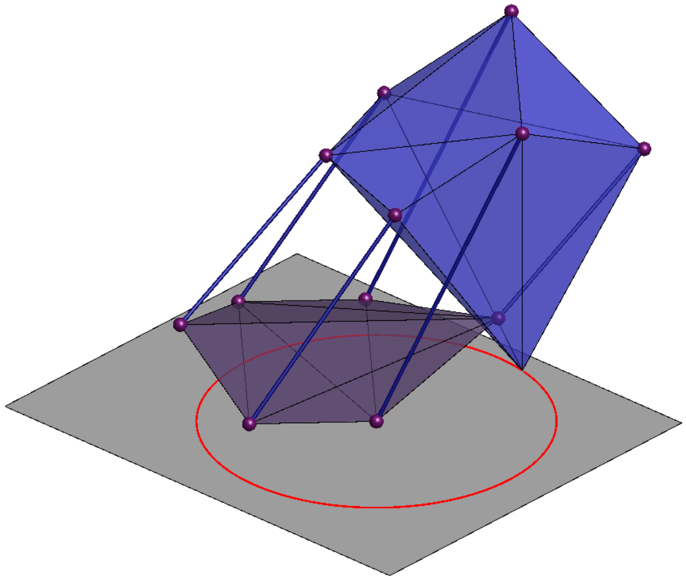


Figure 8: A Stewart-Gough platform. The $z = 0$ plane (gray) contains the circular path (red) corresponding to one point of the end-plate

where each $\square$ represents an independent random complex number. For $f_R(e, g; \hat{p})$, there are two solutions for which the leg constraints in (26) are close to vanishing, consistent with a degree 2 component having 2 witness points, and 38 solutions which are not close to vanishing. Using Theorem 5.1 with $d = 2$ to construct the fiber product system, we apply Remark 3.4 which indicates 4 component systems are needed. The dimensions and corresponding square system sizes via homogenized Lagrange multipliers are reported in Table 11. Tracking the corresponding gradient descent homotopy (6), the recovered parameter values p^* are also reported in Table 12 and the corresponding Stewart-Gough platform has regained its motion.

8.3 Family containing the 6R inverse kinematics problem

The inverse kinematics problem for six-revolute (6R) mechanisms seeks to determine all ways to assemble a loop of six rigid links connected serially by revolute joints. One formulation [34, 48] sets the problem as a member of the following parameterized system of eight quadratics using a 2-homogeneous construction:

$$f(x;p) = \begin{bmatrix} f_0(x;q) \\ f_1(x;q) \\ f_2(x;q) \\ f_3(x;q) \\ x_1^2 + x_2^2 - x_0^2 \\ x_5^2 + x_6^2 - x_0^2 \\ x_3^2 + x_4^2 - x_9^2 \\ x_7^2 + x_8^2 - x_9^2 \end{bmatrix}, \tag{29}$$

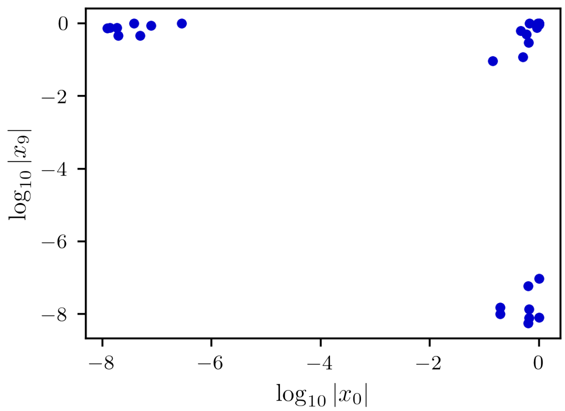


Figure 9: Logarithmic plot of absolute values of homogenizing coordinates for the 64 solutions

where f_j, $j = 0, 1, 2, 3$, has the form

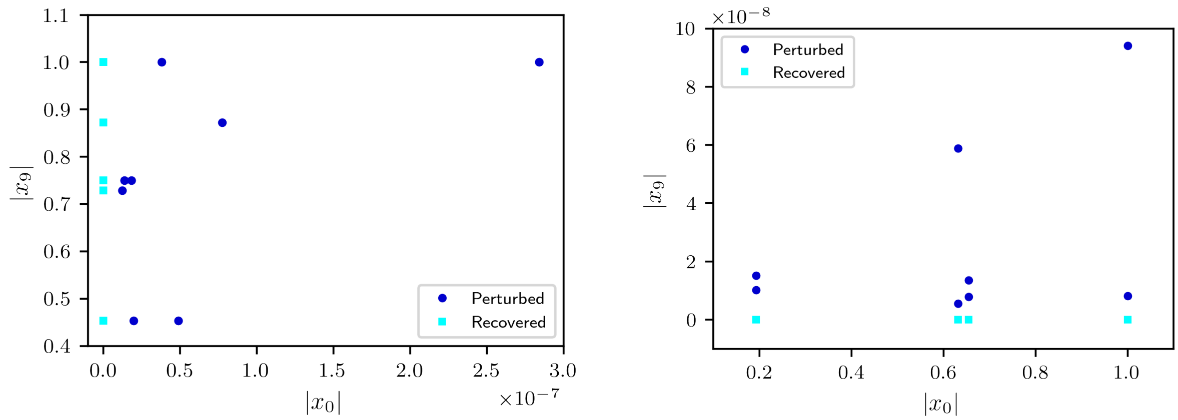
$$\begin{aligned} f_j(x;q) = q_{j0}x_1x_3 + q_{j1}x_1x_4 + q_{j2}x_2x_3 + q_{j3}x_2x_4 + q_{j4}x_5x_7 + q_{j5}x_5x_8 \\ + q_{j6}x_6x_7 + q_{j7}x_6x_8 + q_{j8}x_1x_9 + q_{j9}x_2x_9 + q_{j10}x_3x_0 + q_{j11}x_4x_0 \\ + q_{j12}x_5x_9 + q_{j13}x_6x_9 + q_{j14}x_7x_0 + q_{j15}x_8x_0 + q_{j16}x_0x_9, \end{aligned} \quad (30)$$

with x_0 and x_9 as the homogenizing coordinates. In particular, this system is defined on $\mathbb{P}^4 \times \mathbb{P}^4$ with corresponding variable sets $\{x_0, x_1, x_2, x_5, x_6\} \times \{x_3, x_4, x_7, x_8, x_9\}$. We dehomogenize the system by solving on the affine patches defined by $x_1 = 1$ and $x_3 = 1$.

With q consisting of 68 values, we take the parameters as the 32 values in q associated with monomials that do not vanish at infinity, i.e., $V(x_0) \cup V(x_9)$, namely $q_{j0}, \ldots, q_{j7}$ for $j = 0, \ldots, 3$. The remaining 36 coefficients, namely $q_{j8}, \ldots, q_{j16}$, $j = 0, \ldots, 3$, are fixed as constants because they are associated with monomials that vanish at infinity. Table 13 in Appendix A contains their values. For generic parameters, the resulting system has 64 finite solutions. However, for parameters that correspond with a 6R problem, the system should only have 32 finite solutions. Thus, to use the robustness framework, we truncated $q \in \mathbb{C}^{68}$ corresponding to an actual 6R problem using single precision. Hence, the constants in Table 13 are listed in single precision and the parameter values in Table 14 are listed in both single precision (corresponding to the perturbed parameters) and double precision (corresponding to the initial parameters).[1] Solving the system with the perturbed parameter values results in 64 points corresponding to finite solutions that are clustered into three groups: 16 having $|x_0|$ close to zero, 16 having $|x_9|$ close to zero, and 32 having both $|x_0|$ and $|x_9|$ far from zero as illustrated in Fig. 9.

First, suppose that we aim to recover parameters by forcing the 16 solutions with $|x_0|$ close to zero to be at infinity, i.e., actually satisfy $x_0 = 0$. The fiber product system is constructed following Theorem 4.1. Since the solutions are not necessarily independent of each other, we use Remark 3.4 applied to the 16th-order fiber product system, $\mathcal{F}_{16}$, and observe that the are actually 4 unnecessary conditions, i.e., only $a = 12$ subsystems in the fiber product are necessary. A gradient descent homotopy (6) confirms this result.

[1] See https://bertini.nd.edu/BertiniExamples/inputIPP_1024 for values in 1024-bit precision.



(a) (b)

Figure 10: Absolute values of the homogenizing coordinates for (a) 16 solutions associated with $x_0 = 0$ and (b) 16 solutions associated with $x_9 = 0$

It is the same story if one aims to recover parameters by forcing the 16 solutions with $|x_9|$ close to zero to be at infinity. So, now suppose that we aim to recover parameters by forcing both sets of 16 solutions with either $|x_0|$ or $|x_9|$ close to zero to be at infinity. Then, applying Remark 3.4, we see that these are not independent and only need 23 fiber products. Thus, when pushing these 32 solutions to infinity, we take 12 for one of the infinities and only 11 for the other. After adding homogenized Lagrange multipliers, this results in a square system of size 423. Tracking the gradient descent homotopy (6) yields the recovered parameters reported in Table 14 of Appendix A. Solving with the recovered parameters shows that all 32 of these solutions are pushed back to infinity as illustrated in Fig. 10.

9 Conclusion

Fiber products were brought to bear in numerical algebraic geometry in [46] as a means of finding sets of exceptional dimension. That work showed that the repeated fiber product of $V(f(x,p))$ with itself promotes a set of exceptional dimension into its own irreducible component, which then can be found by numerical irreducible decomposition. The proof in [46], which is summarized in Proposition 3.1, was based on a growth argument that showed that as the order of the fiber product increases, the dimension of the fiber product of the exceptional set grows faster than the unexceptional set containing it. Building on this fiber product framework, we show in Proposition 3.1, that a similar principle applies not only to exceptional dimension, but also to exceptional sets corresponding to reducing the number of finite solutions, decomposing a set into several factors, and imposing conditions for a component to have higher multiplicity. The existing paper most closely related to the present work is [52], but it only addresses single polynomials, not systems of polynomials.

Our fiber product construction is the first stage of a framework for robustness in numerical algebraic geometry wherein computations for a set of parameters near an exceptional set provide a prospective structural element. Starting from the perturbed parameters and its nearly exceptional solution set, the second stage uses local optimization techniques to search for nearby points in parameter space where the special structure exists. Illustrative examples apply the idea to each type of exceptional set and three substantial examples from kinematics are presented.

In all of the examples presented here, we aimed to recover real parameter values using a gradient descent homotopy based on homogenized Lagrange multipliers associated with the Euclidean distance. For nonreal parameter values, one can use isotropic coordinates. Also, there are many other optimization approaches one may use to recover parameter values on exceptional sets and these other approaches could expand the size of the local convergence zone, possibly allowing to recover parameters from larger perturbations. Finally, one can adjust the distance metric used for the objective function, such as one based on knowledge about the relative size of the parameters and their uncertainties.

Figures 3 and 6 provide histograms of the recovered parameters arising from perturbations centered at a parameter value on an exceptional set. In fact, one can analyze the distribution of the recovered parameters in relation to the distribution of the perturbations. As described in Section 5.2, when using perturbations arising from a Gaussian distribution, this will yield a Gaussian distribution when projecting onto a linear space. When projecting onto a nonlinear space, such as in Section 7.2, this yields approximately a Gaussian distribution on the tangent space. Further statistical analysis regarding recovered parameters is warranted.

Finally, we note that the codimension of an exceptional set in parameter space may be less than the number of conditions one seeks to impose on the solution set. For example, in the 6R problem of Section 8.3, one might naively expect the exceptional set for sending 32 points to infinity to be codimension 32, but in fact, it is codimension 23. While we have found this result numerically, it raises the more general question of how such results can be understood using the tools of algebraic geometry.

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A Appendix

The following provides tables associated with Sections 8.2 and 8.3.

Table 1: Solutions to (9) on a randomly selected affine patch for $\hat{p}$ where $\mathrm{i} = \sqrt{-1}$

Solution	x_0			x_1			x_2		
1	0.0000	+	0.0000i	0.0000	+	0.0000i	−0.2235	+	0.8253i
2	0.0020	−	0.0072i	0.0010	−	0.0037i	−0.2263	+	0.8289i
3	−0.0368	+	2.7475i	−0.0682	+	5.0964i	0.0198	−	1.4776i

Table 2: Initial (exact), perturbed (8 decimals), and recovered (8 decimals) parameter values

Parameter	Initial ($\tilde{p}$)	Perturbed ($\hat{p}$)	Recovered (p^*)
p_1	–2.37160000	–2.37284227	–2.36891717
p_2	0.98608803	0.96067280	0.96814820
p_3	–0.53770000	–0.53492792	–0.52506952

Table 3: Initial (exact), perturbed (4 decimals), and recovered (4 decimals) parameter values

Parameter	Initial ($\tilde{p}$)	Perturbed ($\hat{p}$)	Recovered (p^*)
p_1	1	0.9876	1.0992
p_2	–2	–2.2542	–2.1984

Table 4: Summary of solutions (8 decimals) satisfying $f(x;\hat{p}) = 2x_1 - 3x_2 - 1 = 0$

j	w_j	$\dot{w}_j$	$\ddot{w}_j$
1	1.15384590	0.08241763	0.00546655
	0.43589727	−0.27838824	0.00364437
2	−0.99999993	0.07142858	−0.00546648
	−0.99999995	−0.28571428	−0.00364432
3	1.64589862	−0.17082057	0.13416408
	0.76393241	−0.44721371	0.08944272
4	8.35410056	1.17082044	−0.13416415
	5.23606704	0.44721363	−0.08944277

Table 5: Summary for different component systems

Index a	Dimension	System Size	Recovered Components
1	9	35	One component of degree 4
2	8	60	One component of degree 4
3	7	85	One component of degree 4
4	6	110	Two components of degree 2
5	6	135	Two components of degree 2

Table 6: Initial (exact), perturbed (exact), and recovered (7 decimals) parameter values

Parameter	Initial ($\tilde{p}$)	Perturbed ($\hat{p}$)	Recovered (p^*)
p_1	–30	–30.0000000	–30.0000003
p_2	20	20.0000000	19.9999994
p_3	18	18.0000000	18.0000003
p_4	–12	–12.0000000	–11.9999997
p_5	12	12.0000070	12.0000057
p_6	–8	–8.0000000	–8.0000019
p_7	0	0.0000003	–0.0000014
p_8	–5	–5.0000000	–5.0000002
p_9	3	3.0000000	2.9999992
p_{10}	2	2.0000000	2.0000006

Table 7: Comparison of factors (double precision)

Factors from [52]	-30.0000005908641
	$1 - 0.600000000000000x_1^2 - 0.400000158490569x_1x_2$
	$1 - 0.666666625730982x_1 + 0.166666656432746x_2^2$
Factors from p^*	-30.0000003490653
	$1 - 0.600000003568289x_1^2 - 0.400000104966183x_1x_2$
	$1 - 0.666666639555509x_1 + 0.166666672330951x_2^2 - 7.87997843076905 \cdot 10^{-8}x_1x_2$

Table 8: Initial (exact), perturbed (4 decimals), and recovered (4 decimals) parameter values

Parameter	Initial ($\tilde{p}$)	Perturbed ($\hat{p}$)	Recovered (p^*)
p_1	1	1.2346	1.0479
p_2	1	1.0089	1.0980

Table 9: Decomposable coupler curve: dimensions and system size as fiber product index a increases.

Index a	Dimension	System Size
1	3	53
2	2	102
3	2	151
4	2	200

Table 10: Initial (exact), perturbed (4 decimals), and recovered (4 decimals) parameter values

Parameter	Initial ($\tilde{p}$)	Perturbed ($\hat{p}$)	Recovered (p^*)
p_1	1	1.0025	1.0062
p_2	2	2.0101	2.0057
p_3	1	1.0098	1.0062
p_4	2	2.0014	2.0057

Table 11: Stewart-Gough Platform: dimension and system size as fiber product index a increases.

Index a	Dimension	System Size
1	40	72
2	38	102
3	36	132
4	35	162
5	35	192

Table 12: Initial (exact), perturbed (12 decimals), and recovered (12 decimals) parameters

Parameter	Initial ($\tilde{p}$)	Perturbed ($\hat{p}$)	Recovered (p^*)
a_{1_x}	0.0000	0.000000000251	0.000000000320
a_{1_y}	0.0000	0.000000001013	0.000000000982
a_{1_z}	0.0000	0.000000000980	0.000000000477
b_{1_x}	0.0000	−0.000000000200	−0.000000000269
b_{1_y}	0.0000	−0.000000000637	−0.000000000606
b_{1_z}	1.5000	1.499999998979	1.499999999482
d_1	3.2500	3.249999999891	3.249999999724
a_{2_x}	1.0000	1.000000000136	1.000000000272
a_{2_y}	0.0000	−0.000000000753	−0.000000001236
a_{2_z}	0.2500	0.249999998300	0.249999998434
b_{2_x}	1.0000	1.000000001806	1.000000001671
b_{2_y}	0.0000	−0.000000000886	−0.000000000402
b_{2_z}	1.0000	0.999999999658	0.999999999525
d_2	1.5625	1.562499999151	1.562499999240
a_{3_x}	1.0000	0.999999999712	0.999999999839
a_{3_y}	1.0000	0.999999999733	0.999999999918
a_{3_z}	0.0000	0.000000000109	−0.000000000010
b_{3_x}	1.0000	0.999999998769	0.999999998641
b_{3_y}	1.0000	0.999999999125	0.999999998940
b_{3_z}	1.5000	1.499999999413	1.499999999531
d_3	3.2500	3.250000000389	3.250000000350
a_{4_x}	−0.5000	−0.500000000115	−0.499999999750
a_{4_y}	0.5000	0.500000000098	0.500000000171
a_{4_z}	0.0000	0.000000000167	0.000000000893
b_{4_x}	−0.5000	−0.499999998762	−0.499999999127
b_{4_y}	0.5000	0.499999999424	0.499999999351
b_{4_z}	1.0000	1.000000000799	1.000000000073
d_4	2.0000	2.000000000415	2.000000000779
a_{5_x}	0.5000	0.500000000717	0.499999999761
a_{5_y}	1.5000	1.500000000939	1.500000000405
a_{5_z}	0.0000	0.000000000105	0.000000000622
b_{5_x}	0.5000	0.499999998819	0.499999999776
b_{5_y}	1.5000	1.499999999660	1.500000000194
b_{5_z}	1.0000	0.999999999603	0.999999999086
d_5	2.0000	1.999999998435	1.999999998693
a_{6_x}	−0.2500	−0.250000000190	−0.249999999931
a_{6_y}	1.2500	1.249999999364	1.250000000155
a_{6_z}	0.2500	0.250000002270	0.250000001515
b_{6_x}	−0.2500	−0.249999999033	−0.249999999292
b_{6_y}	1.2500	1.250000001020	1.250000000228
b_{6_z}	1.0000	0.999999999682	1.000000000437
d_6	1.5625	1.562500000179	1.562499999676

Table 13: Constant values truncated to single precision

Constant	Single Precision	Constant	Single Precision
q_{08}	$7.4052387 \cdot 10^{-2}$	q_{28}	$1.9594662 \cdot 10^{-1}$
q_{09}	$-8.3050031 \cdot 10^{-2}$	q_{29}	$-1.2280341 \cdot 10^{0}$
q_{010}	$-3.8615960 \cdot 10^{-1}$	q_{210}	$0.0000000 \cdot 10^{0}$
q_{011}	$-7.5526603 \cdot 10^{-1}$	q_{211}	$-7.9034219 \cdot 10^{-2}$
q_{012}	$5.0420168 \cdot 10^{-1}$	q_{212}	$2.6387877 \cdot 10^{-2}$
q_{013}	$-1.0916286 \cdot 10^{0}$	q_{213}	$-5.7131429 \cdot 10^{-2}$
q_{014}	$0.0000000 \cdot 10^{0}$	q_{214}	$-1.1628081 \cdot 10^{0}$
q_{015}	$4.0026384 \cdot 10^{-1}$	q_{215}	$1.2587767 \cdot 10^{0}$
q_{016}	$4.9207289 \cdot 10^{-2}$	q_{216}	$2.1625749 \cdot 10^{0}$
q_{18}	$-3.7157270 \cdot 10^{-2}$	q_{38}	$-2.0816985 \cdot 10^{-1}$
q_{19}	$3.5436895 \cdot 10^{-2}$	q_{39}	$2.6868319 \cdot 10^{0}$
q_{110}	$8.5383480 \cdot 10^{-2}$	q_{310}	$-6.9910317 \cdot 10^{-1}$
q_{111}	$0.0000000 \cdot 10^{0}$	q_{311}	$3.5744412 \cdot 10^{-1}$
q_{112}	$-3.9251967 \cdot 10^{-2}$	q_{312}	$1.2499117 \cdot 10^{0}$
q_{113}	$0.0000000 \cdot 10^{0}$	q_{313}	$1.4677360 \cdot 10^{0}$
q_{114}	$-4.3241927 \cdot 10^{-1}$	q_{314}	$1.1651719 \cdot 10^{0}$
q_{115}	$0.0000000 \cdot 10^{0}$	q_{315}	$1.0763397 \cdot 10^{0}$
q_{116}	$1.3873009 \cdot 10^{-2}$	q_{316}	$-6.9686807 \cdot 10^{-1}$

Table 14: Initial (double precision), perturbed (truncated single precision), and recovered (double precision) parameters

Parameter	Single|Double Precision	Recovered
q_{00}	$-2.4915068\|11232596 \cdot 10^{-1}$	$-2.491506848757833 \cdot 10^{-1}$
q_{01}	$1.6091353\|78745045 \cdot 10^{0}$	$1.609135324728055 \cdot 10^{0}$
q_{02}	$2.7942342\|61384628 \cdot 10^{-1}$	$2.794234123846178 \cdot 10^{-1}$
q_{03}	$1.4348015\|88307759 \cdot 10^{0}$	$1.434801543598025 \cdot 10^{0}$
q_{04}	$0.0000000\|00000000 \cdot 10^{0}$	$2.329107073061927 \cdot 10^{-8}$
q_{05}	$4.0026384\|20852447 \cdot 10^{-1}$	$4.002638399151275 \cdot 10^{-1}$
q_{06}	$-8.0052768\|41704895 \cdot 10^{-1}$	$-8.005276506597172 \cdot 10^{-1}$
q_{07}	$0.0000000\|00000000 \cdot 10^{0}$	$1.339330350134300 \cdot 10^{-8}$
q_{10}	$1.2501635\|03697273 \cdot 10^{-1}$	$1.250163518996785 \cdot 10^{-1}$
q_{11}	$-6.8660735\|90276054 \cdot 10^{-1}$	$-6.866073304900900 \cdot 10^{-1}$
q_{12}	$-1.1922811\|66678474 \cdot 10^{-1}$	$-1.192281095708419 \cdot 10^{-1}$
q_{13}	$-7.1994046\|84195284 \cdot 10^{-1}$	$-7.199404481832083 \cdot 10^{-1}$
q_{14}	$-4.3241927\|30334479 \cdot 10^{-1}$	$-4.324192773933984 \cdot 10^{-1}$
q_{15}	$0.0000000\|00000000 \cdot 10^{0}$	$1.358542627603532 \cdot 10^{-8}$
q_{16}	$0.0000000\|00000000 \cdot 10^{0}$	$-1.039184803095887 \cdot 10^{-9}$
q_{17}	$-8.6483854\|60668959 \cdot 10^{-1}$	$-8.648385383114613 \cdot 10^{-1}$
q_{20}	$-6.3555007\|06536143 \cdot 10^{-1}$	$-6.355500280163283 \cdot 10^{-1}$
q_{21}	$-1.1571992\|24063992 \cdot 10^{-1}$	$-1.157199361445811 \cdot 10^{-1}$
q_{22}	$-6.6640447\|34656436 \cdot 10^{-1}$	$-6.664044436579097 \cdot 10^{-1}$
q_{23}	$1.1036211\|15850889 \cdot 10^{-1}$	$1.103620867759053 \cdot 10^{-1}$
q_{24}	$2.9070203\|22913935 \cdot 10^{-1}$	$2.907020211729024 \cdot 10^{-1}$
q_{25}	$1.2587767\|24480555 \cdot 10^{0}$	$1.258776710166779 \cdot 10^{0}$
q_{26}	$-6.2938836\|22402776 \cdot 10^{-1}$	$-6.293883708977084 \cdot 10^{-1}$
q_{27}	$5.8140406\|45827871 \cdot 10^{-1}$	$5.814040462810132 \cdot 10^{-1}$
q_{30}	$1.4894773\|41316300 \cdot 10^{0}$	$1.489477303748473 \cdot 10^{0}$
q_{31}	$2.3062341\|36720304 \cdot 10^{-1}$	$2.306233954795566 \cdot 10^{-1}$
q_{32}	$1.3281073\|07376312 \cdot 10^{0}$	$1.328107268535429 \cdot 10^{0}$
q_{33}	$-2.5864502\|59957599 \cdot 10^{-1}$	$-2.586450384436285 \cdot 10^{-1}$
q_{34}	$1.1651719\|51133394 \cdot 10^{0}$	$1.165171916593329 \cdot 10^{0}$
q_{35}	$-2.6908493\|58556267 \cdot 10^{-1}$	$-2.690849292497942 \cdot 10^{-1}$
q_{36}	$5.3816987\|17112534 \cdot 10^{-1}$	$5.381698714725988 \cdot 10^{-1}$
q_{37}	$5.8258597\|55666972 \cdot 10^{-1}$	$5.825859575485448 \cdot 10^{-1}$