Certifying isolated singular points and their multiplicity structure

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ABSTRACT

This paper presents two new constructions related to singular solutions of polynomial systems. The first is a new deflation method for an isolated singular root. This construction uses a single linear differential form defined from the Jacobian matrix of the input, and defines the deflated system by applying this differential form to the original system. The advantages of this new deflation is that it does not introduce new variables and the increase in the number of equations is linear instead of the quadratic increase of previous methods. The second construction gives the coefficients of the so-called inverse system or dual basis, which defines the multiplicity structure at the singular root. We present a system of equations in the original variables plus a relatively small number of new variables. We show that the roots of this new system include the original singular root but now with multiplicity one, and the new variables uniquely determine the multiplicity structure. Both constructions are "exact", meaning that they permit one to treat all conjugate roots simultaneously and can be used in certification procedures for singular roots and their multiplicity structure with respect to an exact rational polynomial system.

1. INTRODUCTION

One issue when using numerical methods for solving polynomial systems is the ill-conditioning and possibly erratic behavior of Newton's method near singular solutions. Regularization (deflation) techniques remove the singular structure to restore local quadratic convergence of Newton's method.

Our motivation for the current work is twofold. On one hand, in a recent paper [1], two of the co-authors of the present paper studied a certification method for approximate roots of exact overdetermined and singular polynomial systems, and wanted to extend the method to certify the multiplicity structure at the root as well. Since all these problems are ill-posed, in [1] a hybrid symbolic-numeric approach was proposed, that included the exact computation of a square polynomial system that had the original root with multiplicity one. In certifying singular roots, this exact square system was obtained from a deflation technique that added subdeterminants of the Jacobian matrix to the system iteratively. However, since the multiplicity structure is destroyed by this deflation technique, it remained an open question how to certify the multiplicity structure of singular roots of exact polynomial systems.

Our second motivation is to find a method that simultaneously refines the accuracy of a singular root and a small number of parameters describing the multiplicity structure at the root. The knowledge of the multiplicity structure can be useful in many contexts. It can be used, for instance, to analyze the number of branches of an algebraic curve at a singular point [18]. Coupled with subdivision methods [2], it provides an efficient method to certify the topology of curves or surfaces.

In previous numerical approaches which both describe the multiplicity structure and restore the quadratic convergence of Newton's method, the number of parameters is large which can make computation and certification difficult. Therefore, a method which uses a small number of parameters describing the multiplicity structure, and which uses Newton's method to simultaneously approximate the coordinates of the singular root and the parameters, will improve certification of singular roots and their multiplicity structure.

Contributions

In the present paper, we first give an improved version of a deflation method that can be used in the certification algorithm of [1]. This method reduces the number of added equations at each deflation iteration from quadratic to linear. We prove that applying a single linear differential form to the input system corresponding to a generic kernel element of the Jacobian matrix, reduces both the multiplicity and the depth of the singular root. The deflated system does not involve any approximate coefficients and can therefore be used in certification methods as in [1].

Secondly, to approximate efficiently both the singular point and its multiplicity structure, we propose a new deflation which involves a small number of new variables compared to other approaches that rely on Macaulay multiplication matrices. It is based on a new characterization of the isolated singular point together with its multiplicity structure via inverse systems. The deflated polynomial system exploits the nilpotent and commutation properties of the multiplication matrices in the local algebra of the singular point. We prove that the polynomial system we construct has a root corresponding to the singular root but now with multiplicity one, and the new added coordinates describe the multiplicity structure. In particular, this system completely deflates the system in one step. Moreover, the number of variables and equations in this construction is at most $n+n\delta(\delta-1)/2$ and $N\delta+n(n-1)(\delta-1)(\delta-2)/4$, respectively, where N is the number of input polynomials, n is the number of variables, and δ is the multiplicity of the singular point. This construc-

*Research partly supported by DARPA YFA, NSF grant ACI-1460032, and Sloan Research Fellowship.

†Research partly supported by NSF grant CCF-1217557.

tion is the first approach that completely deflates a singular root and has polynomial number of equations and variables in the input size and the multiplicity. Again, the deflated system does not involve any approximate coefficients and thus can handle conjugate roots simultaneously and also be used in certification techniques of exact polynomials as in [1].

Related work.

The treatment of singular roots is a critical issue for numerical analysis and there is a huge literature on methods which transform the problem into a new one for which Newton-type methods converge quadratically to the root.

Deflation techniques which add new equations in order to reduce the multiplicity were considered in [23, 24]. By triangulating the Jacobian matrix at the (approximate) root, new minors of the polynomial Jacobian matrix are added to the initial system in order to reduce the multiplicity of the singular solution.

A similar approach is used in [8] and [6], where a maximal invertible block of the Jacobian matrix at the (approximate) root is computed and minors of the polynomial Jacobian matrix are added to the initial system. For example, when the Jacobian matrix at the root vanishes, all first derivatives of the input polynomials are added to the system in both of these approaches. Moreover, it is shown in [8] that deflation can be performed at nonisolated solutions in which the process stabilizes to so-called *isosingular sets*. At each iteration of this deflation approach, the number of added equations can be taken to be $(N-r)\cdot(n-r)$, where N is the number of input polynomials, n is number of variables, and r is the rank of the Jacobian at the root.

These methods repeatedly use their constructions until a system with a simple root is obtained.

In [10], a triangular presentation of the ideal in a good position and derivations with respect to the leading variables are used to iteratively reduce the multiplicity. This process is applied for p-adic lifting with exact computation.

In other approaches, new variables and new equations are introduced simultaneously. For example, in [28], new variables are introduced to describe some perturbations of the initial equations and some differentials which vanish at the singular points. This approach is also used in [16], where it is shown that this iterated deflation process yields a system with a simple root.

In [18], perturbation variables are also introduced in relation with the inverse system of the singular point to obtain directly a deflated system with a simple root. The perturbation is constructed from a monomial basis of the local algebra at the multiple root.

In [11, 12], only variables for the differentials of the initial system are introduced. The analysis of this deflation is improved in [4], where it is shown that the number of steps is bounded by the order of the inverse system. This type of deflation is also used in [15], for the special case where the Jacobian matrix at the multiple root has rank $n-1$ (the breadth one case).

In these methods, at each step, both the number of variables and equations are increased, but the new equations are linear in the newly added variables.

The aforementioned deflation techniques usually break the structure of the local ring at the singular point. The first method to compute the inverse system describing this structure is due to F.S. Macaulay [17] and known as the dialytic method. More recent algorithms for the construction of inverse systems are described in [19] which reduces the size of the intermediate linear systems (and exploited in [26]) and further improved in [21] and more recently in [18] using a formal integration method.

The computation of inverse systems has also been used to approximate a multiple root. The dialytic method is used in [29] and the relationship between the deflation approach and the inverse system is analyzed, exploited and implemented in [9]. In [25], a minimization approach is used to reduce the value of the equations and their derivatives at the approximate root, assuming a basis of the inverse system is known. In [27], the inverse system is constructed via Macaulay's method, tables of multiplications are deduced, and their eigenvalues are used to improve the approximated root. They show that the convergence is quadratic at the multiple root. In [14], they show that in the breadth one case the parameters needed to describe the inverse system is small, and use it to compute the singular roots in [13]. The inverse system has further been exploited in deflation techniques in [18]. This is the closest to our approach as it computes a perturbation of the initial polynomial system with a given inverse system, deduced from an approximation of the singular solution. The inverse system is used to transform directly the singular root into a simple root of an augmented system.

Singular solutions of polynomial systems have also been studied by analyzing multiplication matrices (e.g., in [3, 20, 7]), by non local methods, which apply only for zero-dimensional systems.

2. PRELIMINARIES

Let $\mathbf{f} := (f_1,\ldots,f_N) \in \mathbb{K}[\mathbf{x}]^N$ where $\mathbf{x} = (x_1,\ldots,x_n)$ for some field $\mathbb{K} \subset \mathbb{C}$ and $I = (f_1,\ldots,f_N) \subset \mathbb{K}[\mathbf{x}]$. Suppose that $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{C}^n$ is an isolated multiple root of $\mathbf{f}$, $\mathfrak{m}_\xi$ is the maximal ideal at ξ, and Q is the primary component of I at ξ, i.e., $\sqrt{Q} = \mathfrak{m}_\xi$.

Consider the ring of power series $\mathbb{C}[[\boldsymbol{\partial}_\xi]] := \mathbb{C}[[\partial_{1,\xi},\ldots,\partial_{n,\xi}]]$. We will use the notation for $\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n$:

$$\boldsymbol{\partial}_\xi^\beta := \partial_{1,\xi}^{\beta_1}\cdots\partial_{n,\xi}^{\beta_n}.$$

We identify $\mathbb{C}[[\boldsymbol{\partial}_\xi]]$ with the dual space $\mathbb{C}[\mathbf{x}]^*$ by considering $\boldsymbol{\partial}_\xi^\beta$ as derivations and evaluations at ξ, defined by

$$\boldsymbol{\partial}_\xi^\beta(p) := \boldsymbol{\partial}^\beta(p)\Big|_\xi := \frac{d^{|\beta|}p}{dx_1^{\beta_1}\cdots dx_n^{\beta_n}}(\xi) \quad \text{for } p \in \mathbb{C}[\mathbf{x}]. \quad (1)$$

The derivation on $\mathbb{C}[[\boldsymbol{\partial}_\xi]]$ with respect to the variable $\partial_{i,\xi}$ is denoted $d_{\partial_{i,\xi}}$ for $i = 1,\ldots,n$. Note that

$$\frac{1}{\beta!}\boldsymbol{\partial}_\xi^\beta((\mathbf{x}-\xi)^\alpha) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise} \end{cases}$$

where $\beta! = \beta_1!\cdots\beta_n!$.

For $p \in \mathbb{C}[\mathbf{x}]$ and $\Lambda \in \mathbb{C}[[\boldsymbol{\partial}_\xi]] = \mathbb{C}[\mathbf{x}]^*$, let

$$p\cdot\Lambda : q \mapsto \Lambda(p\,q).$$

We check that $p = (x_i - \xi_i)$ acts as a derivation on $\mathbb{C}[[\boldsymbol{\partial}_\xi]]$:

$$(x_i - \xi_i)\cdot\boldsymbol{\partial}_\xi^\beta = d_{\partial_{i,\xi}}(\boldsymbol{\partial}_\xi^\beta).$$

For an ideal $I \subset \mathbb{C}[\mathbf{x}]$, consider

$$I^\perp = \{\Lambda \in \mathbb{C}[[\boldsymbol{\partial}_\xi]] \mid \forall p \in I, \Lambda(p) = 0\}.$$

The vector space $I^\perp$ is naturally identified with the dual space of $\mathbb{C}[\mathbf{x}]/I$. One can easily show that $I^\perp$ is a vector subspace of $\mathbb{C}[[\boldsymbol{\partial}_\xi]]$ which is stable by the derivations $d_{\partial_{i,\xi}}$. Since Q is the $\mathfrak{m}_\xi$-primary of I, we have the following classical result:

Lemma 2.1. *If Q is a $\mathfrak{m}_\xi$-primary component of I, then $Q^\perp = I^\perp \cap \mathbb{C}[\boldsymbol{\partial}_\xi]$.*

This lemma shows that to compute $Q^\perp$, it suffices to compute all polynomials of $\mathbb{C}[\boldsymbol{\partial}_\xi]$ which are in $I^\perp$. Let us denote this set $\mathscr{D} = I^\perp \cap \mathbb{C}[\boldsymbol{\partial}_\xi]$. It is a vector space stable under the derivations $d_{\partial_{i,\xi}}$. Its dimension is the dimension of $Q^\perp$ or $\mathbb{C}[\mathbf{x}]/Q$, that is the *multiplicity* of ξ, denoted by $\delta_\xi(I)$ or simply by δ if ξ and I are clear from the context.

For an element $\Lambda(\boldsymbol{\partial}_\xi) \in \mathbb{C}[\boldsymbol{\partial}_\xi]$ we define the *order*, denoted by $o(\Lambda)$, to be the maximal $|\beta|$ such that $\boldsymbol{\partial}_\xi^\beta$ appears in $\Lambda(\boldsymbol{\partial}_\xi)$ with a non-zero coefficient. For $t \in \mathbb{N}$, let $\mathscr{D}_t$ be the elements of $\mathscr{D}$ of order $\leq t$. As $\mathscr{D}$ is of dimension δ, there exists a smallest $t \geq 0$ such that $\mathscr{D}_{t+1} = \mathscr{D}_t$. Let us call this smallest t, the *nil-index* of $\mathscr{D}$ and denote it by $o_\xi(I)$, or simply by o. As $\mathscr{D}$ is stable by the derivations $d_{\partial_{i,\xi}}$, we easily check that for $t \geq o_\xi(I)$, $\mathscr{D}_t = \mathscr{D}$ and that $o_\xi(I)$ is the maximal degree of the elements in $\mathscr{D}$.

3. DEFLATION USING FIRST DIFFERENTIALS

To improve the numerical approximation of a root, one usually applies a Newton-type method to converge quadratically from a nearby solution to the root of the system, provided it is simple. In the case of multiple roots, deflation techniques are employed to transform the system into another one which has an equivalent root with a smaller multiplicity or even with multiplicity one.

We describe here a construction, using differentials of order one, which leads to a system with a simple root. This construction improves the constructions in [11, 4] since no new variables are added. It also improves the constructions presented in [8, 6] by adding a smaller number of equations at each deflation step.

Consider the Jacobian matrix $J_\mathbf{f}(\mathbf{x}) = [\partial_j f_i(\mathbf{x})]$ of the initial system $\mathbf{f}$. By reordering properly the rows and columns (i.e., polynomials and variables), it can be put in the form

$$J_\mathbf{f}(\mathbf{x}) := \begin{bmatrix} A(\mathbf{x}) & B(\mathbf{x}) \\ C(\mathbf{x}) & D(\mathbf{x}) \end{bmatrix} \quad (2)$$

where $A(\mathbf{x})$ is an $r \times r$ matrix with $r = \operatorname{rank} J_\mathbf{f}(\xi) = \operatorname{rank} A(\xi)$.

Suppose that $B(\mathbf{x})$ is an $r \times c$ matrix. The c columns

$$\det(A(\mathbf{x})) \begin{bmatrix} -A^{-1}(\mathbf{x})B(\mathbf{x}) \\ \mathrm{Id} \end{bmatrix}$$

yield the c elements

$$\Lambda_1^\mathbf{x} = \sum_{i=1}^n \lambda_{1,j}(\mathbf{x})\partial_j, \ \ldots, \ \Lambda_c^\mathbf{x} = \sum_{i=1}^n \lambda_{c,j}(\mathbf{x})\partial_j.$$

Their coefficients $\lambda_{i,j}(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$ are polynomial in the variables $\mathbf{x}$. Evaluated at $\mathbf{x} = \xi$, they generate the kernel of $J_\mathbf{f}(\xi)$ and form a basis of $\mathscr{D}_1$.

Definition 3.1. The family $D_1^\mathbf{x} = \{\Lambda_1^\mathbf{x}, \ldots, \Lambda_c^\mathbf{x}\}$ is the *formal* inverse system of order 1 at ξ. For $\boldsymbol{i} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, c\}$ with $|\boldsymbol{i}| \neq 0$, the $\boldsymbol{i}$*-deflated system* of order 1 of $\mathbf{f}$ is

$$\{\mathbf{f}, \Lambda_{i_1}^\mathbf{x}(\mathbf{f}), \ldots, \Lambda_{i_k}^\mathbf{x}(\mathbf{f})\}.$$

By construction, for $i = 1, \ldots, c$,

$$\Lambda_i^\mathbf{x}(\mathbf{f}) = \sum_{j=1}^n \partial_j(\mathbf{f})\lambda_{i,j}(\mathbf{x}) = \det(A(\mathbf{x}))J_\mathbf{f}(\mathbf{x})[\lambda_{i,j}(\mathbf{x})]$$

has $n - c$ zero entries. Thus, the number of non-trivial new equations added in the $\boldsymbol{i}$-deflated system is $|\boldsymbol{i}| \cdot (N - n + c)$. The construction depends on the choice of the invertible block $A(\xi)$ in $J_\mathbf{f}(\xi)$. By a linear invertible transformation of the initial system and by computing a $\boldsymbol{i}$-deflated system, one obtains a deflated system constructed from any $|\boldsymbol{i}|$ linearly independent elements of the kernel of $J_\mathbf{f}(\xi)$.

Example 3.2. Consider the multiplicity 2 root $\xi = (0, 0)$ for the system $f_1(\mathbf{x}) = x_1 + x_2^2$ and $f_2(\mathbf{x}) = x_1^2 + x_2^2$. Then,

$$J_\mathbf{f}(\mathbf{x}) = \begin{bmatrix} A(\mathbf{x}) & B(\mathbf{x}) \\ C(\mathbf{x}) & D(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 1 & 2x_2 \\ 2x_1 & 2x_2 \end{bmatrix}.$$

The corresponding vector $[-2x_2 \ \ 1]^T$ yields the element

$$\Lambda_1^\mathbf{x} = -2x_2\partial_1 + \partial_2.$$

Since $\Lambda_1^\mathbf{x}(f_1) = 0$, the $\{1\}$-deflated system of order 1 of $\mathbf{f}$ is

$$\{x_1 + x_2^2, \ \ x_1^2 + x_2^2, \ \ -4x_1x_2 + 2x_2\}$$

which has a multiplicity 1 root at ξ.

We use the following to analyze this deflation procedure.

Lemma 3.3 (Leibniz rule). *For $a, b \in \mathbb{K}[\boldsymbol{x}]$,*

$$\boldsymbol{\partial}^\alpha(a\,b) = \sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!}\boldsymbol{\partial}^\beta(a)d_\partial^{\boldsymbol{\beta}}(\boldsymbol{\partial}^\alpha)(b).$$

Proposition 3.4. *Let r be the rank of $J_\mathbf{f}(\xi)$. Assume that $r < n$. Let $\boldsymbol{i} \subset \{1, \ldots, n\}$ with $0 < |\boldsymbol{i}| \leq n - r$ and $\mathbf{f}^{(1)}$ be the* $\boldsymbol{i}$-deflated system *of order* 1 *of* $\mathbf{f}$. *Then, $\delta_\xi(\mathbf{f}^{(1)}) \geq 1$ and $o_\xi(\mathbf{f}^{(1)}) < o_\xi(\mathbf{f})$.*

Proof. By construction, for $i \in \boldsymbol{i}$, the polynomials $\Lambda_i^\mathbf{x}(\mathbf{f})$ vanish at ξ, so that $\delta_\xi(\mathbf{f}^{(1)}) \geq 1$. By hypothesis, the Jacobian of $\mathbf{f}$ is not injective so that $o_\xi(\mathbf{f}) > 0$. Let $\mathscr{D}^{(1)}$ be the inverse system of $\mathbf{f}^{(1)}$ at ξ. Since $(\mathbf{f}^{(1)}) \supset (\mathbf{f})$, we have $\mathscr{D}^{(1)} \subset \mathscr{D}$. In particular, for any non-zero element $\Lambda \in \mathscr{D}^{(1)} \subset \mathbb{K}[\boldsymbol{\partial}_\xi]$ and $i \in \boldsymbol{i}$, we know $\Lambda(\mathbf{f}) = 0$ and $\Lambda(\Lambda_i^\mathbf{x}(\mathbf{f})) = 0$.

Using Leibniz rule, for any $p \in \mathbb{K}[\mathbf{x}]$, we have

$$\begin{aligned}
\Lambda(\Lambda_i^\mathbf{x}(p)) &= \Lambda(\sum_{j=1}^n \lambda_{i,j}(\mathbf{x})\partial_j(p)) \\
&= \sum_{\beta \in \mathbb{N}^n}\sum_{j=1}^n \frac{1}{\beta!}\boldsymbol{\partial}_\xi^{\boldsymbol{\beta}}(\lambda_{i,j}(\mathbf{x}))d_{\partial_\xi}^{\boldsymbol{\beta}}(\Lambda)\partial_{j,\xi}(p) \\
&= \sum_{\beta \in \mathbb{N}^n}\sum_{j=1}^n \frac{1}{\beta!}\boldsymbol{\partial}_\xi^{\boldsymbol{\beta}}(\lambda_{i,j}(\mathbf{x}))\boldsymbol{\partial}_{j,\xi}d_{\partial_\xi}^{\boldsymbol{\beta}}(\Lambda)(p) \\
&= \sum_{\beta \in \mathbb{N}^n} \Delta_{i,\beta}d_{\partial_\xi}^{\boldsymbol{\beta}}(\Lambda)(p)
\end{aligned}$$

where

$$\Delta_{i,\boldsymbol{\beta}} = \sum_{j=1}^n \lambda_{i,j,\boldsymbol{\beta}}\partial_{j,\xi} \in \mathbb{K}[\boldsymbol{\partial}_\xi]$$

with $\lambda_{i,j,\boldsymbol{\beta}} = \frac{1}{\boldsymbol{\beta}!}\partial_\xi^{\boldsymbol{\beta}}(\lambda_{i,j}(\mathbf{x})) \in \mathbb{K}$.

The term $\Delta_{i,\mathbf{0}}$ is $\sum_{j=1}^{n} \lambda_{i,j}(\xi)\partial_{j,\xi}$ which has degree 1 in $\boldsymbol{\partial}_\xi$ since $[\lambda_{i,j}(\xi)]$ is a non-zero element of $\ker J_{\mathbf{f}}(\xi)$. For simplicity, let $\phi_i(\Lambda) := \sum_{\boldsymbol{\beta}\in\mathbb{N}^n} \Delta_{i,\boldsymbol{\beta}} d_\partial^{\boldsymbol{\beta}}(\Lambda)$.

For any $\Lambda \in \mathbb{C}[\boldsymbol{\partial}_\xi]$, we have

$$\begin{aligned} d_{\partial_{j,\xi}}(\phi_i(\Lambda)) &= \sum_{\beta\in\mathbb{N}^n} \lambda_{i,j,\beta} d_\partial^{\boldsymbol{\beta}}(\Lambda) + \Delta_{i,\beta} d_\partial^{\boldsymbol{\beta}}(d_{\partial_{j,\xi}}(\Lambda)) \\ &= \sum_{\beta\in\mathbb{N}^n} \lambda_{i,j,\beta} d_\partial^{\boldsymbol{\beta}}(\Lambda) + \phi_i(d_{\partial_{j,\xi}}(\Lambda)). \end{aligned}$$

Moreover, if $\Lambda \in \mathscr{D}^{(1)}$, then by definition $\phi_i(\Lambda)(\mathbf{f}) = 0$. Since $\mathscr{D}$ and $\mathscr{D}^{(1)}$ are both stable by derivation, it follows that $\forall \Lambda \in \mathscr{D}^{(1)}$, $d_{\partial_{j,\xi}}(\phi_i(\Lambda)) \in \mathscr{D}^{(1)} + \phi_i(\mathscr{D}^{(1)})$. As $\mathscr{D}^{(1)} \subset \mathscr{D}$, this implies that $\mathscr{D}+\phi_i(\mathscr{D}^{(1)})$ is stable by derivation. For any element Λ of $\mathscr{D} + \phi_i(\mathscr{D}^{(1)})$, $\Lambda(\mathbf{f}) = 0$. We deduce that $\mathscr{D} + \phi_i(\mathscr{D}^{(1)}) = \mathscr{D}$. Consequently, the order of the elements in $\phi_i(\mathscr{D}^{(1)})$ is at most $o_\xi(\mathbf{f})$. The statement follows since ϕ_i increases the order by 1, therefore $o_\xi(\mathbf{f}^{(1)}) < o_\xi(\mathbf{f})$. ☐

We consider now a sequence of deflations of the system $\mathbf{f}$. Let $\mathbf{f}^{(1)}$ be the $\boldsymbol{i}_1$-deflated system of $\mathbf{f}$. We construct inductively $\mathbf{f}^{(k+1)}$ as the $\boldsymbol{i}_{k+1}$-deflated system of $\mathbf{f}^{(k)}$ for some choices of $\boldsymbol{i}_j \subset \{1,\ldots,n\}$.

Proposition 3.5. *There exists $k \le o_\xi(\mathbf{f})$ such that ξ is a simple root of $\mathbf{f}^{(k)}$.*

Proof. By Proposition 3.4, $\delta_\xi(\mathbf{f}^{(k)}) \ge 1$ and $o_\xi(\mathbf{f}^{(k)})$ is strictly decreasing with k until it reaches the value 0. Therefore, there exists $k \le o_\xi(I)$ such that $o_{\boldsymbol{\xi}}(\mathbf{f}^{(k)}) = 0$ and $\delta_\xi(\mathbf{f}^{(k)}) \ge 1$. This implies that ξ is a simple root of $\mathbf{f}^{(k)}$. ☐

To minimize the number of equations added at each deflation step, we take $|\boldsymbol{i}| = 1$. Then, the number of non-trivial new equations added at each step is at most $N - n + c$.

We described this approach using first order differentials arising from the Jacobian, but this can be easily extended to use higher order differentials.

4. THE MULTIPLICITY STRUCTURE

Before describing our results, we start this section by recalling the definition of pairs of primal-dual bases for the space $\mathbb{C}[\mathbf{x}]/Q$ and its dual $\mathscr{D}$. The following is a definition:

Definition 4.1 (Primal-dual basis pair)**.** Let $\mathbf{f}$, ξ, Q, $\mathscr{D}$, $\delta = \delta_\xi(\mathbf{f})$ and $o = o_\xi(\mathbf{f})$ be as above. A primal dual basis pair is a basis of $\mathbb{C}[\mathbf{x}]/Q$ of the form

$$B = \{(\mathbf{x}-\xi)^{\alpha_1}, (\mathbf{x}-\xi)^{\alpha_2}, \ldots, (\mathbf{x}-\xi)^{\alpha_\delta}\} \tag{3}$$

with $\alpha_1 = 0$, and a dual basis D of $\mathscr{D}$ of the form:

$$\begin{aligned} \Lambda_{\alpha_1} &= \boldsymbol{\partial}_\xi^{\alpha_1} = 1_\xi \\ \Lambda_{\alpha_2} &= \frac{1}{\alpha_2!}\boldsymbol{\partial}_\xi^{\alpha_2} + \sum_{\substack{|\beta|\le o \\ \beta\notin E}} \frac{\nu_{\alpha_2,\beta}}{\beta!}\,\boldsymbol{\partial}_\xi^{\beta} \\ &\vdots \\ \Lambda_{\alpha_\delta} &= \frac{1}{\alpha_\delta!}\boldsymbol{\partial}_\xi^{\alpha_\delta} + \sum_{\substack{|\beta|\le o \\ \beta\notin E}} \frac{\nu_{\alpha_\delta,\beta}}{\beta!}\,\boldsymbol{\partial}_\xi^{\beta}. \end{aligned} \tag{4}$$

where $E = \{\alpha_1,\ldots,\alpha_\delta\} \subset \mathbb{N}^n$. We also define $E^+ := \bigcup_{i=1}^{n}(E+\mathbf{e}_i)$ with $E+\mathbf{e}_i = \{(\gamma_1,\ldots,\gamma_i+1,\ldots,\gamma_n) : \gamma \in E\}$ and we denote $\partial(E) = E^+ \setminus E$.

We assume that primal-dual basis pair is such that B is *connected to 1* (c.f. [22]) and the orders satisfy $0 = o(\Lambda_{\alpha_1}) \le \cdots \le o(\Lambda_{\alpha_\delta})$ (see e.g. [18]).

Throughout this section we assume that we are given a fixed primal basis B for $\mathbb{C}[\mathbf{x}]/Q$. Note that a primal basis B connected to 1 can be computed numerically from an approximation of ξ as in [5, 9, 21, 18].

Given the primal basis B, the dual basis D can be computed by Macaulay dialytic method and can be used to deflate the root ξ as in [12]. This introduces $n+(\delta-1)\left(\binom{n+o}{n}-\delta\right)$ new variables, which is not polynomial in o. Below, we give a construction of a polynomial system that only depends on at most $n+n\delta(\delta-1)/2$ variables. These variables correspond to the entries of the *multiplication matrices* that we define next.

Let

$$\begin{aligned} M_i : \mathbb{C}[\mathbf{x}]/Q &\to \mathbb{C}[\mathbf{x}]/Q \\ p &\mapsto (x_i-\xi_i)\,p \end{aligned}$$

be the multiplication operator by $x_i - \xi_i$ in $\mathbb{C}[\mathbf{x}]/Q$. Its transpose operator is

$$\begin{aligned} M_i^t : \mathscr{D} &\to \mathscr{D} \\ \Lambda &\mapsto \Lambda\circ M_i = (x_i-\xi_i)\cdot\Lambda = \frac{d}{d\partial_{i,\xi}}(\Lambda) = d_{\partial_{i,\xi}}(\Lambda) \end{aligned} \tag{5}$$

where $\mathscr{D} = Q^\perp \subset \mathbb{C}[\boldsymbol{\partial}_\xi]$. The matrix of M_i in the basis B of $\mathbb{C}[\mathbf{x}]/Q$ is denoted $\mathtt{M}_i$.

As B is a basis of $\mathbb{C}[\mathbf{x}]/Q$, we can identify the elements of $\mathbb{C}[\mathbf{x}]/Q$ with the elements of the vector space $\mathrm{span}_{\mathbb{C}}(B)$. We define the normal form $N(p)$ of a polynomial p in $\mathbb{C}[\mathbf{x}]$ as the unique element b of $\mathrm{span}_{\mathbb{C}}(B)$ such that $p - b \in Q$. Hereafter, we are going to identify the elements of $\mathbb{C}[\mathbf{x}]/Q$ with their normal form in $\mathrm{span}_{\mathbb{C}}(B)$.

For any polynomial $p(x_1,\ldots,x_n) \in \mathbb{C}[\mathbf{x}]$, let $p(\mathbf{M})$ be the operator of $\mathbb{C}[\mathbf{x}]/Q$ obtained by replacing $x_i - \xi_i$ by M_i. By definition of a dual basis, we have the following property:

Lemma 4.2. *For any $p \in \mathbb{C}[\mathbf{x}]$, the normal form of p is $N(p) = p(\mathbf{M})(1)$ and we have*

$$p(\mathbf{M})(1) = \Lambda_{\alpha_1}(p)\,1 + \Lambda_{\alpha_2}(p)\,(\mathbf{x}-\xi)^{\alpha_2} + \cdots + \Lambda_{\alpha_d}(p)\,(\mathbf{x}-\xi)^{\alpha_\delta}.$$

This shows that the coefficient vector $[p]$ of $N(p)$ in the basis B of is $[p] = (\Lambda_{\alpha_i}(p))_{1\le i\le\delta}$.

The following lemma is also well known, but we include it here with proof:

Lemma 4.3. *The values of the coefficients $\nu_{\alpha,\beta}$ for $(\alpha,\beta) \in E\times\partial(E)$ appearing in the dual basis (4) uniquely determine the system of pairwise commuting multiplication matrices $\mathtt{M}_i$, namely, for $i = 1,\ldots,n$*

$$\mathtt{M}_i^t = \boxed{\begin{matrix} 0 & \nu_{\alpha_2,\mathbf{e}_i} & \nu_{\alpha_3,\mathbf{e}_i} & \cdots & \nu_{\alpha_\delta,\mathbf{e}_i} \\ 0 & 0 & \nu_{\alpha_3,\alpha_2+\mathbf{e}_i} & \cdots & \nu_{\alpha_\delta,\alpha_2+\mathbf{e}_i} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{\alpha_\delta,\alpha_{\delta-1}+\mathbf{e}_i} \\ 0 & 0 & 0 & \cdots & 0 \end{matrix}} \tag{6}$$

Moreover,

$$\nu_{\alpha_i,\alpha_k+\mathbf{e}_j} = \begin{cases} 1 & \text{if } \alpha_i = \alpha_k + \mathbf{e}_j \\ 0 & \text{if } \alpha_k + \mathbf{e}_j \in E,\ \alpha_i \ne \alpha_k + \mathbf{e}_j. \end{cases}$$

Proof. As M_i^t acts as a derivation on $\mathscr{D}$ (see (6)) and as the elements Λ_{α_i} are numbered by increasing order, the matrix M_i^t in this basis of $\mathscr{D}$ has an upper triangular form with zero (blocks) on the diagonal.

For an element Λ_{α_j} of order k, its image by M_i^t is

$$\begin{aligned} M_i^t(\Lambda_{\alpha_j}) &= (x_i-\xi_i)\cdot\Lambda_{\alpha_j} \\ &= \sum_{o(\Lambda_{\alpha_l})<k} \Lambda_{\alpha_j}((x_i-\xi_i)(\mathbf{x}-\xi)^{\alpha_l})\Lambda_{\alpha_l} \\ &= \sum_{o(\Lambda_{\alpha_l})<k} \Lambda_{\alpha_j}((\mathbf{x}-\xi)^{\alpha_l+\mathbf{e}_i})\,\Lambda_{\alpha_l} = \sum_{o(\Lambda_{\alpha_l})<k} \nu_{\alpha_j,\alpha_l+\mathbf{e}_i}\Lambda_{\alpha_l}. \end{aligned}$$

This shows that $\mathtt{M}_i^t$ is upper triangular with zeroes on the diagonal, and the entries of $\mathtt{M}_i$ are the coefficients of the dual basis elements corresponding to exponents in $E\times\partial(E)$. The second claim is clear from the definition of $\mathtt{M}_i$. ☐

The previous lemma shows that the dual basis uniquely defines the system of multiplication matrices for $i=1,\dots,n$, so we can combine Lemmas 4.2 and 4.3 to get

$$\mathtt{M}_i^t = \begin{array}{|ccc|}\hline \Lambda_{\alpha_1}(x_i-\xi_i) & \cdots & \Lambda_{\alpha_\delta}(x_i-\xi_i) \\ \Lambda_{\alpha_1}\left((\mathbf{x}-\xi)^{\alpha_2+\mathbf{e}_i}\right) & \cdots & \Lambda_{\alpha_\delta}\left((\mathbf{x}-\xi)^{\alpha_2+\mathbf{e}_i}\right) \\ \vdots & & \vdots \\ \Lambda_{\alpha_1}\left((\mathbf{x}-\xi)^{\alpha_\delta+\mathbf{e}_i}\right) & \cdots & \Lambda_{\alpha_\delta}\left((\mathbf{x}-\xi)^{\alpha_\delta+\mathbf{e}_i}\right) \\ \hline\end{array}$$

$$= \begin{array}{|ccccc|}\hline 0 & \nu_{\alpha_2,\mathbf{e}_i} & \nu_{\alpha_3,\mathbf{e}_i} & \cdots & \nu_{\alpha_\delta,\mathbf{e}_i} \\ 0 & 0 & \nu_{\alpha_3,\alpha_2+\mathbf{e}_i} & \cdots & \nu_{\alpha_\delta,\alpha_2+\mathbf{e}_i} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{\alpha_\delta,\alpha_{\delta-1}+\mathbf{e}_i} \\ 0 & 0 & 0 & \cdots & 0 \\ \hline\end{array}$$

Note that these matrices are nilpotent by their upper triangular structure, and all 0 eigenvalues. As o is the maximal order of the elements of $\mathscr{D}$, we have $\mathtt{M}^\gamma = 0$ if $|\gamma| > o$.

Conversely, the system of multiplication matrices $\mathtt{M}_1,\dots,\mathtt{M}_n$ uniquely defines the dual basis as follows. Consider $\nu_{\alpha_i,\gamma}$ for some (α_i,γ) such that $|\gamma|\le o$ but $\gamma\notin E^+$. We can uniquely determine $\nu_{\alpha_i,\gamma}$ from the values of $\{\nu_{\alpha_j,\beta} : (\alpha_j,\beta)\in E\times\partial(E)\}$ from the following identities:

$$\nu_{\alpha_i,\gamma} = \Lambda_{\alpha_i}((\mathbf{x}-\xi)^\gamma) = [\mathtt{M}_{(\mathbf{x}-\xi)^\gamma}]_{1,i} = [\mathtt{M}^\gamma]_{1,i}. \quad (7)$$

The next definition defines the *parametric multiplication matrices* that we use in our construction.

Definition 4.4 (Parametric multiplication matrices)**.** Let $E=\{\alpha_1,\dots,\alpha_\delta\}\subset\mathbb{N}^n$ be as above. We define the array μ of 0's, 1's and the variables $\mu_{\alpha_i,\beta}$ as follows: for all $\alpha_i,\alpha_k\in E$ and $j\in\{1,\dots,n\}$ the corresponding entry is

$$\mu_{\alpha_i,\alpha_k+\mathbf{e}_j} = \begin{cases} 1 & \text{if } \alpha_i=\alpha_k+\mathbf{e}_j \\ 0 & \text{if } \alpha_k+\mathbf{e}_j\in E,\ \alpha_i\neq\alpha_k+\mathbf{e}_j \\ \mu_{\alpha_i,\alpha_k+\mathbf{e}_j} & \text{if } \alpha_k+\mathbf{e}_j\in\partial(E)\end{cases} \quad (8)$$

Thus the number of variables in μ is $|E\times\partial(E)|\le n\delta(\delta-1)/2$. The *parametric multiplication matrices* are defined for $i=1,\dots,n$ by

$$\mathtt{M}_i^t(\mu) := \begin{array}{|ccccc|}\hline 0 & \mu_{\alpha_2,\mathbf{e}_i} & \mu_{\alpha_3,\mathbf{e}_i} & \cdots & \mu_{\alpha_\delta,\mathbf{e}_i} \\ 0 & 0 & \mu_{\alpha_3,\alpha_2+\mathbf{e}_i} & \cdots & \mu_{\alpha_\delta,\alpha_2+\mathbf{e}_i} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{\alpha_\delta,\alpha_{\delta-1}+\mathbf{e}_i} \\ 0 & 0 & 0 & \cdots & 0 \\ \hline\end{array}, \quad (9)$$

We denote by

$$\mathtt{M}(\mu)^\gamma := \mathtt{M}_1(\mu)^{\gamma_1}\cdots\mathtt{M}_n(\mu)^{\gamma_n},$$

and note that for general parameter values μ, the matrices $\mathtt{M}_i(\mu)$ do not commute, so we fix their order by their indices in the above definition of $\mathtt{M}(\mu)^\gamma$.

Definition 4.5 (Parametric normal form)**.** Let $\mathbb{K}\subset\mathbb{C}$ be a field. We define

$$\begin{aligned} \mathcal{N}_{\mathbf{z},\mu} : \mathbb{K}[\mathbf{x}] &\to \mathbb{K}[\mathbf{z},\mu]^\delta \\ p &\mapsto \mathcal{N}_{\mathbf{z},\mu}(p) := \sum_{\gamma\in\mathbb{N}^n} \frac{1}{\gamma!}\boldsymbol{\partial}_{\mathbf{z}}^\gamma(p)\,\mathtt{M}(\mu)^\gamma[1]. \end{aligned}$$

where $[1]=[1,0,\dots,0]$ is the coefficient vector of 1 in the basis B. This sum is finite since for $|\gamma|\ge\delta$, $\mathtt{M}(\mu)^\gamma=0$, so the entries of $\mathcal{N}_{\mathbf{z},\mu}(p)$ are polynomials in μ and $\mathbf{z}$.

Note that for the specialization at $(\mathbf{z},\mu)=(\xi,\nu)$ the matrices $\mathtt{M}_i(\mu)$ $(i=1,\dots,n)$ are commuting and we have

$$\mathcal{N}_{\xi,\nu}(p) = [\Lambda_{\alpha_1}(p),\dots,\Lambda_{\alpha_\delta}(p)]^t\in\mathbb{C}^\delta.$$

4.1 The multiplicity structure equations of a singular point

We can now characterize the multiplicity structure by polynomial equations.

Theorem 4.6. *Let $\mathbb{K}\subset\mathbb{C}$ be any field, $\mathbf{f}\in\mathbb{K}[\mathbf{x}]^N$ and let $\xi\in\mathbb{C}^n$ be an isolated solution of $\mathbf{f}$. Let $\mathtt{M}_i(\mu)$ for $i=1,\dots n$ be the parametric multiplication matrices as in (9) and $\mathcal{N}_{\xi,\mu}$ be the parametric normal form as in Defn. 4.5 at $\mathbf{z}=\xi$. Then the ideal J_ξ of $\mathbb{C}[\mu]$ generated by the polynomial system*

$$\begin{cases} \mathcal{N}_{\xi,\mu}(f_k) & \text{for } k=1,\dots,N, \\ \mathtt{M}_i(\mu)\cdot\mathtt{M}_j(\mu)-\mathtt{M}_i(\mu)\cdot\mathtt{M}_i(\mu) & \text{for } i,j=1,\dots,n\end{cases} \quad (10)$$

is the maximal ideal

$$\mathfrak{m}_\nu = (\mu_{\alpha,\beta}-\nu_{\alpha,\beta},(\alpha,\beta)\in E\times\partial(E))$$

where $\nu_{\alpha,\beta}$ are the coefficients of the dual basis defined in (4).

Proof. As before, the system (10) has a solution $\mu_{\alpha,\beta}=\nu_{\alpha,\beta}$ for $(\alpha,\beta)\in E\times\partial(E)$. Thus $J_\xi\subset\mathfrak{m}_\nu$.

Conversely, let $C=\mathbb{C}[\mu]/J_\xi$ and consider the map

$$\Phi : C[\mathbf{x}]\to C^\delta,\ \ p\mapsto\mathcal{N}_{\xi,\mu}(p).$$

Let K be its kernel. Since the matrices $\mathtt{M}_i(\mu)$ are commuting modulo J_ξ, we can see that K is an ideal. As $f_k\in K$, we have $\mathcal{I}:=(f_k)\subset K$.

Next we show that $Q\subset K$. By construction, for any $\alpha\in\mathbb{N}^n$ we have modulo J_ξ

$$\mathcal{N}_{\xi,\mu}((\mathbf{x}-\xi)^\alpha) = \sum_{\gamma\in\mathbb{N}^n}\frac{1}{\gamma!}\boldsymbol{\partial}_\xi^\gamma((\mathbf{x}-\xi)^\alpha)\,\mathtt{M}(\mu)^\gamma[1] = \mathtt{M}(\mu)^\alpha[1].$$

Using the previous relation, we check that $\forall p,q\in C[\mathbf{x}]$,

$$\Phi(pq) = p(\xi+\mathtt{M}(\mu))\Phi(q) \quad (11)$$

where $p(\xi+\mathtt{M}(\mu))$ is obtained by replacing $x_i-\xi_i$ by $\mathtt{M}_i(\mu)$. Let $q\in Q$. As Q is the $\mathfrak{m}_\xi$-primary component of $\mathcal{I}$, there exists $p\in\mathbb{C}[\mathbf{x}]$ s.t. $p(\xi)\neq 0$ and $p\,q\in\mathcal{I}$. By (11), we have

$$\Phi(p\,q) = p(\xi+\mathtt{M}(\mu))\Phi(q) = 0.$$

Since $p(\xi) \neq 0$ and $p(\xi + \mathtt{M}(\mu)) = p(\xi)Id + N$ with N lower triangular and nilpotent, $p(\xi+\mathtt{M}(\mu))$ is invertible. We deduce that $\Phi(q) = p(\xi + \mathtt{M}(\mu))^{-1}\Phi(pq) = 0$ and $q \in K$.

Let us show now that Φ is surjective and more precisely, that $\phi((\mathbf{x} - \xi)^{\alpha_k}) = \mathbf{e}_k$ (abusing the notation as here $\mathbf{e}_k$ has length δ not n). Since B is connected to 1, either $\alpha_k = 0$ or there exists $\alpha_j \in E$ such that $\alpha_k = \alpha_j + \mathbf{e}_i$ for some $i \in \{1, \dots, n\}$. Thus the $j - th$ column $\mathtt{M}_i(\mu)$ is $\mathbf{e}_k$ by (8). As $\{\mathtt{M}_i(\mu) : i = 1, \dots, n\}$ are pairwise commuting, we have $\mathtt{M}(\mu)^{\alpha_k} = \mathtt{M}_j(\mu)\mathtt{M}(\mu)^{\alpha_j}$, and if we assume by induction on $|\alpha_j|$ that the first column of $\mathtt{M}(\mu)^{\alpha_j}$ is $\mathbf{e}_j$, we obtain $\mathtt{M}(\mu)^{\alpha_k}[1] = \mathbf{e}_k$. Thus, for $k = 1, \dots, \delta$, $\Phi((\mathbf{x} - \xi)^{\alpha_k}) = \mathbf{e}_k$.

We can now prove that $\mathfrak{m}_\nu \subset J_\xi$. As $M_i(\nu)$ is the multiplication by $(x_i - \xi_i)$ in $\mathbb{C}[\mathbf{x}]/Q$, for any $b \in B$ and $i = 1, \dots, n$, we have $(x_i - \xi_i)\, b = M_i(\nu)(b) + q$ with $q \in Q \subset K$. We deduce that for $k = 1, \dots, \delta$,

$$\Phi((x_i - \xi_i)\,(\mathbf{x} - \xi)^{\alpha_k}) = \mathtt{M}_i(\mu)\Phi((\mathbf{x} - \xi)^{\alpha_k}) = \mathtt{M}_i(\mu)(\mathbf{e}_k) = \mathtt{M}_i(\nu)(\mathbf{e}_k).$$

This shows that $\mu_{\alpha,\beta} - \nu_{\alpha,\beta} \in J_\xi$ for $(\alpha, \beta) \in E \times \partial(E)$ and that $\mathfrak{m}_\nu = J_\xi$. □

In the proof of the next theorem we need to consider cases when the multiplication matrices do not commute. We introduce the following definition:

Definition 4.7. Let $\mathbb{K} \subset \mathbb{C}$ be any field. Let $\mathcal{C}$ be the ideal of $\mathbb{K}[\mathbf{z}, \mu]$ generated by entries of the commutation relations: $\mathtt{M}_i(\mu) \cdot \mathtt{M}_j(\mu) - \mathtt{M}_j(\mu) \cdot \mathtt{M}_i(\mu) = 0$, $i, j = 1, \dots, n$. We call $\mathcal{C}$ the *commutator ideal.*

Lemma 4.8. *For any field $\mathbb{K} \subset \mathbb{C}$ and for any $p \in \mathbb{K}[\mathbf{x}]$, $i = 1, \dots, n$,*

$$\mathcal{N}_{\mathbf{z},\mu}(x_i p) = x_i \mathcal{N}_{\mathbf{z},\mu}(p) + \mathtt{M}_i(\mu)\,\mathcal{N}_{\mathbf{z},\mu}(p) + O_{i,\mu}(p). \qquad (12)$$

where $O_{i,\mu} : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{z}, \mu]^\delta$ is linear with image in the commutator ideal $\mathcal{C}$.

Proof.

$$\begin{aligned}
&\mathcal{N}_{\mathbf{z},\mu}(x_i p)\\
&= \sum_\gamma \frac{1}{\gamma!}\,\partial_{\mathbf{z}}^\gamma(x_i p)\,\mathtt{M}(\mu)^\gamma[1]\\
&= x_i \sum_\gamma \frac{1}{\gamma!}\partial_{\mathbf{z}}^\gamma(p)\,\mathtt{M}(\mu)^\gamma[1] + \sum_\gamma \frac{1}{\gamma!}\gamma_i\,\partial_{\mathbf{z}}^{\gamma - e_i}(p)\,\mathtt{M}(\mu)^\gamma[1]\\
&= x_i \sum_\gamma \frac{1}{\gamma!}\partial_{\mathbf{z}}^\gamma(p)\,\mathtt{M}(\mu)^\gamma[1] + \sum_\gamma \frac{1}{\gamma!}\partial_{\mathbf{z}}^\gamma(p)\,\mathtt{M}(\mu)^{\gamma + e_i}[1]\\
&= x_i\,\mathcal{N}_{\mathbf{z},\mu}(p) + \mathtt{M}_i(\mu)\left(\sum_\gamma \frac{1}{\gamma!}\partial_{\mathbf{z}}^\gamma(p)\,\mathtt{M}(\mu)^\gamma[1]\right)\\
&\qquad + \sum_\gamma \frac{1}{\gamma!}\,\partial_{\mathbf{z}}^\gamma(p)\,O_{i,\gamma}(\mu)[1]
\end{aligned}$$

where $O_{i,\gamma} = \mathtt{M}_i(\mu)\mathtt{M}(\mu)^\gamma - \mathtt{M}(\mu)^{\gamma+e_i}$ is a $\delta \times \delta$ matrix with coefficients in $\mathcal{C}$. Therefore, $O_{i,\mu} : p \mapsto \sum_\gamma \frac{1}{\gamma!}\partial_{\mathbf{z}}^\gamma(p)\,O_{i,\gamma}(\mu)[1]$ is a linear functional of p with coefficients in $\mathcal{C}$. □

The next theorem proves that the system defined as in (10) for general $\mathbf{z}$ has (ξ, ν) as a simple root.

Theorem 4.9. *Let $\mathbf{f} \in \mathbb{K}[\mathbf{x}]^N$ and $\xi \in \mathbb{C}^n$ be as above. Let $\mathtt{M}_i(\mu)$ for $i = 1, \dots n$ be the parametric multiplication matrices defined in (9) and $\mathcal{N}_{\mathbf{x},\mu}$ be the parametric normal form as in Defn. 4.5. Then $(\mathbf{z}, \mu) = (\xi, \nu)$ is an isolated root with multiplicity one of the polynomial system in $\mathbb{K}[\mathbf{z}, \mu]$:*

$$\begin{cases} \mathcal{N}_{\mathbf{z},\mu}(f_k) = 0 & \text{for } k = 1, \dots, N,\\ \mathtt{M}_i(\mu) \cdot \mathtt{M}_j(\mu) - \mathtt{M}_j(\mu) \cdot \mathtt{M}_i(\mu) = 0 & \text{for } i, j = 1, \dots, n. \end{cases} \qquad (13)$$

Proof. For simplicity, let us denote the (non-zero) polynomials appearing in (13) by

$$P_1, \dots, P_M \in \mathbb{K}[\mathbf{z}, \mu],$$

where $M \leq N\delta + n(n-1)(\delta - 1)(\delta - 2)/4$. To prove the theorem, it is sufficient to prove that the columns of the Jacobian matrix of the system $[P_1, \dots, P_M]$ at $(\mathbf{z}, \mu) = (\xi, \nu)$ are linearly independent. The columns of this Jacobian matrix correspond to the elements in $\mathbb{C}[\mathbf{z}, \mu]^*$

$$\partial_{1,\xi}, \dots, \partial_{n,\xi}, \text{ and } \partial_{\mu_{\alpha,\beta}} \text{ for } (\alpha, \beta) \in E \times \partial(E),$$

where $\partial_{i,\xi}$ defined in (1) for $\mathbf{z}$ replacing $\mathbf{x}$, and $\partial_{\mu_{\alpha,\beta}}$ is defined by

$$\partial_{\mu_{\alpha,\beta}}(q) = \frac{dq}{d\mu_{\alpha,\beta}}\Big|_{(\mathbf{z},\mu)=(\xi,\nu)} \qquad \text{for } q \in \mathbb{C}[\mathbf{z}, \mu].$$

Suppose there exist $a_1, \dots, a_n$, and $a_{\alpha,\beta} \in \mathbb{C}$ for $(\alpha, \beta) \in E \times \partial(E)$ not all zero such that

$$\Delta := a_1\partial_{1,\xi} + \cdots + a_n\partial_{n,\xi} + \sum_{\alpha,\beta} a_{\alpha,\beta}\partial_{\mu_{\alpha,\beta}} \in \mathbb{C}[\mathbf{z}, \mu]^*$$

vanishes on all polynomials $P_1, \dots, P_M$ in (13). In particular, for an element $P_i(\mu)$ corresponding to the commutation relations and any polynomial $Q \in \mathbb{C}[\mathbf{x}, \mu]$, using the product rule for the linear differential operator Δ we get

$$\Delta(P_i Q) = \Delta(P_i)Q(\xi, \nu) + P_i(\nu)\Delta(Q) = 0$$

since $\Delta(P_i) = 0$ and $P_i(\nu) = 0$. By the linearity of Δ, for any polynomial C in the commutator ideal $\mathcal{C}$, we have $\Delta(C) = 0$.

Furthermore, since $\Delta(\mathcal{N}_{\mathbf{z},\mu}(f_k)) = 0$ and

$$\mathcal{N}_{\xi,\nu}(f_k) = [\Lambda_{\alpha_1}(f_k), \dots, \Lambda_{\alpha_\delta}(f_k)]^t,$$

we get that

$$(a_1\partial_{1,\xi} + \cdots, a_n\partial_{n,\xi}) \cdot \Lambda_{\alpha_\delta}(f_k) + \sum_{|\gamma| \leq |\alpha_\delta|} p_\gamma(\nu)\ \partial_{\gamma,\xi}(f_k) = 0 \qquad (14)$$

where $p_\gamma \in \mathbb{C}[\mu]$ are some polynomials in the variables μ that do not depend on f_k. If $a_1, \dots, a_n$ are not all zero, we have an element $\tilde{\Lambda}$ of $\mathbb{C}[\boldsymbol{\partial}_\xi]$ of order strictly greater than $\mathrm{ord}(\Lambda_{\alpha_\delta}) = o$ that vanishes on $f_1, \dots, f_N$.

Let us prove that this higher order differential also vanishes on all multiples of f_k for $k = 1, \dots, N$. Let $p \in \mathbb{C}[\mathbf{x}]$ such that $\mathcal{N}_{\xi,\nu}(p) = 0$, $\Delta(\mathcal{N}_{\mathbf{z},\mu}(p)) = 0$. By (12), we have

$$\begin{aligned}
&\mathcal{N}_{\xi,\nu}((x_i - \xi_i)p)\\
&\quad = (x_i - \xi_i)\mathcal{N}_{\xi,\nu}(p) + \mathtt{M}_i(\nu)\mathcal{N}_{\xi,\nu}(p) + O_{i,\nu}(p) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\Delta(\mathcal{N}_{\mathbf{z},\mu}((x_i - \xi_i)p))\\
&= \Delta((x_i - \xi_i)\mathcal{N}_{\mathbf{z},\mu}(p)) + \Delta(\mathtt{M}_i(\mu)\mathcal{N}_{\mathbf{z},\mu}(p)) + \Delta(O_\mu(p))\\
&= \Delta(x_i - \xi_i)\mathcal{N}_{\xi,\nu}(p) + (\xi_i - \xi_i)\Delta(\mathcal{N}_{\mathbf{z},\mu}(p))\\
&\qquad + \Delta(\mathtt{M}_i(\mu))\mathcal{N}_{\xi,\mu}(p) + \mathtt{M}_i(\nu)\Delta(\mathcal{N}_{\mathbf{z},\mu}(p))\\
&\qquad + \Delta(O_{i,\mu}(p))\\
&= 0.
\end{aligned}$$

As $\mathcal{N}_{\xi,\nu}(f_k) = 0$, $\Delta(\mathcal{N}_{\mathbf{z},\mu}(f_k)) = 0$, $i = 1, \ldots, N$, we deduce by induction on the degree of the multipliers and by linearity that for any element f in the ideal I generated by $f_1, \ldots, f_N$, we have

$$\mathcal{N}_{\xi,\nu}(f) = 0 \quad \text{and} \quad \Delta(\mathcal{N}_{\mathbf{z},\mu}(f)) = 0,$$

which yields $\tilde{\Lambda} \in I^{\perp}$. Thus we have $\tilde{\Lambda} \in I^{\perp} \cap \mathbb{C}[\boldsymbol{\partial}_{\xi}] = Q^{\perp}$ (by Lemma 2.1). As there is no element of degree strictly bigger than o in $Q^{\perp}$, this implies that

$$a_1 = \cdots = a_n = 0.$$

Then, by specialization at $\mathbf{x} = \xi$, Δ yields an element of the kernel of the Jacobian matrix of the system (10). By Theorem 4.6, this Jacobian has a zero-kernel, since it defines the simple point ν. We deduce that $\Delta = 0$ and (ξ, ν) is an isolated and simple root of the system (13). $\square$

The following corollary applies the polynomial system defined in (13) to refine the precision of an approximate multiple root together with the coefficients of its Macaulay dual basis. The advantage of using this, as opposed to using the Macaulay multiplicity matrix, is that the number of variables is much smaller, as was noted above.

Corollary 4.10. *Let $\mathbf{f} \in \mathbb{K}[\mathbf{x}]^N$ and $\xi \in \mathbb{C}^n$ be as above, and let $\Lambda_{\alpha_0}(\nu), \ldots, \Lambda_{\alpha_{d-1}}(\nu)$ be its dual basis as in (4). Let $E \subset \mathbb{N}^n$ be as above. Assume that we are given approximates for the singular roots and its inverse system as in (4)*

$$\tilde{\xi} \cong \xi \ \text{ and } \ \tilde{\nu}_{\alpha_i,\beta} \cong \nu_{\alpha_i,\beta} \ \ \forall \alpha_i \in E,\ \beta \notin E,\ |\beta| \leq o.$$

Consider the overdetermined system in $\mathbb{K}[\mathbf{z}, \mu]$ from (13). Then a random square subsystem of (13) will have $\mathbf{z} = \xi, \mu = \nu$ a simple root with high probability. Thus, we can apply Newton's method for this square subsystem to refine $\tilde{\xi}$ and $\tilde{\nu}_{\alpha_i,\beta}$ for $(\alpha_i, \beta) \in E \times \partial(E)$. For $\tilde{\nu}_{\alpha_i,\gamma}$ with $\gamma \notin E^+$ we can use (7) for the update.

Example 4.11. Reconsider the setup from Ex. 3.2 with primal basis $\{1, x_2\}$ and $E = \{(0,0), (0,1)\}$. We obtain

$$\mathtt{M}_1(\mu) = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \quad \text{and} \quad \mathtt{M}_2(\mu) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The resulting deflated system in (13) is

$$F(z_1, z_2, \mu) = \begin{bmatrix} z_1 + z_2^2 \\ \mu + 2z_2 \\ z_1^2 + z_2^2 \\ 2\mu z_1 + 2z_2 \end{bmatrix}$$

which has a nonsingular root at $(z_1, z_2, \mu) = (0, 0, 0)$ corresponding to the origin with multiplicity structure $\{\partial_1, \partial_{x_2}\}$.

5. EXAMPLES

Computations for the following examples can be found at `www.nd.edu/~jhauenst/deflation/`.

5.1 A family of examples

For each $d \geq 2$, we consider $F_d(x_1, x_2, x_3) = \{x_1, x_2^2, x_3^d\}$ having a multiplicity $2d$ root at the origin and primal basis

$$\left\{ \begin{array}{cccc} 1, & x_3, & \ldots, & x_3^{d-1} \\ x_2, & x_2x_3, & \ldots, & x_2x_3^{d-1} \end{array} \right\}.$$

The following compares using our approach described in § 4 with an approach using the null spaces of Macaulay multiplicity matrices for computing both the singular point and multiplicity structure together (see for example [5, 12]).

	New approach		Null space	
d	Poly	Var	Poly	Var
2	17	12	39	21
3	36	25	153	73
4	65	44	423	192
5	104	69	948	417
6	153	100	1851	795

In particular, the number of polynomials and variables grows quadratically in d in our new approach and quartically in d based on using null spaces of Macaulay matrices.

5.2 Caprasse system

Following [5, § 7.7], we consider the Caprasse system

$$f(x_1, x_2, x_3, x_4) = \begin{bmatrix} {x_1}^3 x_3 - 4\,x_1 {x_2}^2 x_3 - 4\,{x_1}^2 x_2 x_4 - 2\,{x_2}^3 x_4 - 4\,{x_1}^2 + \\ \quad 10\,{x_2}^2 - 4\,x_1 x_3 + 10\,x_2 x_4 - 2, \\ x_1 {x_3}^3 - 4\,x_2 {x_3}^2 x_4 - 4\,x_1 x_3 {x_4}^2 - 2\,x_2 {x_4}^3 - 4\,x_1 x_3 + \\ \quad 10\,x_2 x_4 - 4\,{x_3}^2 + 10\,{x_4}^2 - 2, \\ {x_2}^2 x_3 + 2\,x_1 x_2 x_4 - 2\,x_1 - x_3, \\ {x_4}^2 x_1 + 2\,x_2 x_3 x_4 - 2\,x_3 - x_1 \end{bmatrix}$$

at the multiplicity 4 root $\xi = (2, -\sqrt{-3}, 2, \sqrt{-3})$.

We first consider simply deflating the root. Using the approaches of [5, 8, 11], one iteration suffices. For example, using an extrinsic and intrinsic version of [5, 11], the resulting system consists of 10 and 8 polynomials, respectively, and 8 and 6 variables, respectively. Following [8], using all minors results in a system of 20 polynomials in 4 variables which can be reduced to a system of 8 polynomials in 4 variables using the 3×3 minors containing a full rank 2×2 submatrix. The approach of § 3 using an $|\boldsymbol{i}| = 1$ step creates a deflated system consisting of 6 polynomials in 4 variables. In fact, since the null space of the Jacobian at the root is 2 dimensional, adding two polynomials is necessary and sufficient.

Next, we consider the computation both the point and multiplicity structure. Using an intrinsic null space approach via a second order Macaulay matrix, the resulting system consists of 64 polynomials in 37 variables. In comparison, the approach of § 4 using the primal basis $\{1, x_1, x_2, x_1x_2\}$ constructs a system of 30 polynomials in 19 variables.

5.3 Examples with multiple iterations

In our last set of examples, we consider simply deflating a root of the last three systems from [5, § 7] and a system from [10, § 1], each of which required more than one iteration to deflate. These four systems and corresponding points are:

1: $\{x_1^4 - x_2x_3x_4, x_2^4 - x_1x_3x_4, x_3^4 - x_1x_2x_4, x_4^4 - x_1x_2x_3\}$ at $(0, 0, 0, 0)$;

2: $\{x^4, x^2y + y^4, z + z^2 - 7x^3 - 8x^2\}$ at $(0, 0, -1)$;

3: $\{14x + 33y - 3\sqrt{5}(x^2 + 4xy + 4y^2 + 2) + \sqrt{7} + x^3 + 6x^2y + 12xy^2 + 8y^3, 41x - 18y - \sqrt{5} + 8x^3 - 12x^2y + 6xy^2 - y^3 + 3\sqrt{7}(4xy - 4x^2 - y^2 - 2)\}$ at $Z_3 \approx (1.5055, 0.36528)$;

4: $\{2x_1 + 2x_1^2 + 2x_2 + 2x_2^2 + x_3^2 - 1, (x_1 + x_2 - x_3 - 1)^3 - x_1^3, (2x_1^3 + 5x_2^2 + 10x_3 + 5x_3^2 + 5)^3 - 1000x_1^5\}$ at $(0, 0, -1)$.

We compare using the following three methods:

A: intrinsic slicing version of [5, 11];

B: isosingular deflation [8] via a maximal rank submatrix;

C: approach of § 3 using an $|\boldsymbol{i}| = 1$ step.

For each of the four examples above, the following lists the multiplicity δ and depth o as well as the number of nonzero distinct polynomials, variables, and iterations for each of the three deflation methods above.

			Method A			Method B			Method C		
	δ	o	Poly	Var	It	Poly	Var	It	Poly	Var	It
1	131	10	16	4	2	22	4	2	16	4	2
2	16	7	24	11	3	11	3	2	12	3	3
3	5	4	32	17	4	6	2	4	6	2	4
4	18	7	96	41	5	54	3	5	22	3	5

For breath one singular points as in system 3, method B and C give the same deflated system. Except for method B on the second system, all three methods required the same number of iterations to deflate the root. For the first and third systems, our new approach matched the best of the other methods and resulted in a significantly smaller deflated system for the last one.

6. REFERENCES

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