Proceedings of the ASME 2020 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
IDETC/CIE 2020
August 16-August 19, 2020, St. Louis, USA

IDETC2020-22409

CURVE COGNATE CONSTRUCTIONS MADE EASY

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ABSTRACT

Cognate linkages are mechanisms that share the same motion, a property that can be useful in mechanical design. This paper treats planar curve cognates, that is, planar mechanisms whose tracing point draws the same curve. While Roberts cognates for planar four-bars are relatively simple to understand from a geometric drawing, the same cannot be said for planar six-bar cognates, especially the intricate diagrams Dijksman presented in cataloging all the known six-bar curve cognates. The purpose of this article is to show how the six-bar cognates can be easily understood using kinematic equations written using complex vectors, giving a simple method for generating these cognates as alternatives in a mechanical design. The simplicity of the approach enables the derivation of cognates for eight-bars and possibly beyond.

1 Introduction

Cognate linkages are mechanisms that share the same motion, and in particular, curve cognates are distinct mechanisms, each with one degree of freedom, whose respective tracing point draws the same curve. Since cognates may occupy different regions of space and have different transmission characteristics, they can be useful in finding a more suitable mechanical design for the same function.

We will show how curve cognates for general planar six-bar linkages can be generated by a simple sequence of operations: interchange certain link rotations, match coefficients in the kinematic equations, and then solve the resulting linear equations. Given knowledge of which link rotations can be interchanged, the procedure is easy to carry out, easy to interpret graphically, and easy to understand. The simplicity of the approach makes it extensible to eight-bars and possibly beyond. We illustrate this by deriving an eight-bar curve cognate.

The more difficult tasks of determining which rotations can be interchanged and showing that this procedure generates all possible cognates are beyond the scope of this paper: these will be addressed in a companion paper [1] that completes the theory of cognates for general six-bar linkages.

The most famous result in cognate theory is from 1875, when Roberts [2] showed that every four-bar coupler curve is triply generated. That result is sometimes called the Roberts-Chebyshev Theorem in recognition of Chebyshev's independent discovery of it three years later [3]. To our knowledge no results on cognates of six-bar linkages were found until the work of Hartenberg and Denavit [4], followed by Roth [5] and Soni [6]. See Nolle [7] (with reference list in [8]) for a historical review as of 1974. Finally, nearly one hundred years after Roberts, Dijksman [9, 10, 11] compiled cognates for all the six-bar planar linkages. Soni also found cognates for certain eight-bars [12].

Dijksman presented his results by means of intricate geometric constructions. Although correct, these drawings and their explanatory text can be rather difficult to decode, presenting a barrier to understanding and using these results. The purpose of this article is to present a simple method of understanding

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and drawing planar cognates using a complex vector approach. We are by no means the first to approach cognates from this direction; indeed, Nolle [7] states that Schor (1941), Schmid (1950), Meyer zur Capellen (1956), and Wunderlich (1958) all used some version of a complex plane formulation in treating Roberts cognates. No doubt there have been others as well, since the complex plane formulation is arguably the most natural way to treat any planar linkage with all rotational joints. Nonetheless, by taking this approach, we not only confirm Dijksman's results but also reveal the basic principles that explain all of them.

The rest of the paper is organized as follows. Section 2 provides background information involving inversions of linkages, types of cognates, and complex vector notation. Section 3 illustrates our methodology on the Sylvester's skew pantograph. Section 4 revisits Roberts cognates for four-bars. Sections 5 and 6 apply the method to selected Stephenson-type and Watt-type six-bar linkages, respectively. Section 7 demonstrates the method on an eight-bar linkage. A short conclusion is provided in Section 8.

2 Background

This section provides an introductory review to mechanism types and their inversions, types of cognates, and complex vector notation as applied to planar linkages. For this paper, we restrict ourselves to planar mechanisms with rigid links connected by rotational (pin) joints. Each joint connects two links, thereby imposing one vector constraint, equivalent to two scalar constraints, requiring that the respective center points of the joint on the two links must coincide. Furthermore, for the purpose of classifying mechanisms with one degree of freedom, we consider only unexceptional mechanisms, being those whose number of freedoms does not change when the link dimensions are perturbed in a general fashion.

2.1 Mechanism Types, Inversions, and Curve Types

The classification of mechanism types considers only the number of links and a list of which pairs of links are connected by a joint. Since a rigid body moving freely in the plane has three degrees of freedom—two translations and one rotation—and each joint imposes two scalar constraints, the Grashof mobility criterion for unexceptional linkages says that the number of degrees of freedom, a.k.a. the mobility M, of an N-link planar mechanism with one link held stationary as the ground link and J rotational joints is $M = 3(N-1) - 2J$. Accordingly, for mobility $M = 1$, N must be even, so the cases to be considered are two-bars, four-bars, six-bars, etc., having $N = 2L$ links and $J = 3L - 2$ joints, where L is the number of independent loops in the mechanism.

A convenient way to represent and categorize linkage types is by type graphs in which nodes correspond to links and edges correspond to joints. We ban graphs that contain any subgraph with $N > 1$ and mobility $M < 1$, such as a triangle, because a subgraph with $M = 0$ is equivalent to a single rigid link and a general linkage with $M < 0$ cannot be assembled. This leaves a unique two-bar type, a unique four-bar type, two six-bar types (known as Watt and Stephenson six-bars), and sixteen eight-bar types, with the number of types growing rapidly for higher N. Table 1 shows type graphs for $N = 2$, 4, and 6.

Inversions of a linkage are obtained by designating one link as the stationary *ground link*, indicated in Table 1 with a triangle symbol (△). There are two inversions of the Watt six-bar and three for the Stephenson six-bar.

For the purpose of drawing a curve in the plane, we additionally designate a link that will contain the tracing point. In the case of a four-bar, this link is called the *coupler*, and its tracing point draws a *four-bar coupler curve*. By analogy, the curves drawn by six-bars and beyond are often also called *coupler curves*. The last column in Table 1 shows all ways to choose a coupler link, indicated with a square symbol (□), such that the resulting coupler curve type is not already produced with smaller N. For example, for the four-bar, only the link opposite the ground is available since the links adjacent to ground produce the same type of curve as the two-link mechanism, namely circles. Similarly, the Watt-2 six-bar yields no new coupler curve types: it can only produce two-bar and four-bar curves. In all, there is one two-bar curve type (circles), one four-bar curve type, two Watt six-bar curve types (Watt-1A and Watt-1B), and four Stephenson six-bar curve types (suffixed as 1, 2A, 2B, and 3).

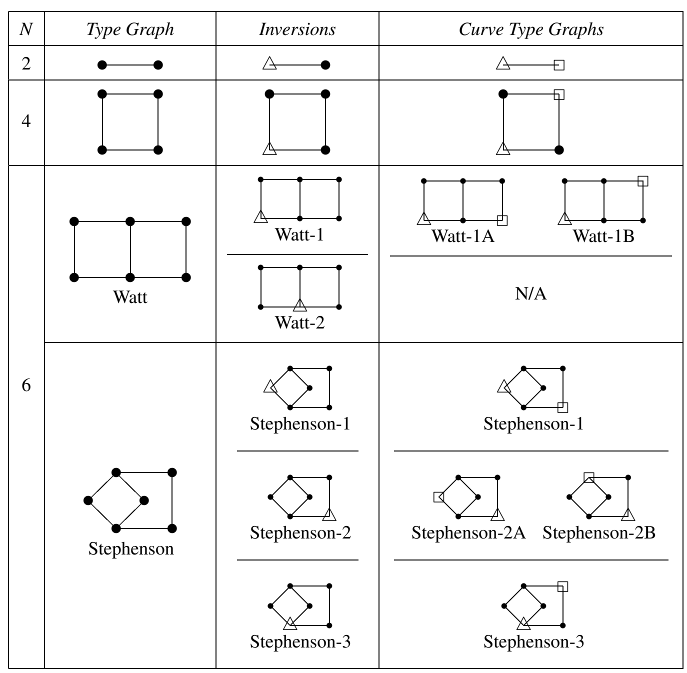


TABLE 1. Mechanism types and coupler curve types up to $N = 6$. A triangle (△) indicates the ground link, and a square (□) indicates the coupler link.

Table 4 of [11, pg. 183] catalogs curve cognates for the four-bar and all the six-bar curve types. A two-bar has no curve cognates since the circle it draws uniquely locates the joint in the ground link at the center of the circle, and the radius of the circle uniquely defines the length of the rotating bar. For the higher linkages, Dijksman reports the number of curve cognates as given in Table 2. We demonstrate how to derive simple formulas for most of these in Sections 4, 5, and 6. Although we omit derivations for the Stephenson-2 cognates due to space considerations, the methodology also applies to these. Furthermore, our methodology also applies to eight-bars and beyond. Since eight-bars are too numerous to beneficially catalog exhaustively, we instead demonstrate the construction of cognates for one illustrative case in Section 7.

Type	*Cognates*	*Type*	*Cognates*
4-bar	3	Stephenson-1	2
Watt-1A	2-D set	Stephenson-2A	4
Watt-1B	4	Stephenson-2B	3
Watt-2	N/A	Stephenson-3	6

TABLE 2. Number of curve cognates for each curve type

2.2 Types of cognates

The cognates under consideration here are curve cognates, that is, linkages which draw the same curve. We only treat cognates of the same curve type, ignoring the possibility that a six-bar might duplicate a four-bar curve or that two types of six-bars might draw the same curve.

Some curve cognates satisfy additional criteria that define subclasses of interest. A *coupler cognate* is a curve cognate where the coupler link maintains the same orientation as the original. Moreover, after selecting an input link, a *timed curve cognate* is a curve cognate with the same functional relationship between the input rotation and the point on the coupler curve. Finally, a *timed coupler cognate* is both a coupler cognate and a timed curve cognate with respect to an input link. In either type of timed cognate, we only allow links adjacent to ground to be selected as the input. Once a curve cognate is found, it is straightforward to check these additional criteria and we will do so.

Another class of cognates that have been considered elsewhere are *function cognates*. These cognates maintain the functional relationship between an input crank and an output link. Since our focus is on curve cognates and some function cognates are not curve cognates, we will not consider function cognates in the present study.

2.3 Complex vector notation

To simplify the mathematical formulas used to represent linkages and compute cognates, we use a complex vector formulation. Thus, a vector $[a\ b]$ in the plane is represented by a complex number $a+bi$ where $i=\sqrt{-1}$ which can always be cast in the form $se^{i\Theta}$ where s is a scalar and Θ is an angle in radians. Complex arithmetic facilitates geometric transformations. In particular, complex addition implements translation, while multiplication by $se^{i\Theta}$ corresponds to a *stretch-rotation*, which stretches by s and performs a complex rotation by angle Θ. Throughout this paper, we use θ to abbreviate the complex rotation, $\theta = e^{i\Theta}$, and more specifically, after numbering the links of a mechanism, θ_j is the complex rotation of link j. By convention, we will always number the ground link as link 0, which does not move.

To illustrate the complex vector notation, we begin with the case of a four-bar linkage. Referring to Figure 1, we have a loop closure equation

$$a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3 = 0 \tag{1}$$

and a coupler point equation

$$p = a_0 + a_1\theta_1 + b_2\theta_2. \tag{2}$$

Note that by subtracting one from the other, we have an alternate coupler point equation

$$p = b_0 + (b_2 - a_2)\theta_2 - a_3\theta_3. \tag{3}$$

This is just the sum of vectors going on a different path from the origin to the coupler point. Although equivalent, one of (2) or (3) will prove more convenient depending on which of the two cognates to the initial linkage one wishes to pursue.

The link dimensions and the placement of the ground pivots in the plane are given by a_0, b_0, a_1, a_2, b_2, a_3. To compute

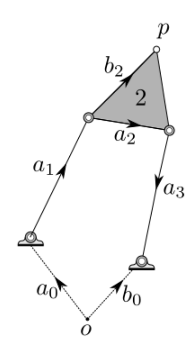


FIGURE 1. Four-bar linkage

the mechanism's motion, consider that given one link rotation, say θ_1, one can solve (1) for θ_2 and θ_3, keeping in mind that the complex loop equation is equivalent to two scalar equations (by taking its real and imaginary parts) and the rotations are each parameterized by a single scalar angle. Then, one can evaluate the coupler point position using (2) or (3). A more facile approach based on using the complex conjugate of the loop equation is presented in [13, 14]. In this paper, we will not need to solve the loop and coupler point equations; instead, we merely need to show that cognate mechanisms satisfy the same equations and therefore produce the same coupler curve.

3 Skew Pantograph

Before proceeding to an analysis of four-bar and six-bar cognates, let us consider the *skew pantograph*, a mechanism discovered by Sylvester [15] that is capable of reproducing a scaled translate of a given curve. Not only are cognates for the four-bar and Stephenson-3 mechanisms directly derivable via the skew pantograph construction, but also our derivation of the mechanism will be the first illustration of our methodology for deriving all of the cognates.

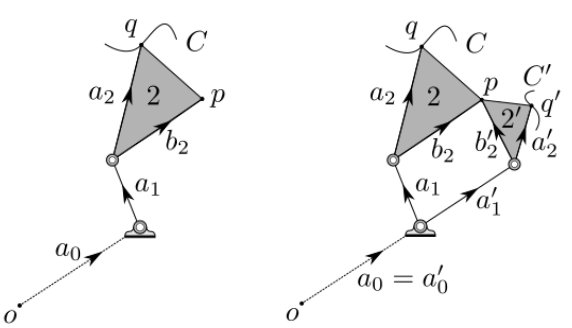
Consider the *dyad coupler mechanism* of Figure 2(a) in which point q of link 2 is constrained to follow curve C. In this manner, link 2 is a coupler between two curves: C and the circle drawn by link 1. As the mechanism moves, point p traces out a coupler curve. We wish to find a curve cognate mechanism of the same type, where curve C is replaced by a similar curve C', that is, C' is a stretch-rotation β of C about a center α:

$$C' = \alpha + \beta(C - \alpha), \tag{4}$$

where α, β are complex numbers.

Theorem 3.1 (Dyad coupler cognate). *For the coupler mechanism as in Figure 2(a) given by link parameters a_0, a_1, a_2, b_2, and curve C, there exists a coupler curve cognate that is another dyad coupler mechanism whose curve C' is similar to C, and whose link parameters are given by*

$$a_0' = a_0, \quad a_1' = b_2, \quad a_2' = \frac{b_2}{a_2}a_1, \quad b_2' = a_1, \quad C' = a_0 + \frac{b_2}{a_2}(C - a_0).$$



(a) (b)

FIGURE 2. Dyad coupler mechanism and skew pantograph

Corollary 3.2 (Skew pantograph (Sylvester [15])). *As in Figure 2(b), by joining two dyads at point p, construct a four-bar linkage with link parameters a_0, a_1, a_2, b_2, $a_1' = b_2$, $a_2' = \frac{b_2}{a_2}a_1$, and $b_2' = a_1$. Then, points q and q' trace out similar curves.*

For clarity, we note that in the skew pantograph, the two triangle links labeled 2 and 2′ in Figure 2(b) are similar.

While the ability of the skew pantograph to duplicate curve C is interesting in itself, our interest will be in viewing it as two coupler mechanisms that are curve cognates for the path traced out by point p. For consistency with the way we present cognates below, we present the mapping between x and $f(x)$ in the form of a table, as follows with $\beta = \frac{b_2}{a_2}$:

	Link Rotations		Mechanism Parameters				
Cognate	link 1	link 2	link 0	link 1	link 2		curve
x	θ_1	θ_2	a_0	a_1	a_2	b_2	C
$f(x)$	θ_2	θ_1	a_0	b_2	βa_1	a_1	$a_0 + \beta(C - a_0)$

Proof of Theorem 3.1 Let θ_1 and θ_2 be the rotations of links 1 and 2, respectively, as q moves along C. Summing complex vectors from the origin to q and from the origin to p, we have

$$q = a_0 + a_1\theta_1 + a_2\theta_2, \qquad p = a_0 + a_1\theta_1 + b_2\theta_2. \tag{5}$$

For the cognate linkage, we similarly have

$$q' = a_0' + a_1\theta_1' + a_2'\theta_2', \qquad p' = a_0' + a_1'\theta_1' + b_2'\theta_2'. \tag{6}$$

Now consider interchanging the rotations, that is, let

$$\theta_1' = \theta_2, \qquad \theta_2' = \theta_1, \tag{7}$$

and enforce the relations

$$p' = p, \qquad q' = \alpha + \beta(q - \alpha). \tag{8}$$

Substituting from (5,6) into (7,8) gives

$$a_0' + a_1'\theta_2 + b_2'\theta_1 = a_0 + a_1\theta_1 + b_2\theta_2 \tag{9}$$
$$a_0' + a_1'\theta_2 + a_2'\theta_1 = \alpha + \beta(a_0 + a_1\theta_1 + a_2\theta_2 - \alpha). \tag{10}$$

We wish to find values of a_0', a_1', a_2', b_2', α, and β such that these equations hold for all values of θ_1 and θ_2. Equating the coefficients of 1, θ_1, and θ_2 from both sides of (9) yields

$$a_0' = a_0, \qquad a_1' = b_2, \qquad b_2' = a_1. \tag{11}$$

Doing the same for (10) yields

$$a_0' = \alpha + \beta(a_0 - \alpha), \qquad a_1' = \beta a_2, \qquad a_2' = \beta a_1. \tag{12}$$

Together, (11,12) are six equations for the six parameters of the cognate. Since substituting values from Theorem 3.1 shows that all six equations are satisfied, $f(x)$ is a cognate of x. *End of proof.*

The key steps for constructing this and all the other coupler cognates to follow are:

1. write complex vector equations for the original mechanism and its cognate,
2. interchange some link rotations between the original and the cognate,
3. set the coupler points equal,
4. set the loop equations equal up to stretch-rotation,
5. solve for the cognate parameters that match coefficients of the rotations.

In Step 2 for the dyad mechanism, there was only one candidate for a pair of rotations to swap. For the mechanisms to follow, there are more possibilities, some of which generate cognates and some that do not.

4 Four-Bar: Roberts Cognates

To further illustrate our procedure for deriving curve cognates, we begin by revisiting Roberts Theorem for four-bars. Here, and in similar statements to follow, the cognates are summarized in a tabular form where the first row, labeled x, contains the link rotations and link vectors for the mechanism as labeled in the relevant figure, and subsequent rows show the new values these take in the cognates. For example, below, to reach a particular location on the coupler curve, for the original mechanism x, link 1 has rotation θ_1, whereas for its cognate $f(x)$, link 1 has the rotation θ_2, which is the rotation that applies to link 2 of x when reaching that same point on the curve. Similarly, the ground pivot located at b_0 for x moves to c_0 for cognate $f(x)$.

The following is Roberts-Chebyshev Theorem [2, 3, 16].

Theorem 4.1 (Four-Bar Cognates). *Every general four-bar coupler curve is triply generated, that is, for general four-bar x, there exist curve cognates $f(x)$ and $g(x)$:*

	Link Rotations			*Link Parameters*					
Cognate	*link 1*	*link 2*	*link 3*	*link 0*		*link 1*	*link 2*		*link 3*
x	θ_1	θ_2	θ_3	a_0	b_0	a_1	a_2	b_2	a_3
$f(x)$	θ_2	θ_1	θ_3	a_0	c_0	b_2	γa_1	a_1	γa_3
$g(x)$	θ_1	θ_3	θ_2	c_0	b_0	ζa_1	ζa_3	$-\gamma a_3$	ζa_2

where

$$\gamma = b_2/a_2, \quad \zeta = 1 - b_2/a_2, \quad \textit{and} \quad c_0 = a_0 + \gamma(b_0 - a_0).$$

Corollary 4.2. *The* timed curve cognate *of x is $g(x)$ if link 1 is designated as the input link and it is $f(x)$ if link 3 is the input. There are no* coupler cognates.

The corollary follows from the theorem just by noting which links keep the same rotation between x and its curve cognates. There are no coupler cognates since the rotation of link 2, the coupler, is not the same for any pair of cognates.

Note that Theorem 4.1 is stated as the existence of curve cognates $f(x)$ and $g(x)$ without claiming that no other curve cognates exist. Although it is true that no other cognates exist, we do not prove that here.

Figure 1 showed a four-bar linkage with Figure 3 illustrating Roberts cognates. Note that links on the opposing sides of a parallelogram, of which there are three, undergo the same rotation.

Proof of Theorem 4.1 As a preparatory step, we note that we can multiply the loop equation (1) by any nonzero complex number without changing its meaning, that is, for $\gamma \neq 0$,

$$\gamma(a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3) = 0 \tag{13}$$

implies the same constraint between the rotations as (1).

Let the vector $(a_0, b_0, a_1, a_2, b_2, a_3)$ describe the original four-bar and let $(a_0', b_0', a_1', a_2', b_2', a_3')$ describe a cognate of it. Likewise, the two linkages have rotations $(\theta_1, \theta_2, \theta_3)$ and $(\theta_1', \theta_2', \theta_3')$, respectively. For these two linkages to be curve cognates, they must have the same coupler point so we denote it simply as p. Accordingly, the loop and coupler point equations for the cognate are

$$a_0' - b_0' + a_1'\theta_1' + a_2'\theta_2' + a_3'\theta_3' = 0, \tag{14}$$
$$p = a_0' + a_1'\theta_1' + b_2'\theta_2'. \tag{15}$$

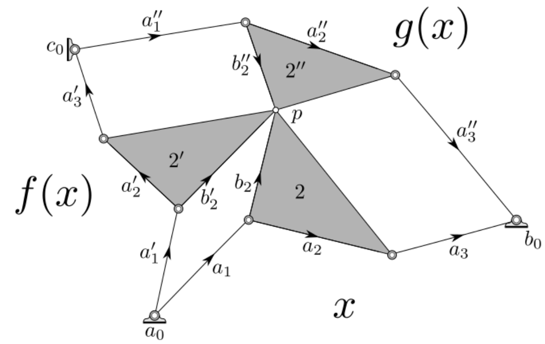


FIGURE 3. Roberts cognates

Curve cognate $f(x)$ is generated by swapping rotations 1 and 2. Substituting $(\theta_1', \theta_2', \theta_3') = (\theta_2, \theta_1, \theta_3)$ into (14,15) and setting these equal to (13,2), respectively, yields

$$\begin{aligned} 0 =& a_0' - b_0' + a_1'\theta_2 + a_2'\theta_1 + a_3'\theta_3 \\ & \gamma(a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3), \end{aligned} \tag{16}$$

$$p = a_0' + a_1'\theta_2 + b_2'\theta_1 = a_0 + a_1\theta_1 + b_2\theta_2. \tag{17}$$

Equating the coefficients of the rotations 1, θ_1, θ_2, θ_3 on both sides of these equations yields a set of seven linear equations in seven unknown parameters, these being $(a_0', b_0', a_1', a_2', b_2', a_3')$ for the cognate linkage and the factor γ. Listing these out in detail, the coupler point equation (17) yields

$$a_0' = a_0, \qquad b_2' = a_1, \qquad a_1' = b_2, \tag{18}$$

while the loop equation (16) gives

$$a_0' - b_0' = \gamma(a_0 - b_0), \quad a_2' = \gamma a_1, \quad a_1' = \gamma a_2, \quad a_3' = \gamma a_3. \tag{19}$$

We notice that expressions for a_1' appear in both (18) and (19). Setting these equal, we obtain $\gamma = b_2/a_2$, which is the stretch-rotation that transforms a_2 into b_2. Rearranging the first entry of (19) shows that b_0' is a new ground pivot, which we rename as $c_0 := b_0' = a_0 + \gamma(b_0 - a_0)$. By these calculations, we see that at each configuration of x, the linkage $f(x)$ gives the same coupler point p, and hence it is a coupler cognate of x.

We may derive $g(x)$ in a similar fashion to our proof of $f(x)$. This time, since $g(x)$ interchanges θ_2 with θ_3, it is advantageous to use the alternative coupler point equation (3). Denoting the link parameters for $g(x)$ as a_0'', b_0'', etc., and this time rescaling the loop equation (1) by ζ, we have

$$\begin{aligned} 0 =& a_0'' - b_0'' + a_1''\theta_1 + a_2''\theta_3 + a_3''\theta_2 \\ &= \zeta(a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3), \end{aligned} \tag{20}$$

$$p = b_0'' + (b_2'' - a_2'')\theta_3 - a_3''\theta_2 = b_0 + (b_2 - a_2)\theta_2 - a_3\theta_3. \tag{21}$$

Matching term-by-term yields $g(x)$ as given in Theorem 4.1.

End of proof.

Second proof of $g(x)$: Although the following derivation of $g(x)$ does not generalize for use in the treatment of six-bars, it shows the group structure of four-bar cognates. It relies on the fact that links 1 and 3 play equivalent roles in the four-bar, and so swapping the labeling of the links gives two ways of writing the same four-bar. Since we traverse the loop in the opposite direction, the signs of some parameters reverse:

	Link Rotations			Link Parameters					
Cognate	link 1	link 2	link 3	link 0		link 1	link 2		link 3
x	θ_1	θ_2	θ_3	a_0	b_0	a_1	a_2	b_2	a_3
$s(x)$	θ_3	θ_2	θ_1	b_0	a_0	$-a_3$	$-a_2$	$b_2 - a_2$	$-a_1$

Given $f(x)$ and $s(x)$, one finds that $g(x) = s(f(s(x)))$.
End of second proof for $g(x)$.

One may wonder what happens if the transformations $f(x)$ and $s(x)$ are repeated in various combinations. It is easy to see that each is its own inverse: $x = f(f(x)) = s(s(x))$. Moreover, if they are repeated in alteration, no new cognates result, because $x = s(f(s(g(x))))$. These facts show that repeated application of the transformations $f(x)$ and $s(x)$ results in a closed group of order 6, being the three curve cognates each written two ways (i.e., x and $s(x)$ are the same linkage). This same six-fold grouping appears in solutions to the nine-point coupler curve synthesis problem for four-bars [17].

A final alternative proof is via Theorem 3.1.

Alternative proof: The four-bar can be regarded as a dyad coupler mechanism of the type illustrated in Figure 2(a) where curve C is a circle. Applying Theorem 3.1 to dyad 1-2 with curve C being the circle drawn by link 3 gives cognate $f(x)$, while applying it to dyad 3-2 with curve C being the circle drawn by link 1 gives $g(x)$.
End of alternative proof.

Finally, we note that in the mapping from x to $f(x)$, one fixed pivot stays in place at a_0, while the new one at c_0 is such that triangle a_0, b_0, c_0 is similar to the coupler triangles 2, 2′, and 2″ in Figure 3. Moreover, a_0, b_0, and c_0 are the singular foci of the coupler curve [18]. These are well-known classical results.

Example 4.3. The below table lists the parameters for the four-bar linkage along with the parameters (to 4 decimal places) for the two cognates derived in Theorem 4.1 and drawn in Figure 3.

	x	$f(x)$	$g(x)$
a_0	$0.0 + 0.0i$	$0.0000 + 0.0000i$	$-0.6549 + 2.2196i$
b_0	$3.0 + 0.8i$	$-0.6549 + 2.2196i$	$3.0000 + 0.8000i$
a_1	$0.8 + 0.8i$	$0.2000 + 0.9000i$	$1.4118 + 0.2196i$
a_2	$1.2 - 0.3i$	$-0.6118 + 0.5804i$	$1.2431 - 0.4392i$
b_2	$0.2 + 0.9i$	$0.8000 + 0.8000i$	$0.2431 - 0.7392i$
a_3	$1.0 + 0.3i$	$-0.2431 + 0.7392i$	$1.0000 - 1.2000i$

5 Stephenson-type six-bar cognates

The following applies the techniques provided in Sections 3 and 4 to derive curve cognates for Stephenson-1 and Stephenson-3 six-bar curves. The methodology also applies to the Stephenson-2A and -2B curves, omitted here to save space.

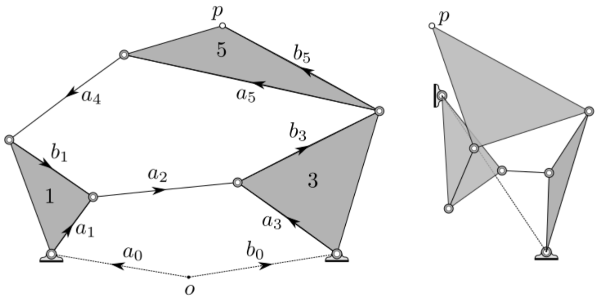


FIGURE 4. Stephenson-1 mechanism and a cognate

5.1 Stephenson-1

Figure 4 displays the notation for the Stephenson-1 six-bar.

Theorem 5.1 (Stephenson-1 Cognates). *Every general Stephenson-1 coupler curve is doubly generated by x and $f(x)$:*

Cognate	*Link Rotations* 1	2	3	4	5	*Link Parameters* 0		1		2	3		4	5	
x	θ_1	θ_2	θ_3	θ_4	θ_5	a_0	b_0	a_1	b_1	a_2	a_3	b_3	a_4	a_5	b_5
$f(x)$	θ_2	θ_1	θ_3	θ_4	θ_5	c_0	b_0	$\gamma_1 a_2$	$\gamma_2 a_2$	$\gamma_1 a_1$	$\gamma_1 a_3$	$\gamma_2 b_3$	$\gamma_2 a_4$	$\gamma_2 a_5$	b_5

such that

$$\gamma_1 = \frac{b_1(a_3+b_3)}{b_1 a_3 + a_1 b_3}, \quad \gamma_2 = \frac{a_1(a_3+b_3)}{b_1 a_3 + a_1 b_3}, \quad c_0 = b_0 + \gamma_1(a_0 - b_0).$$

Corollary 5.2. *Since the rotations of links 3 and 5 are both preserved, if link 3 is the input, this is a* timed coupler cognate*. Otherwise, if link 1 is the input, this is a* coupler cognate*, but its timing is not preserved.*

Proof: Similar to the way we treated curve cognates for the four-bar, we denote a cognate with a'_0 in place of the original a_0, etc. We set loop equations, of which there are now two, for the cognate equal to scaled versions of the loop equations for the original six-bar, and we equate their coupler points. Swapping the rotations of links 1 and 2 in the cognate, we have

$$\begin{aligned} 0 &= a'_0 - b'_0 + a'_1\theta_2 + a'_2\theta_1 - a'_3\theta_3 \\ &\qquad = \gamma_1(a_0 - b_0 + a_1\theta_1 + a_2\theta_2 - a_3\theta_3), \qquad (22) \\ 0 &= b'_1\theta_2 + a'_2\theta_1 + b'_3\theta_3 + a'_4\theta_4 + a'_5\theta_5 \\ &\qquad = \gamma_2(b_1\theta_1 + a_2\theta_2 + b_3\theta_3 + a_4\theta_4 + a_5\theta_5), \qquad (23) \\ p &= b'_0 + (a'_3 + b'_3)\theta_3 + b'_5\theta_5 = b_0 + (a_3+b_3)\theta_3 + b_5\theta_5. \qquad (24) \end{aligned}$$

Equating coefficients in each equation yields the following:

$$\begin{gathered} a'_0 - b'_0 = \gamma_1(a_0 - b_0), \quad a'_1 = \gamma_1 a_2, \quad a'_2 = \gamma_1 a_1, \quad a'_3 = \gamma_1 a_3, \\ b'_1 = \gamma_2 a_2, \quad a'_2 = \gamma_2 b_1, \quad b'_3 = \gamma_2 b_3, \quad a'_4 = \gamma_2 a_4, \quad a'_5 = \gamma_2 a_5, \\ b'_0 = b_0, \quad a'_3 + b'_3 = a_3 + b_3, \quad b'_5 = b_5. \end{gathered}$$

This is a system of linear equations in the unknowns $a'_0, \ldots, b'_5$ and γ_1, γ_2. There are 12 equations for 12 unknowns, having the unique answer given in Theorem 5.1.
End of proof.

One may confirm that $x = f(f(x))$, so this is a group of just two curve cognates.

To demonstrate what would happen if one tries to perform an interchange which is not permitted, consider swapping rotations 2 and 3. To simplify the computation, we replace (24) with

$$a_0 + (a_1 + b_1)\theta_1 - a_4\theta_4 + (b_5 - a_5)\theta_5 = p.$$

Applying the two stretch-rotation factors and equating coefficients in each equation, as above, yields

$$\begin{gathered} a'_0 - b'_0 = \gamma_1(a_0 - b_0), \quad a'_1 = \gamma_1 a_1, \quad a'_2 = -\gamma_1 a_3, \quad a'_3 = -\gamma_1 a_2, \\ b'_1 = \gamma_2 b_1, \quad a'_2 = \gamma_2 b_3, \quad b'_3 = \gamma_2 a_2, \quad a'_4 = \gamma_2 a_4, \quad a'_5 = \gamma_2 a_5, \\ a'_0 = a_0, \quad a'_1 + b'_1 = a_1 + b_1, \quad a'_4 = a_4, \quad b'_5 - a'_5 = b_5 - a_5. \end{gathered}$$

This is a set of 13 linear equations in 12 unknowns. For general choices of the Stephenson-1 parameters $(a_0, b_0, a_1, b_1, a_2, a_3, b_3, a_4, a_5, b_5)$, this system is inconsistent, showing that swapping rotations 2 and 3 does not generally lead to a curve cognate. In fact, performing elimination on the linear system, one finds that there exists a solution if and only if $a_1 a_4(a_3 + b_3) = 0$. But any Stephenson-1 linkage satisfying that condition is degenerate: its coupler point is either stationary or moves on a circle.

It would be laborious to check that no other permutation of the rotations besides the swap of θ_1 and θ_2 leads to a valid cognate. Furthermore, it is not obvious that permuting rotations is the only way to generate a cognate. We address completeness of the cognate theory in a companion paper [1].

Example 5.3. The below table lists the parameters for the Stephenson I linkage along with the parameters (to 4 decimal places) for the cognate derived in Theorem 5.1 and drawn in Figure 4.

	x	$f(x)$		x	$f(x)$
a_0	$-2.0+0.0i$	$-0.7323+1.1043i$	a_3	$-0.7+0.5i$	$0.0198+0.5696i$
b_0	$0.0+0.0i$	$0.0000+0.0000i$	b_3	$1.0+0.5i$	$0.2802+0.4304i$
a_1	$0.3+0.4i$	$0.4214-0.5155i$	a_4	$-0.8-0.6i$	$-0.1777-0.4236i$
b_1	$0.6-0.4i$	$0.3731+0.2719i$	a_5	$-1.8+0.4i$	$-0.8063-0.2595i$
a_2	$1.0+0.1i$	$0.3307-0.0192i$	b_5	$-1.1+0.6i$	$-1.1000+0.6000i$

5.2 Stephenson-3

Figure 5 displays the notation for the Stephenson-3 six-bar.

Theorem 5.4 (Stephenson-3 Cognates (Roth [5])). *Every general Stephenson-3 coupler curve is sextuply generated by x, $f(x)$, $g(x)$,* skew(x)*, $f($*skew$(x))$*, and $g($*skew$(x))$ *where $f(x)$ and $g(x)$ are the same as in Theorem 4.1 leaving the rotations and parameters of links 4 and 5 undisturbed, while*

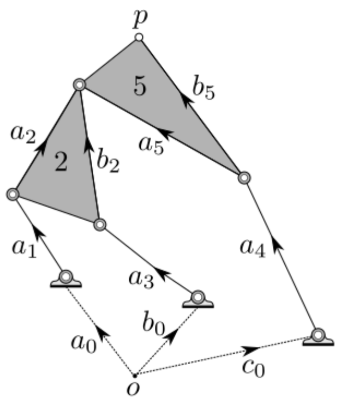


FIGURE 5. Stephenson-3 mechanism

skew(x) *is the result of applying Theorem 3.1 to links 4-5 with curve C being the four-bar curve drawn by links 0,1,2,3. For convenience, we tabulate* skew(x)*:*

Cognate	*Link Rotations* 1	2	3	4	5	*Link Parameters* 0			1	2		3	4	5	
x	θ_1	θ_2	θ_3	θ_4	θ_5	a_0	b_0	c_0	a_1	a_2	b_2	a_3	a_4	a_5	b_5
skew(x)	θ_1	θ_2	θ_3	θ_5	θ_4	d_0	e_0	c_0	βa_1	βa_2	βb_2	βa_3	b_5	βa_4	a_4

where

$$\beta = b_5/a_5, \qquad d_0 = c_0 + \beta(a_0 - c_0), \qquad e_0 = c_0 + \beta(b_0 - c_0).$$

Corollary 5.5. *Both $f(x)$ and $g(x)$ are* coupler cognates *of x.* Timed coupler cognates *are $f(x)$ if link 3 is the input, $g(x)$ if link 1 is the input, and both $f(x)$ and $g(x)$ if link 4 is the input. Meanwhile,* skew(x) *does not preserve the rotation of the coupler, but it is a* timed curve cognate *if either of links 1 or 3 is the input. Timing is preserved for $f(\text{skew}(x))$ if link 3 is the input, and it is preserved for $g(\text{skew}(x))$ if link 1 is the input.*

No further proof is necessary, since these cognates are defined in terms of Theorems 3.1 and 4.1.

6 Watt-type six-bar cognates

As mentioned in Section 2.1, Watt-2 only draws a four-bar curve and will not be considered. The following summarizes cognates for Watt-1A and Watt-1B inversions of six-bar linkages.

6.1 Watt-1A

For all of the six-bars of Stephenson topology considered in Section 5, the resulting system of linear equations had a unique solution for each valid interchange of rotations. However, for the Watt-1A, illustrated in Figure 6, the companion paper [1] shows that no nontrivial interchanges are valid. Nonetheless, it happens that there is a two-dimensional family of cognates that includes the original mechanism, as detailed in the following theorem.

Theorem 6.1 (Watt-1A cognates). *For every general Watt-1A six-bar, there exists a two-dimensional family of cognate mechanisms, summarized as*

Cognate	*Link Rotations* 1	2	3	4	5	*Link Parameters* 0		1	2		3		4	5	
x	θ_1	θ_2	θ_3	θ_4	θ_5	a_0	b_0	a_1	a_2	b_2	a_3	b_3	a_4	a_5	b_5
$f(x,a_0')$	θ_1	θ_2	θ_3	θ_4	θ_5	a_0'	b_0	$\gamma_1 a_1$	$\gamma_1 a_2$	$\gamma_2 b_2$	$\gamma_1 a_3$	$\gamma_2 b_3$	$\gamma_2 a_4$	$\gamma_2 a_5$	b_5

such that

$$\gamma_1 = \frac{a_0' - b_0}{a_0 - b_0}, \qquad \gamma_2 = 1 + \frac{(a_0 - a_0')a_3}{(a_0 - b_0)b_3}$$

and where a_0' can be chosen freely in the complex plane.

Corollary 6.2. *Every member of $f(x,a_0')$ in Theorem 6.1 is a* timed coupler cognate.

If one chooses $a_0' = a_0$, then $f(x,a_0)$ returns the original mechanism, i.e., $x = f(x,a_0)$. The set of cognates is two-dimensional since a_0' has real and imaginary parts.

The corollary holds because the rotation of every link is preserved between x and each member of $f(x,a_0')$.

Proof: All of the link rotations are preserved. Thus, the equations to match term-by-term are

$$\begin{aligned}
0 =& a_0' - b_0' + a_1'\theta_1 + a_2'\theta_2 + a_3'\theta_3 \\
&= \gamma_1(a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3), \\
0 =& b_2'\theta_2 + b_3'\theta_3 + a_4'\theta_4 + a_5'\theta_5 \\
&= \gamma_2(b_2\theta_2 + b_3\theta_3 + a_4\theta_4 + a_5\theta_5), \\
p =& b_0' - (a_3' + b_3')\theta_3 + b_5'\theta_5 = b_0 - (a_3 + b_3)\theta_3 + b_5\theta_5.
\end{aligned}$$

Matching coefficients of the rotations yield 11 linear equations in 12 unknowns, these being 10 link parameters along with γ_1 and γ_2. Letting a_0' be free and solving for the remaining unknowns gives the link parameters as listed in Theorem 6.1.
End of proof.

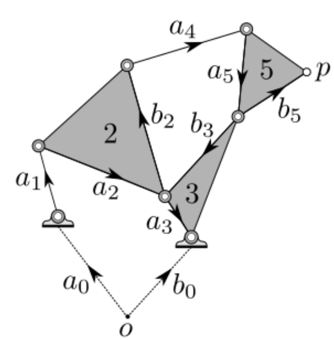


FIGURE 6. Watt-1A mechanism

6.2 Watt-1B

We complete our investigation of six-bars by computing cognates of the Watt-1B, which is shown in Figure 7.

Theorem 6.3 (Watt-1B cognates). *Every general Watt-1B coupler curve is quadruply generated by x, $f(x)$, $g(x)$, and $h(x)$ which are described as follows*

Cognate	*Link Rotations* 1	2	3	4	5
x	θ_1	θ_2	θ_3	θ_4	θ_5
$f(x)$	θ_1	θ_3	θ_2	θ_4	θ_5
$g(x)$	θ_1	θ_2	θ_3	θ_5	θ_4
$h(x)$	θ_1	θ_3	θ_2	θ_5	θ_4

Link Parameters 0		1	2		3		4		5
a_0	b_0	a_1	a_2	b_2	a_3	b_3	a_4	b_4	a_5
c_0	b_0	$\gamma_1 a_1$	$\gamma_1 a_3$	$-a_3$	$-b_2$	$\gamma_2 b_2$	$\gamma_2 a_4$	c_4	$\gamma_2 a_5$
d_0	b_0	$\zeta_1 a_1$	$\zeta_1 a_2$	$\zeta_2 b_2$	$\zeta_1 a_3$	$\zeta_2 b_3$	$\zeta_2 a_5$	$-a_5$	$-b_4$
e_0	b_0	$\lambda_1 a_1$	$\lambda_1 a_3$	$\lambda_2 b_3$	$\lambda_1 a_2$	$\lambda_2 b_2$	$\lambda_2 a_5$	$a_3 a_5/b_3$	$\lambda_2 a_4$

where

$$\gamma_1 = -b_2/a_2, \quad \gamma_2 = -a_3/b_3,$$
$$c_0 = \gamma_1 a_0 + (1-\gamma_1)b_0, \quad c_4 = (1-\gamma_2)a_4 + b_4,$$
$$\zeta_1 = \frac{a_3a_4 + b_3a_4 + b_3b_4}{a_3a_4}, \quad \zeta_2 = -\frac{b_4}{a_4}, \quad d_0 = \zeta_1 a_0 + (1-\zeta_1)b_0,$$
$$\lambda_1 = \frac{b_2b_4}{a_2a_4}, \quad \lambda_2 = -\frac{a_3a_4 + b_3a_4 + b_3b_4}{b_3a_4}, \quad e_0 = \lambda_1 a_0 + (1-\lambda_1)b_0.$$

Corollary 6.4. *Curve cognate $f(x)$ is a* coupler cognate *of x. If link 1 is the input, it is also a* timed coupler cognate *of x. Curve cognate $g(x)$ is a* timed curve cognate *of x for either input, link 1 or link 3. Curve cognate $h(x)$ is a* timed curve cognate *if link 1 is the input.*

Proof: To prove $f(x)$, use loops 0-1-2-3 and 2-3-5-4 and use the path 0-3-2-4 from o to p. Interchange the rotations of links 2 and 3, and follow the usual procedure, scaling the first loop by γ_1 and the second by γ_2.

To prove $g(x)$, use the same two loops, but take the path 0-3-5-4 from o to p. Interchange the rotations of links 4 and 5 and introduce scalings ζ_1 and ζ_2 on the loops.

Curve cognate $h(x)$ is the result of applying both the former mappings: $h(x) = f(g(x))$. *End of proof.*

7 An eight-bar cognate

So far, we have used our approach to provide simple proofs of known results in cognate theory. However, the beauty of the approach is that it easily generates cognates for more complex linkages. To show this, we generate a novel cognate of the eight-bar linkage shown in Figure 8.

The relevant loop and coupler point equations are:

$$a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3 = 0, \tag{25}$$
$$b_1\theta_1 - a_2\theta_2 - b_3\theta_3 + a_4\theta_4 - b_5\theta_5 = 0, \tag{26}$$
$$b_4\theta_4 - a_5\theta_5 + a_6\theta_6 + a_7\theta_7 = 0, \tag{27}$$
$$b_0 + (b_3 - a_3)\theta_3 + (a_5 + b_5)\theta_5 + b_7\theta_7 = p. \tag{28}$$

Consider interchanging rotations 1 and 2; and introduce three stretch-rotation factors, one for each loop equation (25)-(27). After equating coefficients from each kinematic equation, one obtains the cognate parameters which are summarized as:

Cognate	Link Rotations 1	2	3	4	5	6	7
x	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7
$f(x)$	θ_2	θ_1	θ_3	θ_4	θ_5	θ_6	θ_7

Link Parameters 0		1		2	3		4		5		6	7	
a_0	b_0	a_1	b_1	a_2	a_3	b_3	a_4	b_4	a_5	b_5	a_6	a_7	b_7
$\gamma_1 a_0 + (1-\gamma_1)b_0$	b_0	$\gamma_1 a_2$	$-\gamma_2 a_2$	$\gamma_1 a_1$	$\gamma_1 a_3$	$\gamma_2 b_3$	$\gamma_2 a_4$	$\gamma_3 b_4$	$\gamma_3 a_5$	$\gamma_2 b_5$	$\gamma_3 a_6$	$\gamma_3 a_7$	b_7

where

$$\gamma_1 = \frac{b_1(a_3 - b_3)}{a_1b_3 + b_1a_3}, \qquad \gamma_2 = \frac{a_1(b_3 - a_3)}{a_1b_3 + b_1a_3}, \tag{29}$$
$$\gamma_3 = \frac{a_1(a_3b_5 + b_3a_5) + b_1a_3(a_5 + b_5)}{a_5(a_1b_3 + a_3b_1)}. \tag{30}$$

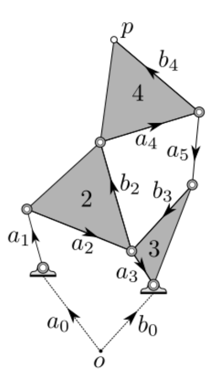


FIGURE 7. Watt-1B mechanism

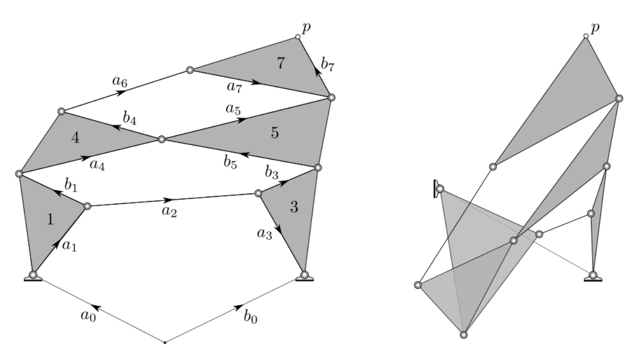


FIGURE 8. An eight-bar mechanism and a cognate

Since rotation 7 is preserved, this is a coupler cognate. When link 3 is the input crank, this is a timed curve and timed coupler cognate. However, it is not a timed curve cognate when link 1 is the input crank.

The companion paper [1] shows that no other permutations of the link rotations are possible.

Example 7.1. The below table lists the parameters for the eight-bar linkage along with the parameters (to 4 decimal places) for the cognate derived above and drawn in Figure 8.

	x	$f(x)$		x	$f(x)$
a_0	$-4.0+0.0i$	$-2.2665+1.2640i$	a_4	$2.1+0.5i$	$0.7494+1.4184i$
b_0	$0.0+0.0i$	$0.0000+0.0000i$	b_4	$-1.5+0.4i$	$-1.4238-0.6638i$
a_1	$0.8+1.0i$	$1.4797-0.6767i$	a_5	$2.5+0.6i$	$1.5503+2.0892i$
b_1	$-1.0+0.5i$	$-1.1130-1.4949i$	b_5	$-2.3+0.4i$	$-1.3503-1.0892i$
a_2	$2.5+0.2i$	$0.7693+0.3138i$	a_6	$1.9+0.6i$	$1.0847+1.6995i$
a_3	$0.7-1.2i$	$0.0174-0.9011i$	a_7	$2.1-0.4i$	$1.8894+1.0534i$
b_3	$0.9+0.4i$	$0.2174+0.6989i$	b_7	$-0.5+0.9i$	$-0.5000+0.9000i$

8 Conclusion

We have presented a method of deriving planar curve cognates and illustrated its application to the four-bar, several six-bars, and one eight-bar linkage. Cognates are found by interchanging link rotations in a complex vector formulation of loop and coupler-point equations, resulting in formulas that are easy to apply in a computer graphics environment. Beyond giving simple derivations for known cognates, the procedure also allows one to find new cognates, as we show by finding a cognate of an example eight-bar linkage. Although for organizational clarity we have reported all the results in a theorem-proof format, our true intent is to present the procedure used inside the proofs.

The procedure depends on the selection of valid permutations of link rotations. As we saw in Section 5.1, not all permutations lead to valid cognates. Thus, at the level of development presented here, some trial-and-error would be required to find all cognates of a given linkage type. Moreover, although all the known planar cognates are obtained by interchanging link rotations, there is no *a priori* reason to believe that this is the only way a cognate can arise. A companion paper [1] addresses these issues, allowing one to confidently find all possible cognates.

9 ACKNOWLEDGMENT

SNS and JDH were partially supported by NSF CCF-181274. SNS was also supported by the Schmitt Leadership Fellowship in Science and Engineering.

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