Sixth-order singularities of the 3RR planar pentad mechanism

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Abstract. The direct kinematics of several planar mechanisms, including 3RPR planar platform robots, reduce to the problem of assembling a 3RR planar structure, also known as the pentad mechanism. A general pentad has at most six isolated real assemblies, which are the solutions of a system of polynomial equations. As the parameters of the mechanism are specialized, the nonsingular roots can merge to form singular roots of multiplicity up to six. While classical results concerning four-bars can be used to derive 3RR mechanisms with multiplicity as high as four, the results presented here are the first to provide a complete solution to the case of multiplicity six, thereby solving a problem formulated by the second author. In such a configuration, the idealized mechanism still has an isolated root but, in practice, when small deformations or joint clearance are allowed, the mechanism can move through a large displacement.

Keywords: pentad mechanism, singularities, numerical algebraic geometry, Dixon determinant, parameter homotopy

1 Introduction

The direct kinematics problem for an n-DOF robot requires solving for the assembly configuration of the device when n inputs are held fixed. Assuming that the device is built with rigid links and algebraic joints, when the n inputs are held constant, the device generally becomes a zero-DOF structure with a finite number of assembly configurations. If the device is in a singular configuration, even though ideally it has just one local assembly configuration, in practice small deflections of the links or small clearances in the joints allow substantial movement, sometimes referred to as "shaky" degree(s) of freedom. Similarly, if one builds an ideally zero-DOF structure in a singular configuration, it may accommodate substantial local deflections. This is not always an undesirable situation, as one might intentionally build such a device and make use of its motion capability. Compared to using a true 1-DOF mechanism, the singular structure may have fewer joints or may be stiffer in directions complementary to the allowed motion. This may be particularly relevant to designing mechanisms that use compliant joints as these must always have a limited range of motion.

In particular, this article analyzes the singular configurations of the 3RR planar pentad mechanism consisting of a coupler link supported relative to a ground link by three legs, all joints being revolute "R" joints (see Figure 1). For brevity, in the rest of this article, we shall refer to this structure as simply the "pentad." Beyond its existence as a structure in its own right, the pentad also arises in the direct kinematics analysis of several planar robot devices, including the 3RPR planar platform robot where the prismatic "P" joints are the inputs. It also arises in the input/output analysis of Stephenson II and III six-bar mechanisms when the input is the dyadic link to ground.

Singularities can be higher-order. If the direct kinematics problem is reduced to a univariate polynomial, the multiplicity of a singularity is the algebraic degree of the corresponding root, e.g., a double root of the polynomial is a multiplicity-two singularity. Since the direct kinematics of a general pentad has six roots, pentads can have singularities of multiplicity from 2 up to 6. Higher-order singularities are of interest since they will generally have a larger shaky motion than one with a lower-order.

Pentads with multiplicity up to four can be constructed using classical results for four-bars. This is because the assembly configurations of a pentad can be found as the intersection of the four-bar coupler curve defined by four of the pentad's links with the circle defined by the final leg. Multiplicity three can be achieved at any point along a four-bar coupler curve by placing the final ground pivot at the center of curvature. Multiplicity four occurs where that center lies on the cubic of stationary curvature.

The case of multiplicity six, the highest possible order, was considered in [7]. Examples for the general case and symmetric case having multiplicity six were demonstrated. Moreover, the problem of counting the number of solutions in the general case was formulated, which is solved below using techniques from numerical algebraic geometry (e.g., see the books [2,8] for a general overview). In particular, we find that the coupler plane of a general four-bar has 72 points that can be connected to a corresponding center point in the ground link to form a multiplicity six pentad structure. The final result is a 72-path parameter homotopy for finding these points, reducible to 36 paths through the use of a two-way symmetry. Furthermore, we study a subcase in which an additional symmetry reduces the homotopy to 24 paths.

The rest of the article is structured as follows. Section 2 summarizes known results for singularities of pentads. Section 3 presents an eliminant for the loop closure equations of the pentad, which is used in Section 4 to formulate a polynomial system describing the multiplicity six conditions. A summary of computational results is provided in Section 5 with Section 6 considering the special case of symmetric pentads. A short conclusion is provided in Section 7.

Notation: Throughout this article, when considering the number of solutions of a polynomial system, we are counting them in the complex number field, which is an upper bound on the number of real roots. We formulate the problem in isotropic coordinates wherein a point P in the plane with Cartesian coordinates (x, y) is represented by the pair of isotropic coordinates $(p, \bar{p}) = (x + iy, x - iy)$,

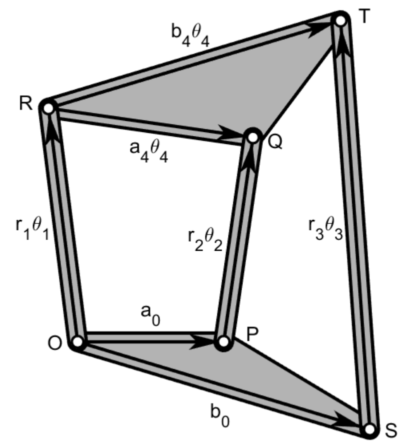

Fig. 1: Pentad structure such that a_0, b_0, a_4, b_4 are vectors in the complex plane, r_1, r_2, r_3 are real leg lengths, and $\theta_1, \theta_2, \theta_3, \theta_4$ are rotations (complex numbers with unit magnitude).

where $i^2 = -1$. It follows that $(x, y) \in \mathbb{R}^2$ if and only if $\bar{p}$ is the complex conjugate of p, which becomes the definition of "real" in isotropic coordinates. Rotation of P about the origin $(0, 0)$ by angle Θ gives the point $(\theta p, \bar{\theta}\bar{p})$, where $\theta = e^{i\Theta}$ and $\bar{\theta} = e^{-i\Theta}$. For real angle Θ, $|\theta| = |\bar{\theta}| = 1$ and $\bar{\theta} = 1/\theta$ so that $\theta\bar{\theta} = 1$.

2 Multiplicity 1 to 6 and ∞

Figure 1 shows a pentad consisting of grounded points O, P, S, connected to points R, Q, T in a coupler link via legs of length $r_1, r_2, r_3 \in \mathbb{R}$. Considering T to be the coupler point of four-bar $OPQR$, we see that the assembly configurations of the pentad correspond to the intersections of that coupler curve with the circle of radius r_3 centered on S. This viewpoint allows one to draw on classical results to construct singular pentads with multiplicity up to 4. Reference texts for the classical results include [4,6], while [7] illustrates their use for constructing singular pentads.

In [7], instances of pentads with multiplicity 6 were found by using Newton's method from an initial guess. It also demonstrated full solutions by Gröbner basis calculations for a subcase where a high degree of symmetry was imposed. Prior to our current contribution, the general case remained open.

Beyond multiplicity six, the only possibility is for the pentad to have a 1-dimensional motion. The only nontrivial case is the double parallelogram mechanism, wherein the whole coupler plane translates around a circle without rotating. There are two additional trivial cases that we will encounter in our study here: when $T = R$ and $S = O$ the pentad degenerates into a four-bar, and the same happens when $T = Q$ and $S = P$.

3 Pentad Eliminant

To prepare for solving the multiplicity 6 case, we first form an eliminant for the loop closure equations of the pentad. We could form the eliminant for any of the link rotations $\theta_1, \theta_2, \theta_3, \theta_4$. We choose to form it for θ_4, the rotation of the coupler triangle. Using complex vector notation as indicated in Figure 1, the loop closure equations are:

$$a_0 - r_1\theta_1 + r_2\theta_2 - a_4\theta_4 = 0, \tag{1}$$
$$b_0 - r_1\theta_1 + r_3\theta_3 - b_4\theta_4 = 0, \tag{2}$$

Applying the Main Result of [9] gives an eliminant for θ_4 as

$$f(\theta_4) = \det \begin{bmatrix} D_1\theta_4 + D_2 & A^T \\ A & -\bar{D}_1\theta_4^{-1} - \bar{D}_2 \end{bmatrix} = 0, \tag{3}$$

$$D_1 = \begin{bmatrix} b_4 & 0 & 0 \\ 0 & (-b_4 + a_4) & 0 \\ 0 & 0 & -a_4 \end{bmatrix}, D_2 = \begin{bmatrix} -b_0 & 0 & 0 \\ 0 & (b_0 - a_0) & 0 \\ 0 & 0 & a_0 \end{bmatrix}, A = \begin{bmatrix} 0 & -r_3 & -r_1 \\ r_3 & 0 & r_2 \\ r_1 & -r_2 & 0 \end{bmatrix},$$

where the entries of $\bar{D}_1$ are the complex conjugates of those of D_1 and similarly for $\bar{D}_2$ and D_2. This means that in $\bar{D}_1$, we replace a_4 with $\bar{a}_4$, etc.

Expanding the determinant gives the eliminant in the form

$$f(\theta_4) = c_0\theta_4^3 + c_1\theta_4^2 + c_3\theta_4 + c_4 + \bar{c}_3\theta_4^{-1} + \bar{c}_2\theta_4^{-2} + \bar{c}_0\theta_4^{-3} = 0. \tag{4}$$

The notation $\bar{c}_i$ indicates a symmetry wherein the last three coefficients are complex conjugates of the first three, while the middle one, c_4, is its own complex conjugate. Multiplying by θ_4^3 clears the reciprocals to turn this into a sextic polynomial in θ_4. To simplify the presentation, let $x = \theta_4$ and rewrite (4) as

$$f(x) = c_0x^6 + c_1x^5 + c_2x^4 + c_3x^3 + c_4x^2 + c_5x + c_6 = 0. \tag{5}$$

Expressions for the coefficients in terms of the pentad parameters are quite long, so we do not report them here. Nonetheless, they are easily generated from (3) using a symbolic mathematical program.

It is significant to note that the link lengths r_1, r_2, and r_3 only appear in the coefficients c_i with even powers, so the degree of the equations can be reduced by making the replacements $q_i = r_i^2$, $i = 1, 2, 3$.

4 Formulation for Multiplicity 6 Singularities

The coefficients of the polynomial (4) are functions of all the parameters of the pentad. To find multiplicity 6 pentads, we take the four-bar $OPQR$ as given, and we solve for points S and T and the leg length r_3. This means that $a_0, \bar{a}_0, a_4, \bar{a}_4, q_1, q_2$ are given, and we wish to find $b_0, \bar{b}_0, b_4, \bar{b}_4, q_3$. Since reflecting the four-bar through line $\overleftrightarrow{OP}$ preserves lengths, the solutions will appear in pairs.

Our first task is to derive the conditions for (4) to have a root of multiplicity 6. This requires $f(x)$ to have the form

$$f(x) = (ux+v)^6 = (u^6)x^6 + (6u^5v)x^5 + (15u^4v^2)x^4 + (20u^3v^3)x^3 + (15u^2v^4)x^2 + (6uv^5)x + (v^6). \quad (6)$$

Comparing this to (5), we see that the coefficients c_i, $i = 0, \ldots, 6$ must satisfy

$$6vc_0 = uc_1, \quad 15vc_1 = 6uc_2, \quad 20vc_2 = 15uc_3, \quad 15vc_3 = 20uc_4, \quad 6vc_4 = 15uc_5, \quad vc_5 = 6uc_6. \quad (7)$$

This results in a polynomial system of six equations in the five unknown pentad parameters, $(b_0, \bar{b}_0, b_4, \bar{b}_4, q_3) \in \mathbb{C}^5$, and $[u, v] \in \mathbb{P}^1$. A solution to this system gives the parameters of the pentad along with the rotation of the coupler link at the singular position, namely $\theta_4 = x = -v/u$.

Examination of the coefficient formulas show that if we divide the mechanism parameters into two groups as $\{q_3, b_0, b_4\}$ and $\{\bar{b}_0, \bar{b}_4\}$, then the coefficients have bidegree $(2, 2)$. Hence, placing $\{u, v\}$ in a third group, each equation in the system (7) has 3-homogeneous degree $(2, 2, 1)$ and the system has a 3-homogeneous Bézout number [8] that is the coefficient of $\alpha^3\beta^2\gamma$ in $(2\alpha+2\beta+\gamma)^6$, which equals 1920. This is an upper bound on the number of solutions of the system, and is equal to the number of paths that will be tracked in a 3-homogeneous homotopy to solve the system.

In Section 2, for multiplicity ∞, we noted that the pentad degenerates into a four-bar with a one-dimensional motion if S, T are matched to either O, P or R, Q. These correspond to the vanishing of all the coefficients c_i, $i = 0, \ldots, 6$. Some of the 1920 paths of a 3-homogeneous homotopy will terminate on these sets in singularities that slow down computation. Moreover, the one-dimensional motion of the pentad corresponds to $[u, v]$ being undetermined. In the case that we try to solve the system symbolically with a Gröbner basis algorithm, these curves prevent finding the isolated solutions we seek without additional computations, e.g., saturation. To remove these one-dimensional components, it suffices to perform a so-called probabilistic saturation, e.g., see [5]. In particular, by appending an equation of the form

$$1 - w \sum_{i=0}^{6} \xi_i c_i = 0, \quad (8)$$

where $\xi_0, \ldots, \xi_6$ are chosen randomly in $\mathbb{C}$ and w is a new variable, these components are removed but the isolated points remain with probability one. Whenever all $c_i = 0$, $w \to \infty$ sending the problematic components to infinity. This lets a Gröbner basis computation find the isolated roots directly and also allows the homotopy software Bertini [1] to abandon the paths as they diverge to infinity without wasting computational effort approximating their endpoints precisely.

5 Computational Results

Solving the system (7,8) using a 4-homogeneous homotopy (with w in a group of its own) requires tracking 1920 paths. The results for generic, complex parameters $(a_0, \bar{a}_0, a_4, \bar{a}_4, r_1, r_2)$ are summarized as follows:

- 72 nonsingular isolated solutions,
- 32 singular solutions of multiplicity 6,
- 8 singular solutions of multiplicity 15,
- the remaining 1536 paths diverge to infinity.

Counting multiplicities, there are $72 \cdot 1 + 32 \cdot 6 + 8 \cdot 15 = 384$ finite solutions. Separately, we computed the degree of the system using Macaulay2 [3] in finite fields of characteristic 7919 and 31991. Each of these returned a polynomial of degree 384 whose radical had degree $112 = 72 + 32 + 8$. The agreement between these finite-field computations and the homotopy results lends credence to the conclusion that we have found the generic number of solutions over the complex numbers.

All of these solutions satisfy (6) in that $f(\theta_4) = (u\theta_4 + v)^6 = 0$ for a coupler rotation of $\theta_4 = -v/u$. However, only the 72 nonsingular solutions actually give multiplicity 6 configuration of the pentad. The singular solutions correspond to situations where the circle swept by T around center S touches the four-bar coupler curve generated by $OPQR$ with coupler point T at more than one point, all with the same coupler rotation. For example, as shown in Figure 2, the coupler curve and circle may touch with multiplicity 5 at one point and multiplicity 1 at another, or in two points with multiplicities 4 and 2 (always with the same coupler rotation, $-v/u$).

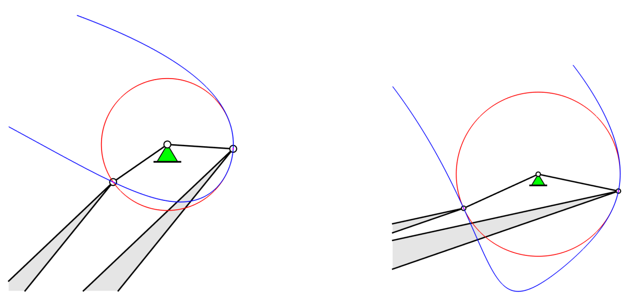

Fig. 2: For singular solutions of the multiplicity 6 problem, the coupler curve and circle touch in two points.

Accordingly, the only true multiplicity 6 assemblies of the pentad are those given by the 72 nonsingular roots. Hence, having solved the problem for generic parameters, we may solve any target problem using a 72-path parameter homotopy that moves from those generic values to the target values. Note that for a real example, the target parameters have $\bar{a}_0$ the complex conjugate of a_0, $\bar{a}_4$ the complex conjugate of a_4, and q_1, q_2 real and positive. Similarly, the "real" assemblies are the solutions where $(b_0, \bar{b}_0)$ and $(b_4, \bar{b}_4)$ are pairs of complex conjugates, q_3 is real and positive, and $\theta_4 = -v/u$ has unit magnitude. Due to reflection symmetry through line $\overleftrightarrow{OP}$, solutions appear in symmetric pairs. Hence, a 36-path homotopy tracking just one of each pair suffices to find all of them.

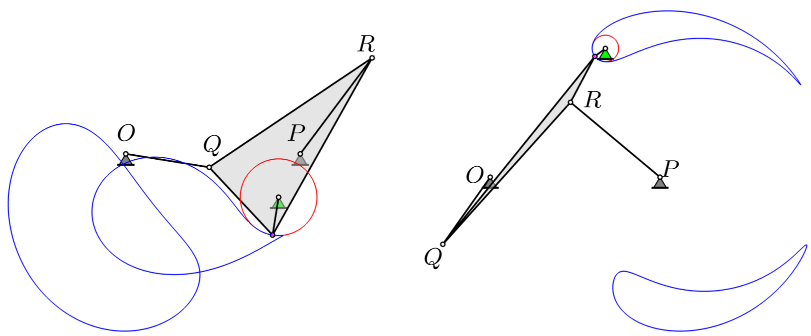

Fig. 3: Two multiplicity 6 solutions for the same four-bar $OPQR$

Figure 3 shows several solutions computed in this manner. It is unknown at this time what the maximum number of real solutions can be, only that it will be an even number less than or equal to 72.

6 Symmetric Case

The problem can be specialized to symmetric four-bars, where $r_1 = r_2$. For this subcase, the perpendicular bisector of $\overleftrightarrow{OP}$ is a second line of symmetry. Using the 72-path homotopy of the general case still yields 72 solutions, as follows:

- 6 symmetric pairs with S on the perpendicular bisector of $\overleftrightarrow{OP}$,
- 6 symmetric pairs with S on $\overleftrightarrow{OP}$,
- 12 groups of 4 with S off the lines of symmetry.

The study in [7] was restricted to only the first type. As in the general case, by taking advantage of symmetries, one can find all solutions by tracking just one in each symmetry orbit, for 24 paths in all. Figure 4 shows two solutions.

7 Conclusion

We have solved the problem formulated in [7] of counting the number of 3RR planar pentad mechanisms with a sixth-order singularity. Our method results in a 72-path homotopy for finding all solutions, reducible by symmetry to 36 paths for the general case and 24 paths for the symmetric case.

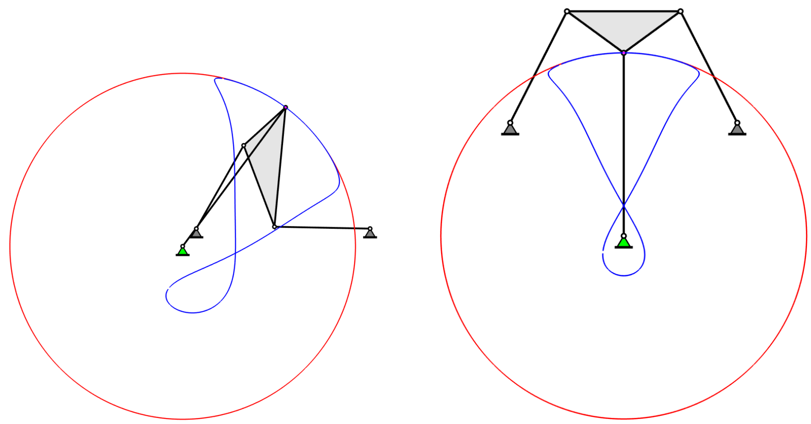

Fig. 4: Solutions for the symmetric case

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