CHAPTER 10

Passivity Based Control

Passivity-based control or PBC is a popular method for synthesizing stabilizing controllers. The popularity of this method is based on the following facts about passive systems

- Many different types of systems (in particular mechanical systems) satisfy an energy conservation principle in which the energy function which ensures the passivity of these systems. The energy functions for these systems provide a natural choice for the storage function which is an important first step in the design of stable passivating controls.
- Passive systems can be easily stabilized through output feedback, the origin can be asymptotically stabilized if we can also ensure the passivated system satisfies a detectability condition.
- The passivity theorem ensures that any feedback interconnection of passive systems is again passive. This provides a modular method for the construction of large-scale passive networked systems while also providing some degree of robustness to unmodeled passive dynamics.

In spite of these benefits, however, passivity-based control initially faced a number of hurdles. Initial feedback passivation methods were limited to systems that had relative degree one with minimum phase zero dynamics. These obstacles can be overcome through recursive design procedures like backstepping for strict feedback systems and forwarding for strict feedforward systems. These design methods, however, all require some knowledge about a reasonable storage function for the passive plant to be controlled. While the total kinetic and potential energy provide reasonable storage functions for mechanical systems, this choice is less obvious for more systems in electronics and biology. The hurdle was addressed by focusing on systems that have a Hamiltonian structure. A number of interesting and important applications can be modeled in this manner, thereby providing additional motivation for the interest in passivity based control.

This chapter consists of three parts. In the first part, we review fundamental passivity concepts for feedback interconnections. We then examine feedback passivation methods for the global stabilization of cascaded systems. As noted above, these methods are limited to passive systems that have relative degree one and minimum phase dynamics. We will examine the use of backstepping to bypass the issue regarding relative degree one systems. The minimum phase issue may be addressed using another recursive design method known as forwarding. These recursive design methods all require passive systems in which we already know the storage function. For systems where the storage function needs to be determined, we examine the use of energy-balancing or shaping methods. These methods are well suited for a special class of system; port-controlled Hamiltonian (PCH) systems. PCH realizations can be constructed for numerous networked dynamical systems consisting of interconnected energy-storing elements. Examples of such systems are found in mechanics, electrical circuits, chemical and biological systems. We will investigate a particular energy-shaping approach for PCH systems known as *interconnection and damping assignment* (IDA). This method has been successfully used to construct a stabilizing controllers for a wide range of applications.

1. Excess and Shortage of Passivity

This section reviews many of the concept originally presented in chapter 6, and introduces methods for shifting an excess of passivity in one subsystem to another non-passive system (i.e. it has a shortage) to passivate a feedback loop [**SJK12**]. This is similar to loopshifting strategies used to ensure satisfaction of the small gain condition in $\mathcal{L}_p$ stable systems. Indices measuring the shortage or excess of passivity have sometimes been referred to as *passivity indices* [**MA10**].

Basic Definitions: We consider a dynamical system of the form,

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

where $f(0,0) = 0$ and $h(0,0) = 0$. For convenience, we assume u and y have the same dimension, m.

We assume there is associated with the system a function $r : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ called the *supply rate*. This system is *dissipative* with supply rate $r(u,y)$ if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(x) \geq 0$ and

$$V(x(t)) - V(x(0) \leq \int_0^t r(u(s), y(s))ds \tag{226}$$

for all $t \geq 0$. The function V is called a *storage function*. We say the system is *passive* if the supply rate $r(u,y) = u^T y$. If the storage function is C^1, then we may rewrite the dissipative inequality in differential form as

$$\dot{V}(x(t)) \leq r(u(t), y(t))$$

for all t.

If the system is dissipative, we may associate with it a function $V_a : \mathbb{R}^n \mathbb{R}$ called the *available storage*. This function takes values

$$V_a(x) = \sup_{u,t\geq 0} \left\{ -\int_0^t r(u(t), y(t))dt \, : \, x(0) = x \right\}$$

The available storage may be interpreted as the maximum amount of "energy" that can be extracted from a system given its initial condition.

Passivity Indices: One of the main results (theorem 81) regarding passive systems is that the feedback interconnection of two passive systems is again passive. Such a feedback interconnection is shown on the left side of Fig. 1. When one of the systems in the interconnection is not passive, however, it is possible to still ensure the passivity of the interconnection by using an "excess" of capacity in one system to compensate for the "shortage" of capacity in the other systems.

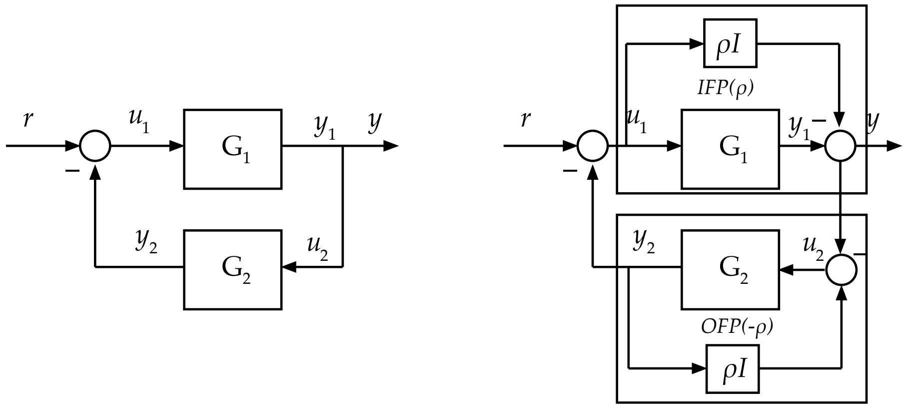

FIGURE 1. (left) feedback interconnection of plant G_2 with controller G_1 (right) Loop transformation on the feedback interconnection

The following definitions formalize the notion of "excess" and "shortage" of capacity. We say the system is

- *Output Feedback Passive* (OFP) if it is dissipative with respect to $r(u, y) = u^T y - \rho y^T y$ for some $\rho \in \mathbb{R}$.
- *Input Feedforward Passive* (IFP) if it is dissipative with respect to $r(u, y) = u^T y - \nu u^T u$ for asome $\nu \in \mathbb{R}$.

We quantify the "excess" and "shortage" properties with the notation IFP(ν) and OFP(ρ), respectively. In particular a positive sign of ρ or ν means that the system has an excess of passivity and if these constants are negative then the system has a shortage of passivity. These constants are sometimes referred to as *passivity indices* [**MA10**].

Passivity indices can be used to establish the passivity of a feedback interconnection between plant G_2 and controller G_1 even if the plant is unstable. In particular, if the plant, G_2, is known to be $OFP(-\rho)$ where $\rho > 0$, then this system is not passive and has a shortage of passivity that can be compensated for by a negative ρ-feedback around the plant. To preserve the overall feedback interconnection unchanged, a feedforward $-\rho I$ is connected in parallel with the controller. If the controller G_1 is $IFP(\rho)$, in other words it has an excess of passivity ρ, then its parallel connection when $-\rho I$ is passive. This *loop transformation* of the feedback interconnection is shown in the right hand pane of Fig. 1. Thus a shortage of passivity (and a lack of stability) of the plant G_2 can be compensated for by the excess of passivity in the controller G_1. The net effect is the same as in a feedback interconnection of two passive systems. We will formally establish this result below after reviewing some results regarding the relationship between stability and passivity.

Stability and Passivity: The definitions of dissipativity and passivity do not require the storage function, V, to be positive definite. They only require that V is positive semidefinite. As a consequence, it is possible for the presence of an unobservable unstable part of the system to destabilize the equilibrium $x = 0$. For dissipativity to exclude such situations, we need to impose a condition known as *zero-state detectability*.

Consider a system with zero input, (i.e. $u = 0$) and let Z be the largest positively invariant set contained in $\{x \in \mathbb{R}^n : y = h(x, 0) = 0\}$. We say the system is *zero-state detectable* (ZSD) if $x = 0$ is asymptotically stable whenever $x_0 \in Z$. Ad was shown in chapter 6, if the system is passive with a C^1 storage function V, then if the system is also ZSD we can prove that the equilibrium $x = 0$ is stable with $u = 0$.

Note that ensuring a passive system is ZSD only establishes "stability", it does not establish *asymptotic stability*. Asymptotic stability can be assured if we introduce a feedback loop $u = -y$ as is stated and proven in the following theorem

THEOREM 109. **(Passivity and Stabilizing Feedback)** *Consider a system, G, with state equations $\dot{x} = f(x, u)$ and $y = h(x, u)$. Assume G is passive with storage function V. If $y = h(x)$ (only a function of x) then the feeback $u = -y$ achieves asymptotic stability of $x = 0$ if and only if the system is ZSD.*

Proof: Because h is independent of u, the feedback loop with $u = -y$ is well-posed. For $u = -y$, the time derivative of V satisfies

$$\dot{V}(x) \leq -y^T y \leq 0$$

Stability is then established using the Invariance principle and the assumption that the system is ZSD.

Conversely, if the equilibrium $x = 0$ of $\dot{x} = f(x, -y)$ is asymptotically stable, then it is asymptotically stable conditional to any subset Z. In particular, this is the case when Z is the largest positively invariant set contained in $E = \{x : y = h(x) = 0\}$ which proves that the system is ZSD. $\diamondsuit$

We may now extend the preceding theorem to feedback interconnections.

THEOREM 110. **(Stability of Feedback Interconnections)** *Consider the feedback interconnection shown on the left side of Fig. 1 with systems G_i ($i = 1, 2$) that are dissipative with supply rates*

$$r_i(u_i, y_i) = u_i^T y_i - \rho_i^T(y_i) y_i - \nu_i^T(u_i) u_i \tag{227}$$

where $\rho_i : \mathbb{R}^m \to \mathbb{R}^m$ have component functions such that $\rho_{ij}(0) = 0$ and $\rho_{ij}(y_j) y_j > 0$ for $j = 1, 2 \ldots, m$ and similarly for ν_i. Furthermore if the two systems are ZSD and their respective storage functions $V_1(x_1)$ and $V_2(x_2)$ are C^1, then the equilibrium $(x_1, x_2) = (0, 0)$ of the feedback interconnection with command input $r \equiv 0$ is

- *stable if $\nu_1^T(v)v + \rho_2^T(v)v \geq 0$ and $\nu_2^T(v)v + \rho_1^T(v)v \geq 0$ for all $v \in \mathbb{R}^m$*
- *asymptotically stable if $\nu_1^T(v)v + \rho_2^T(v)v > 0$ and $\nu_2^T(v)v + \rho_1^T(v)v > 0$ for all $v \in \mathbb{R}^m/\{0\}$.*

Proof: The first assertion is established as follows. We consider the $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ as a candidate storage function for the interconnected system. Using the interconnection identities, $u_1 = -y_2$,

$u_2 = y_1$, the time derivative of V is

$$\dot{V} \leq -(\nu_2 + \rho_1)^T(y_1)y_1 - (\nu_1 + \rho_2)^T(y_2)y_2 \leq 0$$

If V is positive definite, then this establishes stability. If V is only semidefinite, we know that because $V = 0$ implies $V_1 = V_2 = 0$ that $h_1(x_1, 0) = h_2(x_2, 0) = 0$. This assures the loop is well-posed and that h_1 and h_2 are both independent of the input. So we can without loss of generality presume $h_1(x_1, u_1) = h_1(x_1)$. Since $V(x) = 0$ implies $y_1 = h_1(x_1) = 0$ and $y_2 = h_2(x_2, u_2) = h_2(x_2, y_1) = h_2(x_2, 0) = 0$, we can then use the interconnection identities to obtain

$$V = 0 \quad \Rightarrow \quad y_1 = y_2 = u_1 = y_2 = 0$$

The largest positively invariant set of $\dot{x}_1 = f(x_1, 0)$, and $\dot{x}_2 = f_2(x_2, 0)$ is also included in $\{(x_1, x_2) : y_1 = y_2 = 0\}$. Because G_1 and G_2 are ZSD, the origin is asymptotically stable conditional to Z and so is stable.

To establish the theorem's second assertion, note that if $\nu_1^T(v)v + \rho_2^T(v)v > 0$ and $\nu_2^T(v)v + \rho_1^T(v)v > 0$ for all $v \neq 0$, then all bounded solutions converge to the set $\{(x_1, x_2) : y_1 = y_2 = 0\}$. By the Invariance principle every bounded solution converges to the largest invariant set in the set $E = \{(x_1, x_2) : V(x_1, x_2) = 0\}$ which is $(0, 0)$ since the two systems are ZSD. $\diamondsuit$

The following theorem formalizes our earlier assertion that one can compensate for a shortage of passivity in one subsystem of a feedback interconnection by having an excess of passivity in the other subsystem. It may be seen as a corollary of the preceding theorem with $\rho(y) = \rho y$ and $\nu(u) = \nu u$.

THEOREM 111. **(Passivity Indices Theorem)** *Consider the feedback interconnection on the left side of Fig. 1. Assume that the system G_1 is globally asymptotically stable and IFP(ν) and that system G_2 is zero-state detectable (ZSD) and OFP(ρ). Then the origin is asymptotically stable if $\nu + \rho > 0$.*

Proof: So under the assumptions that G_2 is OFP(ρ) and G_1 is IFP(ν), we know that

$$\begin{aligned} V_2(x_2) &\leq u_2^T y_2 - \rho y_2^T y_2 \\ V_1(x_1) &\leq u_1^T y_1 - \nu u_1^T u_1 \end{aligned}$$

So we can apply theorem 110 with $\rho(y) = \rho y$ and $\nu(u) = \nu u$. Let us take $V = V_1 + V_2$ as a candidate storage function for the interconnected system. Using the interconnection relations (with $r = 0$) that $u_1 = -y_2$ and $u_2 = y_1$, we can see that

$$\begin{aligned} V(x_1, x_2) &= V_1(x_1) + V_2(x_2) \\ &\leq u_1^T y_1 - \nu u_1^T u_1 + u_2^T y_2 - \rho y_2^T y_2 \\ &= -y_2^T y_1 + y_1^T y_2 - (\rho + \nu) y_2^T y_2 \\ &= -(\rho + \nu) y_2^T y_2 \end{aligned}$$

If we then apply the preceding theorem we obtain the desired result. $\diamondsuit$

2. Feedback Passivation

We now consider the problem of stabilizing the origin of the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{228}$$

with u as the input and y as the output. The feedback control u will be computed assuming we only have access to the output y. This problem can be solved if we are free to select the output function $y = h(x)$ to ensure the system is passive. If we can use this selection of the output to ensure the system is passive then theorem 109 asserts we can use the feedback control $u = -y$ to stabilize the system. If we can further assure that the system is ZSD, then this interconnection will be globally asymptotically stable.

So our task is to search for an output function $y = h(x)$ so the system is passive with a positive definite storage function. This requires, of course, that the system is stable when $u = 0$. This can be overly restrictive and so we seek a more flexible way of passivating the plant that does not require the original plant to be stable. Instead we simply assume the uncontrolled system is stabilizable and use a feedback law in conjunction with a selected output function, h, to passivate the system. We therefore need to find an output function $y = h(x)$ *and* a feedback transformation

$$u = \alpha(x) + \beta(x)v$$

with $\beta(x)$ invertible such that

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x) + g(x)\beta(x)v \\ y &= h(x) \end{aligned}$$

is passive. If such a transformation can be used to passiate the plant then we say the original system is *feedback passive*. The selection of the output $y = h(x)$ and the construction of the feedback transformation $u = \alpha(x) + \beta(x)v$ is usually called *feedback passivation*. If we can further establish that the passivated system is ZSD, then asymptotic stability of the passivated system can be achieved with a simple additional feedback $v = -ky$ where $k > 0$ is any positive feedback gain.

The crucial limitation in feedback passivation is that the output y that we define must 1) have a relative degree no greater than one and 2) the zero-dynamics of the system must be stable (i.e. minimum phase). This is particularly restrictive because neither property can be modified by feedback. So it essentially confines the utility of feedback passivation methods (as described above) to minimum-phase nonlinear plants with a relative degree of one. The following theorem states that any passive input-affine system has a relative degree of one at $x = 0$.

THEOREM 112. **(Relative degree of passive system)** *Consider the system (228) and assume $g(0)$ and $\frac{\partial h}{\partial x}(0)$ have full rank. If this system is passive with a C^2 storage function $V(x)$, then it has relative degree one at $x = 0$.*

Proof: It can be shown that the passive affine system's storage function and output must satisfy the relations

$$L_f V(x) \leq 0, \quad [L_g V]^T(x) = h(x) \tag{229}$$

Multiplying both sides of (229) by $g(x)$ yields

$$\frac{\partial}{\partial x}\left[g^T(x)\frac{\partial V^T}{\partial x}(x)\right] g(x) = \frac{\partial h}{\partial x}(x)g(x) \tag{230}$$

At $x = 0$, $\frac{\partial V}{\partial x}(0) = 0$ and (230) becomes

$$g^T(0)\frac{\partial^2 V}{\partial x^2}(0)g(0) = L_g h(0)$$

The Hessian $\frac{\partial^2 V}{\partial x^2}$ at $x = 0$ is symmetric positive semidefinite and can therefore be factored as $R^T R$. This yields

$$L_g h(0) = g^T(0)R^T R g(0) \tag{231}$$

The matrix $R^T R$ need not be positive definite and so we need one additional condition that we obtain by differentiating (229) to get

$$\frac{\partial h}{\partial x}(0) = g^T(0)R^T R \tag{232}$$

With this, one can use (231) to conclude that $L_g h(0)$ is nonsingular, which means the system has relative degree one. ♢

If the system has relative degree one at $x = 0$, one can define a local coordinate change $(z, \xi) = (T(x), h(x))$ and rewrite the system equations in normal form

$$\begin{array}{rcl} \dot{z} & = & q(z,\xi) + \gamma(z,\xi)u \\ \dot{\xi} & = & a(z,\xi) + b(z,\xi)u \\ y & = & \xi \end{array} \tag{233}$$

where $b(z, \xi) = L_g h(x)$ is locally invertible near $x = 0$. For the normal form system in (233) , the requirement $y \equiv 0$ is satisfied with the feedback law

$$u = -b^{-1}(z, 0)a(z, 0)$$

and so the zero-dynamics subsystem exists locally with the differential equation

$$\dot{z} = q(z, 0) - \gamma(z, 0)b^{-1}(z, 0)a(z, 0) := \phi(z) \tag{234}$$

We will now use this representation of the system in normal form to examine the zero-dynamics of a passive system.

Recall that the system is *minimum phase* if the equilibrium of its zero dynamics (234) is asymptotically stable. It is *weakly minimum phase* if it is Lyapunov stable and there exists a C^2 positive definite function $W(z)$ such that $L_\phi W \leq 0$ in a neighborhood of $z = 0$. The following theorem asserts that if a system is passive it is also weakly minimum phase.

THEOREM 113. **(Weak minimum phase of passive systems)** *If the affine system $\dot{x} = f(x) + g(x)u$ with $y = h(x)$ is passive with a C^2 positive definite storage function $V(x)$, then it is weakly minimum phase.*

Proof: By definition, the zero dynamics of the system evolve in the manifold $\xi = h(x) = 0$. In this manifold the second passivity condition , $[L_g V]^T(x) = h(x)$ implies that $L_g V = 0$. Because $\dot{V} \leq u^T y = 0$, we have

$$\dot{V} = L_f V + L_g V u = L_f V \leq 0$$

So $V(x)$ is non-increasing along solutions in the manifold $h(x) = 0$ and the equilibrium $z = 0$ of (234) is stable. $\diamondsuit$

The relative degree and zero dynamics of the affine system are invariant under feedback transformations $u = \alpha(x) + \beta(x)v$. This is because $L_g h(0)$ is simply multiplied by $b(0)$ and so the relations in (234) are unchanged by the transformation. This also means that relative degree one and weak minimum phase conditions are necessary for feedback passivity. These observations can be summarized in the following theorem.

THEOREM 114. **(Feedback Passivity)** *Assume that rank $\frac{\partial h}{\partial x}(0) = m$. Then the affine input system is feedback passive with a C^2 positive definite storage function $V(x)$ if and only if it has relative degree one at $x = 0$ and is weakly minimum phase.*

This theorem is clearly of major interest for feedback passivation designs because it provides a significant limit on when this method can be used.

Example: By selecting the output $y = x_2$ for

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= u \end{aligned}$$

we obtain a relative degree one system which is already in its normal form. Its zero dynamics subsystem $\dot{x}_1 = 0$ is only stable so that the system is weakly minimum phase. A feedback transformation is

$$u = v + x_1^3$$

renders the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= -x_1^3 + v \\ y &= x_2 \end{aligned}$$

which is passive with storage function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Since $y(t) \equiv v(t) \equiv 0$ implies $x_1(t) = x_2(t) = 0$, the additional output feedback $v = -y$ achieves global asymptotic stability of $(x_1, x_2) = (0, 0)$.

Example: The nonlinear equations for an m-link robot take the form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$$

where q is an m-dimensional vector of generalized coordinates representing joint positions, u is an m-dimensional control input, and $M(q)$ is a symmetric inertia matrix that is positive definite for all $q \in \mathbb{R}^m$. The term $C(q,\dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces. The matrix C has the property that $\dot{M} - 2C$ is a skew-symmetric matrix for all $q, \dot{q} \in \mathbb{R}^m$, where $\dot{M}$ is the total derivative of $M(q)$ with respect to time. The term $D(q)$, where D is a positive semidefinite symmetric matrix, accounts for viscous damping. The term $g(q)$, which accounts for gravitational forces is given by $g(q) = \left[\frac{\partial P(q)}{\partial q}\right]^T$ where $P(q)$ is the total potential energy of the links due to gravity. We consider the regulation problem of designing a state feedback control law so that q asymptotically tracks a constant reference q_r. Let $e = q - q_r$ denote the position tracking error. Then e satisfies the differential equation

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

The objective is to stabilize this error system about $e = 0$ and $\dot{e} = 0$.

Since $(e, \dot{e}) = (0, 0)$ may not be an equilibrium of the system, we consider a control of the form

$$u = g(q) - K_p e + v$$

where K_p is a positive definite symmetric matrix and v is the new control variable. This control compensates for the gravitational force $g(q)$ that disturbs the system away from the origin. Substituting this control into the error differential equation yields,

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v$$

thereby ensuring that $(e, \dot{e}) = (0, 0)$ is an equilibrium point for the system.

We take as a candidate storage function

$$V(e, \dot{e}; q) = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T K_p e$$

This function is positive definite and when we compute its directional derivative we see that

$$\begin{aligned}
\dot{V} &= \dot{e}^T M \ddot{e} + \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e} \\
&= \frac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \\
&\leq \dot{e}^T v
\end{aligned}$$

So we take the output function to be $y = \dot{e}$ to render the system from input v to output y passive with storage function V.

Based on our earlier theorems, this choice passivates the system, but to see whether passivity implies the asymptotic stability of the origin, we need to check to see if the system is zero-state observable. For $v = 0$, we see that $y(t) = 0$ for all t if and only if $\dot{e}(t) = 0$ for all t. This implies that $\ddot{e}(t) = 0$ for all t, which implies that $K_p e(t) = 0$ and so $e(t) = 0$. We may therefore conclude that the system is zero-state observable and so it can be globally stabilized by the control $v = -\phi(\dot{e})$ with any function ϕ such that $\phi(0) = 0$ and

$y^T\phi(y) > 0$ for all $y \neq 0$. The choice of $v = -K_d\dot{e}$ with a positive definite symmetric matrix K_d results in the control

$$u = g(q) - K_p(q - q_r) - K_d\dot{q}$$

which is a classical PD contoller with a gravity compensation term.

3. Passivation of Cascades

This section examines feedback stabilization designs for the cascade of two nonlinear systems with subsystem states, z and ξ as shown in Fig. 2. In this cascade, we see that the control enters only the ξ subystem. The interconnected subsystems are assumed to satsify the following differential equations

$$\begin{aligned}\dot{z} &= f(z) + \psi(z,\xi)\\ \dot{\xi} &= a(\xi,u)\end{aligned}$$

The important thing to note is that the driving ξ subsystem only depends on the control, u, and the state ξ. The way in which the output, ξ, of the driving subsystem impacts the driven z system is through the interconnection function $\psi(z,\psi)$.

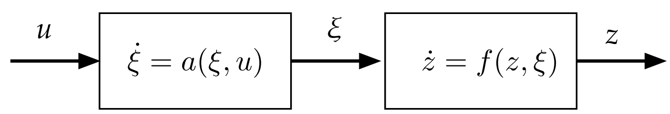

FIGURE 2. Cascade System

As we demonstrated in earlier chapters, the stability of the driving and driven system need not imply the stability of the cascade. This means, of course, that if the two subsystems are only passive, we cannot ensure the passivity of the cascade as well. Our problem, therefore, is to use feedback passivation to stabilize the cascade in Fig.2. The approach we use depends on identifying two passive subsystems of the cascade and then use the control to form their feedback interconnection, thereby ensuring through the passivity theorem 81 the passivity (and the stabilizability) of the entire cascade.

The main assumption we use to achieve this objective is that the equilibrium of the unforced driven systemm $\dot{z} = f(z)$, is globally stable with a C^2 radially unbounded positive definite function $W(z)$ such that $L_fW \leq 0$. So we are only assuming that the driven system $\dot{z} = f(z)$ is globally Lyapunov stable.

To examine how the passivation design is done, we first consider a cascade in which the driving system is linear.

$$\begin{aligned}\dot{z} &= f(z) + \psi(z,\xi)\\ \dot{\xi} &= A\xi + Bu\end{aligned} \tag{235}$$

To identify two passive subsystems, G_1 and G_2, we factor the interconnection function as

$$\psi(z,\xi) = \tilde{\psi}(z,\xi)C\xi \tag{236}$$

We have thus created the linear block G_1 with the transfer function

$$G_1(s) = C(sI - A)^{-1}B$$

For this block to be passive, the choice of the output matrix C must render $G_1(s)$ a positive real transfer function. The block G_2 is the nonlinear system

$$\dot{z} = f(z) + \tilde{\psi}(z, \xi)u_2$$

with the input $u_2 = y_1$ and the output y_2 has yet to be defined. We are free to select the output $y_2 = h_2(z, \xi)$ to guarantee passivity. Using $W(z)$ as a positive definite storage function for G_2, we require

$$\dot{W} = \frac{\partial W}{\partial z}(f(z) + \tilde{\psi}(z, \xi)y_1) \leq y_2^T u_2 \tag{237}$$

Knowing from our assumption that $L_f W \leq 0$, we satisfy the dissipative relation in (237) by selecting

$$y_2 = h_2(z, \xi) := (L_{\tilde{\psi}}W)^T(z, \xi) = \tilde{\psi}^T \left[\frac{\partial W}{\partial z}\right]^T \tag{238}$$

The block G_2 that we just constructed is, therefore, passive. Next with the feedback transformation $u = -h_2(z, \xi) + v$ we create the feedback interconnection if Fig. 3 which by the passivity theorem is passive from v to y_1. By Theorem 109, we can therefore achieve global stability with the control $v = -y_1$.

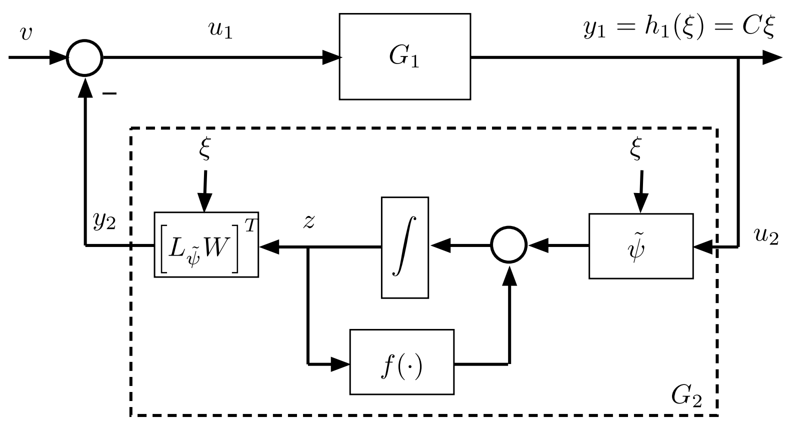

FIGURE 3. Rendering the cascade in (235) passive from v to y_1

We can apply the previous construction in an analogous manner to the cascade with a nonlinear ξ-subsystem to obtain the following theorem

THEOREM 115. **(Feedback Passivation of Cascade)** *Suppose that for the cascade*

$$\begin{aligned} \dot{z} &= f(z) + \psi(z, \xi) \\ \dot{\xi} &= a(\xi) + b(\xi)u \end{aligned} \tag{239}$$

in which the equilibrium $z = 0$ of $\dot{z} = f(z)$ is globally stable with a C^2 radially unbounded positive definite function $W(z)$ such that $L_f W \leq 0$. Suppose there exists an output $y = h(\xi)$ such that

- *the interconnection $\psi(z, \xi)$ can be factored as $\psi(z, \xi) = \tilde{\psi}(z, \xi)y$,*

- *the subsystem*

$$\begin{aligned} \dot{\xi} &= a(\xi) + b(\xi)u \\ y &= h(\xi) \end{aligned} \tag{240}$$

is passive with a C^1 positive definite, radially unbounded storage function $U(\xi)$.

Then the entire cascade in (239) is rendered passive with the feedback transformation

$$u = -(L_{\tilde{\psi}}W)^T(z,\xi) + v \tag{241}$$

and $V(z,\xi) = W(z) + U(\xi)$ is its storage function. If, with the new input v and the output y, the cascade is ZSD, then $v = -ky$ with $k > 0$ achieves global asymptotic stability of the equilibrium $(z,\xi) = (0,0)$.

Example: Let us consider the system

$$\begin{aligned} \dot{z} &= -z + z^2\xi \\ \dot{\xi} &= u \end{aligned} \tag{242}$$

We will consider two strategies for stabilizing this system. In the first strategy, we use partial feedback of ξ to force the driving ξ-subsystem to go to zero. The idea is that by driving ξ to zero that the stability of the unforced driving system $\dot{z} = -z$ will be sufficient to ensure the stability of the cascade. The second strategy will use full state feedback to implement a feedback passivating control.

While both strategies achieve local stabilization of the origin, there is a subtle difference in global nature of the stability achieved. In particular, we will show that the first partial feedback strategy only achieves what is called *semiglobal* asymptotic stability, by which we mean that there is a control for any initial condition that assures asymptotic stability, but the gain required to achieve that convergence grows with the distance of the initial state from the equilibrium. On the other hand, the passivating control is able to achieve *global* asymptotic stability in the sense that there is a fixed control law that assures convergence to the origin for *any* initial condition.

Let us first look at the partial feedback approach. In this case we use a linear feedback law $u = -k\xi$ with $k > 0$ to achieve asymptotic stability of $(z,\xi) = (0,0)$. We use the Lyapunov function $V(z,\xi) = z^2 + \xi^2$ to estimate the region of attraction. The derivative of V is

$$\dot{V} = -2(z^2 + k\xi^2 - \xi z^3) = -\begin{bmatrix} z & \xi \end{bmatrix} \begin{bmatrix} 2 & -z^2 \\ -z^2 & 2k \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix} \tag{243}$$

negative for $z^2 < 2\sqrt{k}$. An estimate of the region of attraction is the largest set $V = c$ in which $\dot{V} < 0$. This shows that with a feedback gain $k > \frac{c^2}{4}$ that we can guarantee any prescribed c. In other words, we can guarantee convergence from an initial condition, x_0, with $V(x_0) > c$ only if the gain $k > c^2/4$. So this what we mean when we say the system is *semi-globally* asymptotically stable.

Let us now consider a passivating design that employs full-state feedback to achieve *global* stabilization. We use $y_1 = \xi$ to first create a linear passive system G_1. Then by selecting $W(z) = \frac{1}{2}z^2$ as a storage function, we establish that the first equation in (242) defines a passive system G_2 with $u_2 = \xi$ as the input and $y_2 = z^3$ as the output. Hence with the feedback transformation

$$u = -y^2 + v = -z^3 + v$$

the cascade (242) becomes a feedback connection of two passive systems. The ZSD property is also satisfied because in the set $y_1 = \xi = 0$, the system reduces to $\dot{z} = -z$. Therefore a linear feedback control $v = -ky_1$ with $k > 0$ will render the whole cascade globally asymptotically stable (GAS).

When the subsystem (240) is *feedback passive* rather than passive, theorem 115 applies after a passivating feedback transformation. In particular, when the ξ-subsystem is linear as in (235), we know the system (A, B, C) is feedback passive if and only if it is weakly minimum phase and has relative degree one. After a linear change of coordinates, the system (A, B, C) can be written as

$$\begin{aligned} \dot{\xi}_0 &= Q_{11}\xi_0 + Q_{12}y \\ \dot{y} &= Q_{21}\xi_0 + Q_{22}y + CBu \end{aligned} \tag{244}$$

and the feedback transformation

$$u = (CB)^{-1}(-2Q_{12}^T P_{11}\xi_0 - Q_{11}\xi_0 - Q_{22}y + v) := F\xi + Gv \tag{245}$$

renders the system passive with the storage function $U = \xi_0^T P_{11}\xi + \frac{1}{2}y^T y$. The results of the preceding discussion are formalized in the following theorem

THEOREM 116. **(Passivation of partially linear cascades)** *Suppose that for the cascade*

$$\begin{aligned} \dot{z} &= f(z) + \psi(z, \xi) \\ \dot{\xi} &= A\xi + Bu \end{aligned} \tag{246}$$

that the origin of $\dot{z} = f(z)$ is globally stable with Lyapunov function $W(z)$ and assume there exists an output $y = C\xi$ such that

- *the interconnection $\psi(z, \xi)$ can be factored as $\psi(z, \xi) = \tilde{\psi}(z, \xi)y$,*
- *the system (A, B, C) has relative degree one and is weakly minimum phase.*

Then the entire cascade (246) with $y = C\xi$ as the output is feedback passive. Its passivity from v to y is achieved with the feedback transformation

$$u = F\xi = G(L_{\tilde{\psi}}W)^T(z, \xi) + Gv \tag{247}$$

where F and G are defined as in (245). The feedback control $v = -ky$ for $k > 0$ guarantees GAS of $(z, \xi) = (0, 0)$ if 1) $\dot{z} = f(z)$ is GAS and (A, B) is stabilizable or 2) the cascade with output y and input v is ZSD.

Example: Consider the cascade,

$$\begin{aligned} \dot{z} &= -qz^3 + (c\xi_1 + \xi_2)z^3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{248}$$

the z-subsystem $\dot{z} = -qz^3$ is GAS when $q > 0$ and only GS when $q = 0$. With $y_1 = c\xi_1 + \xi_2$, the interconnection term $\psi(z, \xi)$ is factored as $\psi(z, \xi) = y_1 z^3$. The resulting ξ-subsystem is

$$\begin{aligned} \dot{\xi}_1 &= -c\xi_1 + y_1 \\ \dot{y}_1 &= -c^2\xi_1 + cy + u \end{aligned} \tag{249}$$

This subsystem has relative degree one and its zero dynamics subsystem is $\dot{\xi}_1 = -c\xi_1$. Hence the ξ-subsystem is minimum phase if $c > 0$ and nonminimum phase if $c < 0$. For $c \geq 0$, this linear block G_1 is rendered passive by a feedback transformation

$$u = -(1 - c^2)\xi_1 - (1 + c)y_1 + v \tag{250}$$

which achieves $\dot{U} \leq vy_1$ with storage function $U(\xi) = \frac{1}{2}(\xi_1^2 + y_1^2)$. To render the nonlinear block G_2 passive, we select $W(z) = \frac{1}{2}z^2$ and let the output be $y_2 = L_{\tilde{\psi}} W(z) = z^4$. Then closing the loop with

$$v = -y_2 + w = -z^4 + w \tag{251}$$

we render the entire system passive from w to y_1. The remaining step is to verify whether the feedback law for $w = -y_1$ achieves GAS. When $q > 0$, GAS is by condition 1) in theorem 116. When $q = 0$, the ZSD property requires $c > 0$, that is the linear subsystem must be strictly minimum phase: in the set where $y_1 \equiv w \equiv 0$, which implies $\dot{y}_1 = c^2\xi_1 - z^4 \equiv 0$. It is then clear that $(z, \xi_1) = (0, 0)$ is the only invariant set of $\dot{z} = 0, \dot{\xi}_1 = -c\xi_1$ only if $c > 0$.

Example: Let us now consider a nonlinear cascade

$$\begin{aligned} \dot{z} &= f(z) + \psi(z, \xi) \\ \dot{\xi} &= a(z, \xi) + b(z, \xi)u \end{aligned} \tag{252}$$

and assume that $\dot{z} = f(z)$ is globally stable with Lyapunov function W. We also assume that $b^{-1}(z, \xi)$ exists for all (z, ξ). We can make (252) passive by selecting $y = \xi$ and letting $\psi = \tilde{\psi}(z, \xi)\xi$. The feedback transformation

$$u = b^{-1}(z, y)\left(v - a(z, y) - L_{\tilde{\psi}} W(z, y)\right) \tag{253}$$

renders the entire cascade (252) passive with the storage function

$$V(z, y) = W(z) + \frac{1}{2}y^T y \tag{254}$$

Example: The rotational motion of a rigid body subject to three independent scalar control torques can be modeled by

$$\begin{aligned} \dot{\rho} &= \frac{1}{2}\left(I_3 + S(\rho) + \rho\rho^T\right)\omega \\ M\dot{\omega} &= -S(\omega)M\omega u \end{aligned}$$

where $\omega \in \mathbb{R}^3$ is the velocity vector and $\rho \in \mathbb{R}^3$ is a particular choice of parameteric parameters that lead to a three-dimensional representation of the rotation group. The matrix $S(x)$ is a skew-symmetric matrix

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

M is a positive definite symmetric inertia matrix and I_3 is a 3×3 identity matrix.

We take the output $y = \omega$ and note that the system may be viewed as a cascade of a driving system,

$$\begin{aligned} M\dot{\omega} &= -S(\omega)M\omega + u \\ y &= \omega \end{aligned}$$

with the driven system

$$\dot{\rho} = \frac{1}{2}\left[I_3 + S(\rho) + \rho\rho^T\right]\omega$$

Let us take $V(\omega) = \frac{1}{2}\omega^T M\omega$ and note that

$$\dot{V} = \omega^T M\dot{\omega} = -\omega^T S(\omega)M\omega + \omega^T u = y^T u$$

where we used the property that $\omega^T S(\omega)\omega = 0$. The driving system therefore is passive.

The unforced driven system $\dot{\rho} = 0$ has a stable equilibrium point at $\rho = 0$ and any radially unbounded positive definite C^1 function $W(\rho)$ will serve as a Lyapunov function. So all of the assumptions for feedback passivation are satisfied and the entire cascade can be rendered passive by the control

$$u = -\left\{\frac{\partial W}{\partial \rho}\frac{1}{2}\left[I_3 + S(\rho) + \rho\rho^T\right]\right\}^T + v$$

Taking $W(\rho) = k\ln(1 + \rho^T\rho)$ with $k > 0$ yields,

$$\begin{aligned} u &= -\left\{\frac{k\rho^T}{1+\rho^T\rho}\left[I_3 + S(\rho) + \rho\rho^T\right]\right\}^T + v \\ &= -k\rho + v \end{aligned}$$

where we used the property that $\rho^T S(\rho) = 0$.

To establish asymptotic stability of the origin, we still need to check zero-state observability (detectability) of the passive system

$$\begin{aligned} \dot{\rho} &= \frac{1}{2}\left[I_3 + S(\rho) + \rho\rho^T\right]\omega \\ M\dot{\omega} &= -S(\omega)M\omega - k\rho + v \\ y &= \omega \end{aligned}$$

With $v = 0$, we see that $y(t) \equiv 0$ if and only if $\omega(t) \equiv 0$. This implies that $\dot{\omega}(t) \equiv 0$, which in turn allows one to deduce that $\rho(t) = 0$ or all t. So the system is zero-state observable and we can globally stabilize it by the control

$$u = -k\rho - \phi(\omega)$$

with any locally Lipschitz function ϕ such that $\phi(0) = 0$ and $y^T \phi(y) > 0$ for all $y \neq 0$.

4. Backstepping Feedback Passivation

The feedback passivation designs discussed in the preceding section for cascade structure will now be extended to a larger class of nonlinear systems. This section shows how the recursive design procedure known as backstepping from chapter 8 can be used to bypass the relative degree-one obstacle that limited the passivation designs discussed in the preceding section.

Backstepping is applicable to nonlinear systems in lower-triangular form and while it does bypass the relative degree obstacle, it still requires the system to be minimum phase. A complementary recursive design procedure known as *forwarding* can be applied to system in upper-triangular form and this provides a method for avoiding the minimum phase requirement. A brief description of forwarding will be found in the next section.

We will discuss the use of backstepping in feedback passivation by first presenting an example and then formalizing what we see in this example. In particular, we consider the following strict-feedback system

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_1^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \tag{255}$$

where θ is an uncertain parameter known to belong to the interval $\theta \in [-1, 1]$. This system is shown in the block diagram of Fig. 4 where there is a feedback loop and the absence of forward paths other than the integrator chain. When $u = 0$, the system has two types of instability: a linear instability due to the double integrator (x_2, x_3) and a nonlinear instability occurring in the subsytm $\dot{x}_1 = \theta x_1^2$. The objective is to achieve global asymptotic stability (GAS) through a systematic passivation design.

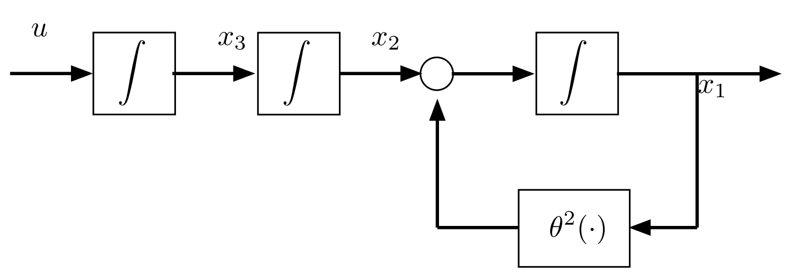

FIGURE 4. The block diagram of a strict feedback system

To use the feedback passivation methods of the preceding sections we need to identify a passivating output and a storage function to be used as a Lyapunov function. The two requirements of the passivating output are : relative degree one and weak minimum phase. For an output of (255) to be relative degree one, it must be a function of x_3; so we let $y_3 = x_3 - \alpha_2(x_1, x_2)$. We then need to select $\alpha_2(x_1, x_2)$ to satisfy the minimum phase requirement; namely that the zero dynamics are GAS. Setting $y_3 \equiv 0$, we see that the zero-dynamics subsystem is

(256)
$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_1^2 \\ \dot{x}_2 &= \alpha_2(x_1, x_2) \end{aligned}$$

For this subsystem, we need to find a stabilizing control law $\alpha_2(x_1, x_2)$. This feedback stabilization problem, however, is for a lower order subsystem than the original third order system in (255). So we've reduced the original problem's complexity to the stabilization of a second order system

(257)
$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_1^2 \\ \dot{x}_2 &= x_3 \end{aligned}$$

in which x_3 serves the role of the "control". To solve this lower order problem we again need to construct a new relative degree one passivating output $y_2 = x_2 - \alpha_1(x_1)$ and design $\alpha_1(x_1)$ to achieve GAS of the zero-dynamics subsystem

(258)
$$\dot{x}_1 = \alpha_1(x_1) + \theta x_1^2$$

Once more the problem has been reduced, but now to the stabilization of the first order subsystem

(259)
$$\dot{x}_1 = x_2 + \theta x_1^2$$

in which x_2 is the "control" and $y_1 = x_1$ is the output.

Note that our construction of the passivating outputs y_1, y_2, and y_3 proceed in a *bottom-up direction*: from y_3 to y_2 to y_1. These outputs were obtained by constructing functions $\alpha_1(x_1)$ and $\alpha_2(x_1, x_2)$ each playing the role of a "control law": $\alpha_1(x_1)$ for x_2 as a "virtual control" of equation (259) and $\alpha_2(x_1, x_2)$ for x_3 as a "virtual control" of (257). This shows that the recursive design procedure must proceed in the *top-down* direction, by first designing $\alpha_1(x_1)$, then $\alpha_2(x_1, x_2)$ and finally $\alpha_3(x_1, x_2, x_3)$ which is the actual control u to be used. In this top-down direction, we start from the scalar subsystem (259), then augment it by one equation to (257) and again by one more equation to the original system (255). On a block diagram, we move "backward" starting from the integrator furthest from the control input, hence the term *backstepping*.

This is, of course, the same backstepping procedure we introduced earlier in chapter 8. The only major difference now is that the construction of the controls is based on a feedback passivating technique, rather than the universal formulae discussed in chapter 8.

Let us now reinterpret the construction of the passivating outputs as a backstepping construction of the "control laws" $\alpha_1(x_1)$, $\alpha_2(x_1, x_2)$, and $\alpha_3(x_1, x_2, x_3)$. In the first step, the subsystem (259) with output y_1 and input x_2 is rendered passive by the "control law" $\alpha_1(x_1)$. At the second step the subsystem (257) with the output $y_2 = x_2 - \alpha_1(x_1)$ and input x_3 is erndered passive by the control law $\alpha_2(x_1, x_2)$. At the third and final step, the original system (255) with output $y_3 = x_3 - \alpha_2(x_1, x_2)$ and input u is rendered passive and GAS by the control law $u = \alpha_3(x_1, x_2, x_3)$. At each step a Lyapunov function is constructed that also serves as a storage function.

One may also assert that backstepping circumvents the relative degree obstacle to passivation. For the output $y = x_1$, the original system has relative degree three. However, at each design step, the considered subsystem only has relative degree one with the zero dynamics rendered GAS at the preceding step.

The basic step described above is now formalized in the following theorem

THEOREM 117. **(Backstepping as recursive feedback passivation)** *Assume that for the system*

$$\dot{z} = f(z) + g(z)u \tag{260}$$

a C^1 feedback transformation $u = \alpha_0(z) + v_0$ and a C^2 positive definite radially unbounded storage function $W(x)$ are known such that this system is passive from the input v_0 to the output $y_0 = [L_g W]^T(z)$. In other words, $\dot{W} \leq y_0^T v_0$. Then the augmented system

$$\begin{aligned} \dot{z} &= f(z) + g(z)\xi \\ \dot{\xi} &= a(z,\xi) + b(z,\xi)u \end{aligned} \tag{261}$$

where $b^{-1}(z,\xi)$ exists for all $(z,1)$, is feedback passive with respect to the output $y = \xi - \alpha_0(z)$ and the storage function $V(z,y) = W(z) + \frac{1}{2}y^T y$. A particular control law ("exact backstepping") that renders (261) passive is

$$u = b^{-1}(z,\xi)\left(-a(z,\xi) - y_0 + \frac{\partial \alpha_0}{\partial z}\left(f(z) + g(z)\xi\right) + v\right) \tag{262}$$

The system (261) with (262) is ZSD for the input v if and only if the system (260) is ZSD for the input v_0.

Proof: Substituting $\xi = y + \alpha_0(z)$, we rewrite (261) as

$$\begin{aligned} \dot{z} &= f(z) + g(z)(\alpha_0(z) + y) \\ \dot{y} &= a(z, y + \alpha_0(z)) + b(z, y + \alpha_0(z))u - \dot{\alpha}_0(z,y) \end{aligned} \tag{263}$$

After the feedback transformation in (262), this system becomes

$$\begin{aligned} \dot{z} &= f(z) + g(z)(\alpha_0(z) + y) \\ \dot{y} &= -y_0 + v \end{aligned} \tag{264}$$

The passivity property from y to v is established with the storage function $V = W(z) + \frac{1}{2}y^T y$. Its time derivative satisfies

$$\dot{V} = \dot{W} + y^T(-y_0 + v) \leq y^T v$$

where we used the passivity assumption $\dot{W} \leq y_0^T v_0$ and the fact that $v_0 = y$.

To verify the ZSD property of (264), we set $y \equiv v \equiv 0$ which implies $y_0 \equiv 0$. Hence the system (264) is ZSD if and only if $z = 0$ is attractive conditionally to the the largest invariant set of $\dot{z} = f(z) + g(z)\alpha_0(z)$ in the set where $y_0 = (L_g W)^T = 0$. This is equivalent to the ZSD property of the original system (260) for the input v_0 and the output y_0. $\diamondsuit$

5. Forwarding

Forwarding is a recursive procedure that removes the weak minimum phase obstacle to feedback passivation and can be used on systems to which backstepping cannot be used. Backstepping, for example, cannot be used on the cascade

$$
\begin{aligned}
\dot{z} &= f(z) + \tilde{\psi}(z, \xi_i)\xi_i, \quad i \in \{1, \ldots, n\} \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_n &= u
\end{aligned}
\tag{265}
$$

We cannot use backstepping here because the zero-dynamic subsystem is an unstable chain of integrators.

Forwarding circumvents this obstacle in a recursive manner. The method was introduced in [**JSK96**] and the monograph [**SJK12**] contains a more thorough treatment of the method than I provide in this section.

Forwarding starts with the cascade,

$$
\begin{aligned}
\dot{z} &= f(z) + \tilde{\psi}(z, \xi_n)\xi_n \\
\dot{\xi}_n &= u_n \\
y_n &= \xi_n
\end{aligned}
\tag{266}
$$

which ignores the unstable part of the zero dynamics. This subsystem satisfies the passivation requirements so that a Lyapunov function $V_n(z, \xi_n)$ and a stabilizing feedback $u_n = \alpha_n(z, \xi_n)$ are easy to construct. The true control input is denoted by u_n to indicate that the first step of forwarding starts with the ξ_n-equation. The second step moves "forward" from the input, that is it includes the ξ_{n-1}-equation:

$$
\begin{aligned}
\dot{\xi}_{n-1} &= \xi_n \\
\dot{z} &= f(z) + \tilde{\psi}(z, \xi_n)\xi_n \\
\dot{\xi}_n &= u_n(z, \xi_n)
\end{aligned}
\tag{267}
$$

This new subsystem is the cascade of a stable system $\dot{\xi}_{n-1} = 0$ with the globally asymptotically stable system (z, ξ_n), the interconnection term being just the state ξ_n. We can then construct a Lyapunov function $V_{n-1}(z, \xi_n, \xi_{n-1})$ that is is non-increasing along trajectories of (267). This means that the system

$$
\begin{aligned}
\dot{\xi}_{n-1} &= \xi_n \\
\dot{z} &= f(z) + \tilde{\psi}(z, \xi_n)\xi_n \\
\dot{\xi}_n &= u_n(z, \xi_n) + u_{n-1} \\
y_{n-1} &= L_g V_{n-1}
\end{aligned}
\tag{268}
$$

with input-output pair (u_{n-1}, y_{n-1}) is passive and the damping control $u_{n-1} = -y_{n-1}$ can be used to achieve global asymptotic stability

By recursively adding a new state equation to an already stabilized subsystem we are able to recursively build a Lyapunov function $V_1(z, \xi_n, \ldots, \xi_1)$ such that the entire cascade is rendered feedback passive with respect to the output $y = L_g V_1$. This output is the last one in a sequence of outputs constructed at each step.

With respect to each of these outputs, the entire system has relative degree one, but the weak minimum phase requirement is satisfied only at the last step. At each intermediate step, the zero dynamics of the entire system are unstable.

The preceding description shows that with *forwarding* the weak minimum phase requirement of feedback passivation is relaxed by allowing instability of the zero dynamics, characterized by repeated eigenvalues on the imaginary axis. Because of the peaking obstacle, however, this weak nonminimum phase requirement cannot be relaxed further without imposing additional restrictions.

6. Energy-Balancing Passivity-Based Control

The passivity based control methods discussed above all presume the existence of a storage function that is known ahead of time. Determination of this storage function directly from the $\dot{x} = f(x) + g(x)u$ description of the system is often difficult, as there are many choices. When we go back to the underlying physics behind the model, however, the storage function can often be identified from energy considerations. This is, in fact, one of the great benefits of passivity-based control methods. PBC methods can use first principle modeling knowledge to obtain and ultimately re-shape the total energy of the system in a meaningful and useful manner. The rest of this chapter discusses methods that have been developed for directly using energy concepts in passivity based control.

Let us consider a system that is connected to the external world through *port power variables* u and y, which when multiplied together has units of power. In electrical systems, u and y correspond to current and voltage. In mechanical systems these port power variables correspond to force and velocity. We confine our attention to systems that satisfy an energy balance relation

$$H(x(t)) - H(x(0)) = \int_0^T u^T(s)y(s)ds - d(t) \tag{269}$$

where H is the total energy function and $d(t)$ is a non-negative dissipation term.

We consider the problem of stabilizing this ”energy” system about a desired equilibrium point x^*. Note that by the energy balance assumptions, it should be apparent that the energy of the uncontrolled system (i.e. $u = 0$) is nonincreasing and so will actually decrease in the presence of dissipation. If the energy function is bounded below, then the system must eventually stop at a point of minimum energy. That minimum energy point, however, may not be the same as our desired operating point. So the fundamental problem we are concerned with is finding a control u that stabilizes the system's operation about a desired equilibrium point x^*.

The strategy we will use to solve this stabilization problem involves selecting a control action $u = \alpha(x) + v$ and an output z such that the redefined system with the new input v and output z satisfy a desired energy balancing equation of the form

$$H_d(x(t)) - H_d(x(0)) = \int_0^t v^T(s)z(s)ds - d_d(t)$$

where H_d is a *desired* total energy function whose minimum lies at the desired operating point x^* and whose dissipation rate equals a *desired* dissipation (i.e. convergence) rate, $d_d(t)$. Note that this strategy selects the new control and output to reshape the energy function in a way that ensures passivity and thereby allows us to stabilize through a selection of feedback control v.

One particular class of systems for which this approach works very well are mechanical systems. Let us assume our uncontrolled system has an energy function H that satisfies the dissipation relation in equation (269). Let us assume we can find a function $\alpha(x)$ such that

$$-\int_0^t \alpha^T(x(s))y(s)ds = H_a(x(t)) + k$$

for a function $H_a(x)$ such that the desired energy function

$$H_d(x) = H(x) + H_a(x)$$

has its minimum at the desired equilibrium x^*. Then the control $u = \alpha(x) + v$ will ensure the map $v \mapsto y$ is passive with storage function H_d whose minimum is at the desired equilibrium x^*. Since this transformed nonlinear system is passive, we know we can stabilize the set point x^* through a simple output feedback law. This idea of selecting α to *reshape* the energy function in a manner that assures stabilizability about a desired setpoint is sometimes called *energy-shaping* or *energy-balancing* [**OVDSMM01**].

Let us apply this energy-balancing approach to a passive system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{270}$$

From our earlier discussion we know that passivity from u to y is equivalent to the existence of a non-negative function $H : \mathbb{R}^n \to \mathbb{R}$ (i.e. the storage function) that satisfies the relations (theorem **??**)

$$\left[\frac{\partial H(x)}{\partial x}\right]^T f(x) \leq 0, \quad h(x) = \left[\frac{\partial H(x)}{\partial x} g(x)\right]^T$$

From the preceding relation, we can then establish the following

THEOREM 118. *Consider the passive system in equation (270) with storage function $V(x)$. Let $H_a(x)$ be a function such that $H_d(x) = H(x) + H_a(x)$ has a minimum at a desired equilibrium x^*. If there exists a vector function $\alpha(x)$ that satisfies the partial differential equation*

$$\left[\frac{\partial H_a(x)}{\partial x}\right]^T (f(x) + g(x)\alpha(x)) = -h^T(x)\alpha(x) \tag{271}$$

then the control input $u = \alpha(x) + v$ renders the original system in equation (270) passive.

Remark: The main point of this theorem is to highlight the fact that the ability to "shape" the storage function about a desired equilibrium depends on our ability to solve the partial differential equation in (271).

Proof: This conclusion follows immediately after noting that the left hand side of equation (271) equals $\dot{H}_a$ while the right hand side is $-y^T u$. So we see that

$$\dot{H}_a = -y^T u$$

we then integrate from 0 to t to establish the passivity of the resulting system. $\diamondsuit$

A necessary condition for the solvability of the PDE in equation (271) is that $h^T(x)\alpha(x)$ vanishes at all zeros of $f(x)+g(x)\alpha(x)$. In other words for any $\hat{x}$ such that $f(\hat{x})+g(\hat{x})\alpha(\hat{x})=0$ we can guarantee that $h^T(\hat{x})\alpha(\hat{x})=0$. Note that $f(x)+g(x)\alpha(x)$ is clearly zero at the equilibrium and so the right hand side $-y^Tu$ should also be zero at the equilibrium. This right hand side is the energy extracted from the controller and so one can conclude that we can use this energy-balancing method only if the system can be stabilized by *extracting a finite amount of energy from the controller*. This will always be the case for the regulation of mechanical systems where the extracted power is the product of force and velocity and we want to drive the velocity to zero. It may be hard to enforce for an electrical or electromechanical system where power is the product of voltages and currents and the equilibria are usually non-zero.

The preceding discussion showed that energy-balancing PBC is only applicable to systems with finite dissipation. A natural question is how one might characterize these cases where energy-balancing PBC can be used? This question may be addressed by adopting an "interconnection" viewpoint to control. Namely, including the way the "controller" is interconnected to the "plant" as a way of increasing design flexibility. In particular we view the controller Σ_c, as a one port system that will be coupled with the plant to be controlled, Σ, through a two-port interconnection system, Σ_I. This interconnection is shown in Fig. 5. The interconnection in Fig. 5 is said to be *power preserving* if the two-port subsystem, Σ, is lossless. In other words, we require

$$\int_0^t \begin{bmatrix} y^T(s) & y_c^T(s) \end{bmatrix} \begin{bmatrix} u(s) \\ u_c(s) \end{bmatrix} ds = 0$$

The following theorem provides conditions under which the interconnection in Fig. 5 is passive.

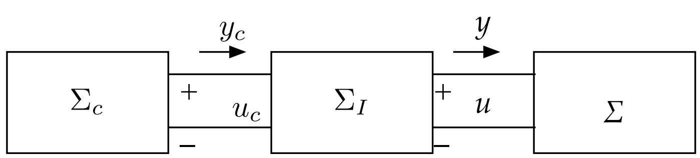

FIGURE 5. Control as Interconnection

THEOREM 119. *Consider the interconnection in Fig. 5 with some external inputs (v, v_c) as*

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = \Sigma_I \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ v_c \end{bmatrix}$$

Assume Σ and Σ_c are passive with states $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^{n_c}$ and storage functions $H(x)$ and $H_c(\zeta)$, respectively. Then the system $\begin{bmatrix} v \\ v_c \end{bmatrix} \mapsto \begin{bmatrix} y \\ y_c \end{bmatrix}$ is also passive with storage function $H(x)+H_c(\zeta)$.

Proof: This is proven by the following calculation:

$$\begin{aligned} \int_0^T \begin{bmatrix} v^T(s) & v_c^T(s) \end{bmatrix} \begin{bmatrix} y(s) \\ y_c(s) \end{bmatrix} ds &= \int_0^t u^T(s)y(s)ds + \int_0^T u_c^T(s)y_c(s)ds \\ &\geq H(x(t)) + H_c(\zeta(t)) - H(x(0)) - H_c(\zeta(0)) \end{aligned} \tag{272}$$

where the first equation follows from the lossless property of Σ_i and the last inequality is obtained from the passivity of each subsystem. $\diamondsuit$.

The preceding theorem suggests that we can use the interconnection of passive controllers to the plant through a power-preserving interconnection to shape the closed-loop total energy. While we have great freedom in assigning the controller's storage function, $H_c(\zeta)$, the system energy function $H(x)$ is given and it is not immediately apparent how we can reshape the desired energy function to have a minimum at x^*, when it is a function of both x and ζ. One way of addressing this issue is to assume that the motion of the closed-loop system is restricted to the set

$$\Omega \stackrel{\Delta}{=} \{(x,\zeta) \,:\, \zeta = F(x) + k\} \tag{273}$$

By doing this we force a functional relationship between x and ζ and so the closed-loop total energy can be expressed as a function of x only

$$V_d(x) = H(x) + H_c(F(x) + k)$$

Unfortunately, finding this F again involves solving a partial differential equation. There are, however, special classes of systems for which this approach is more tractable. This is the class of so-called *port-controlled Hamiltonian Systems* (PCHS). The following section introduces this important class of systems and then proceeds to show one can apply energy-balancing PBC to such systems. The importance of the PCHS class is that it is actually relatively large. It can be used to model a variety of networked control systems and therefore provides a framework that allows the application of PBC methods to the important class of networked control systems.

7. Port-Controlled Hamiltonian Systems

This section describes the class of port-controlled Hamiltonian (PCH) systems, derives some of the model properties and provides some applications where PCH realizations arise in a natural manner.

PCH systems are Hamiltonian systems that have been extended to include input/output ports modeling their interconnection to the external environment. These models have their origin in analytical mechanics. It is well known that one may obtain a physical system's equations of motion by invoking a principle of least action which asserts all physical processes seek to minimize their expended energy. This leads to a calculus of variations problem whose solutions must necessarily satisfy a set of *Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau \tag{274}$$

where $q = (q_1, \ldots, q_k)^T$ are generalized configuration coordinates for the system with k degrees of freedom, the Lagrangian, L, equals the difference $K - P$ between kinetic energy and potential energy, and $\tau = (\tau_1, \ldots, \tau_k)^T$ is a vector of generalized forces. In standard mechanical systems the kinetic energy is of the form $K(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, where $M(q)$ is a $k \times k$ symmetric positive definite matrix.

We define the vector of generalized momenta $p = (p_1, \ldots, p_k)^T$ as $p = \frac{\partial L}{\partial \dot{q}}$ and by taking the state vector $(q_1, \ldots, q_k, p_1, \ldots, p_k)^T$, one tranforms the k second order Euler-Lagrange equations (274) into $2k$ first order differential equations

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \tau \end{aligned} \tag{275}$$

where

$$H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + P(q) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$$

is the total energy of the system. The equations (275) are called the *Hamiltonian equations of motion* and the function H is called the *Hamiltonian*.

An immediate consequence of this system is that it satisfies *energy balance* in that

$$\dot{H} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \dot{q}^T \tau$$

The quantity $\dot{q}^T \tau$ (product of velocity and force) represents mechanical power and the time rate of change in the total energy H, is therefore seen to be equal to the amount of supplied work. So these systems satisfy a conservation of energy principle. Moreover, if the Hamiltonian $H(q,p)$ is the sum of a positive kinetic and potential energy which is bounded from below, then it follows that the Hamiltonian system with inputs $u = \tau$ and outputs $y = \dot{q}$ is *passive* (in fact lossless) with a storage function $V(q,p) = H(q,p) - C \geq 0$ in which C is a constant such that $P(q) \geq C$. Note that the difference between the storage function V and the Hamiltonian H is only a constant, so in the following we can let the storage function be represented by the Hamiltonian, H.

Port controlled Hamiltonian or PCH systems are a generalization of Hamiltonian systems that can be written as

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\,\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x} \end{aligned} \tag{276}$$

where $J(x)$ is an $n \times n$ matrix whose entries depend smoothly on x and which is assumed to be skew-symmetric

$$J(x) = -J^T(x)$$

and $R(x) = R^T(x) \geq 0$ represents the dissipation within the system. The particular form of PCH system given above is sometimes called a PCH with dissipation because of the R matrix. If $R = 0$ (i.e. no dissipation) then the the skew-symmetric nature of J, allows us to recover the energy-balance relation $\dot{H} = u^T(t)y(t)$ by showing that the system (276) is lossless if $H \geq 0$ and $R = 0$.

Port-controlled Hamiltonian realizations arise systematically from network models of physical systems that view the system as an interconnection of energy storing elements. This particular type of system arises in biological, chemical, electrical, and mechanical systems, thereby providing a physical basis for the selection of storage function upon which to develop passivity-based controls. A few examples can be used to illustrate the breadth of potential applications.

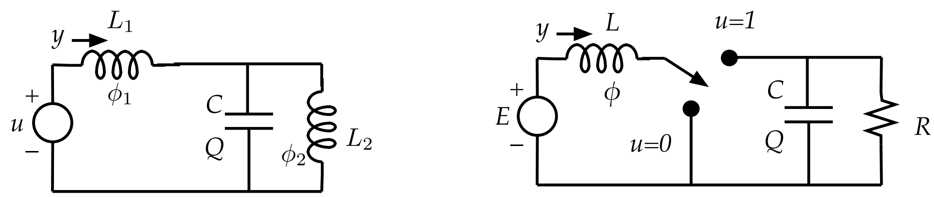

FIGURE 6. (left) controlled LC-circuit - (rigth) Boost Converter

Consider a controlled LC-circuit in Fig. 6 consisting of two inductors with magnetic energies $H_1(\phi_1)$ and $H_2(\phi_2)$ having magnetic flux linkages ϕ_1 and ϕ_2, respectively, and a capacitor with electrical energy $H_3(Q)$ (Q being the charge). If the elements are linear then $H_1(\phi_1) = \frac{1}{2L_1}\phi_1^2$, $H_2(\phi_2) = \frac{1}{2L_2}\phi_2^2$, and $H_3(Q) = \frac{1}{2C}Q^2$. We let u denote a voltage source. Using Kirchoff's laws one can obtain the following state equations

$$\begin{aligned} \begin{bmatrix} \dot{Q} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \phi_1} \\ \frac{\partial H}{\partial \phi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \frac{\partial H}{\partial \phi_1} = \text{current through first inductor} \end{aligned}$$

with $H(Q, \phi_1, \phi_2) = H_1(\phi_1) + H_2(\phi_2) + H_3(Q)$ being the total energy stored in the circuit. The skew symmetric matrix $J = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ takes its structure from the interconnections between the circuit element, thereby providing a way of writing any electrical circuit as a port-controlled Hamiltonian system.

Example: Consider a rigid body spinning about its center of mass in the absence of gravity. The energy variables are the three components of the body angular momentum p along the three principal axes $p = (p_x, p_y, p_x)$, and the energy is the kinetic energy

$$H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right)$$

where I_x, I_y, and I_z are the principal moments of inertia. Euler's equations describing the dynamics are

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{bmatrix} + g(p)u$$

Note that the skew-symmetric $J = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$ is a function of the system momenta.

Example: The next example shows how PCH models with dissipation can be used to model switching circuits. The right pane of Fig. 6 shows the circuit diagram for an idealized boost converter. Boost converters are important switching circuits that allow the near lossless stepping up or down of input voltages. In a real boost converter, the ”switch” is realized by a transistor that is being toggled back and forth between its cutoff and saturation regions. The system equations for this circuit are

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \left(\begin{bmatrix} 0 & -u \\ u & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1/R \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} E \\ y &= \frac{\partial H}{\partial x_1} \end{aligned}$$

with x_1 being the magnetic flux linkage of the inductor, x_2 being the charge of the capacitor, and the total energy $H(x_1, x_2) = \frac{1}{2L}x_1^2 + \frac{1}{2C}x_2^2$. The control input u lies in $[0, 1]$ represents the PWM switching duty ratio. The internal interconnection matrix J is either $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, depending on the ideal switch position. Clearly both of these J's are skew symmetric. Since port-controlled Hamiltonian systems are passive, we can deduce that the boost circuit may be modeled as switching between two ”passive” systems.

8. Interconnection and Damping Assignment

Consider the PCH system in equation (276) with storage function H and a desired constant equilibrium point x^*. The objective is to find a control $u = \alpha(x) + v$ so that the closed-loop dynamics is a PCH system satisfying the energy-balancing relation

$$H_d(x(t)) - H_d(x(0)) = \int_0^t v^T(s)z(s)ds - d_d(t)$$

where $H_d(x)$ is the desired total energy with a strict minimum at x^*, z is a new passive output and the dissipation term has been set equal to a desired dissipation level $d_d(t) \geq 0$. When we let $v = 0$, then the control that solves the passivation will stabilize x^* with Lypaunov function $H_d(x)$.

The procedure we will use for designing the control was presented in [**OVDSME02**] and is called interconnection damping assignment (IDA) passivity based control (PBC). The IDA-PBC design proceeds as follows. Recall that in PCH systems the internal energy exchanges are captured by the interconnection and damping matrices. So we first *fix* the desired structure of these matrices and then we derive a PDE parameterized by the chosen matrices whose solutions characterize all of the energy functions that can be assigned. Finally, we select, from this family of solutions, one that satisfies the requirements and use that solution to compute the control law. IDA-PBC therefore provides a systematic procedure for the design of passivity-based controls when the system can be realized as a port-controlled Hamiltonian system. The following theorem describes the IDA-PBC method.

THEOREM 120. **(IDA-PBC Theorem)** *Given $J(x, u)$, $R(x)$, $H(x)$, $g(x, u)$, and the desired equilibrium to be stabilized $x^* \in \mathbb{R}^n$. Assume there exist functions $\alpha(x)$, $J_a(x)$, $R_a(x)$ and a vector function $K(x)$ that* *satisfy*

$$[J(x,\alpha(x)) + J_a(x) - (R(x) + R_a(x))]\, K(x) = -\,[J_a(x) - R_a(x)]\,\frac{\partial H}{\partial x}(x) + g(x,\alpha(x)) \tag{277}$$

such that the following conditions hold

- *the structure of J_d and R_d is preserved, i.e.*

$$J_d(x) := J(x,\alpha(x)) + J_a(x) = -J_d^T(x)$$
$$R_d(x) := R(x) + R_a(x) = R_d^T(x) \geq 0$$

- *The underlying PDE is integrable, which can be assured if*

$$\frac{\partial K}{\partial x}(x) = \left[\frac{\partial K}{\partial x}(x)\right]^T \tag{278}$$

- *The equilibrium is properly assigned, i.e. $K(x)$ at x^* satisfies*

$$K(x^*) = -\frac{\partial H}{\partial x}(x^*) \tag{279}$$

- *We can assure x^* is Lyapunov stability, i.e. the Jacobian of $K(x)$ at x^* satisfies the bound*

$$\frac{\partial K}{\partial x}(x^*) > -\frac{\partial^2 H}{\partial x^2}(x^*) \tag{280}$$

Under these conditions, the closed loop system with $u = \alpha(x)$ will be a PCH system with dissipation and a total energy function H_d

$$H_d(x) := H(x) + H_a(x) \tag{281}$$

with

$$\frac{\partial H_a}{\partial x}(x) = K(x) \tag{282}$$

Furthermore x^* will be a locally stable equilibrium of the closed-loop. It will be asymptotically stable if x^* is the largest invariant set within the set of $x \in \mathbb{R}^n$ such that $\left[\frac{\partial H_d}{\partial x}(x)\right]^T R_d(x)\frac{\partial H_d}{\partial x}(x) = 0$.

Proof: For every given $\alpha(x)$, $J_a(x)$, $R_a(x)$, the solution of equation (277) is a gradient of the form (282) if and only if the integrability condition in equation (278) is satisfied. We can then use equation (281) to see that the closed-loop system is a PCH system with total energy (281).

We now prove that under (279) and (280) that the desired equilibrium is stable. Notice that the equilibrium assignment condition (279) ensures $H_d(x)$ has an extremum at x^*, while the Lyapunov stability condition (280) show that the extremum is actually an isolated minimum. On the other hand, from (270) (with $v = 0$ and a suitably defined $d_d(t)$) we have that along trajectories of the closed loop, $H_d(x(t))$ is non-increasing and therefore qualifies as a Lyapunov function. So we can conclude $x*$ is a stable equilibrium. Asymptotic stability follows from the Invariance principle. $\diamondsuit$

Theorem 120 itemizes the relationships that must occur in designing a stabilizing control for a PCH system. For systems

$$
\begin{aligned}
\dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x)\frac{\partial H}{\partial x}
\end{aligned}
$$

we can fix $J_a(x)$ and $R_a(x)$ and then look for a solution of the PDE

$$
g^{\perp}\left[J(x) + J_a(x) - (R(x) + R_a(x))\right]\frac{\partial H_a}{\partial x} = -g^{\perp}(x)\left[J_a(x) - R_a(x)\right]\frac{\partial H}{\partial x}
$$

in terms of $H_a(x)$ where $g^{\perp}$ is a left annihilator of $g(x)$. This yields a linear PDE of the form $A(x)\frac{\partial H_a}{\partial x} = b(x)$ which one can solve using the method of characteristics. The control may then be directly computed from the formula

$$
\alpha(x) = \left[g^T(x)g(x)\right]^{-1} g^T(x)\left[(J + J_a - (R + R_a))\frac{\partial H_a}{\partial x} + (J_a - R_a)\frac{\partial H}{\partial x}\right]
$$

Notice that this construction does not require us to ”guess” a candidate energy function. In many of the Lyapunov based redesign methods, one must first have a candidate Lyapunov function to start the design. For PCH systems, that choice of the Lyapunov function arises directly from the physical structure of the system.

We now illustrate the design of IDA-PBC controllers for the DC-to-DC boost power converter whose PCH model we gave in the preceding section. Recall that this system is described by a PCH model where $x = [x_1, x_2]^T$ and $g = [E, 0]^T$ with $J(s) := \begin{bmatrix} 0 & -u \\ u & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & 0 \\ 0 & 1/R \end{bmatrix}$ with $H(x) = \frac{1}{2L}x_1^2 + \frac{1}{2C}x_2^2$ as the total energy. The input $u \in [0, 1]$ represents the duty ratio of the PWM switching.

Following the method proposed in theorem 120, we initially set $J_a(x) = R_a(x) = 0$ and define the vector $K(x)$ as

$$
\begin{aligned}
K(x) = \begin{bmatrix} k_1(x) \\ k_2(x) \end{bmatrix} &= (J(\alpha(x)) - R)^{-1}\begin{bmatrix} E \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{E}{R\alpha^2(x)} \\ -\frac{E}{\alpha(x)} \end{bmatrix}
\end{aligned} \tag{283}
$$

The integrability condition (278) then becomes

$$
\frac{\partial k_2}{\partial x_1}(x) = \frac{\partial k_1}{\partial x_2}(x)
$$

which yields the nonlinear PDE

$$
\frac{\partial k_2}{\partial k_1}(x) + \frac{2}{RE}k_2\frac{\partial k_2}{\partial x_2}(x) = 0
$$

A solution to this PDE may be obtained to see that

$$
k_2(x) = \frac{c_1 x_2 + c_3}{(2/RE)c_1 x_1 + c_2} \tag{284}
$$

where c_i for $i = 1, 2, 3$ are arbitrary constants. Replacing (284) into (283) defines the control law

(285) $$\alpha(x) = -E\left(\frac{(2/RE)c_1x_1 + c_2}{c_1x_2 + c_3}\right)$$

We will now check the remaining conditions in theorem 120 to determine intervals for the constants c_i, $i = 1, 2, 3$, so that the stabilization objective is achieved with the control $u = \alpha(x)$.

- To enforce (279), we evaluate the control (285) at the equilibrium point. This gives the following linear function that relates the constants

(286) $$c_2 = -\left(\frac{2LV_d^2}{R^2E^2} + C\right)c_1 - \frac{c_3}{V_d}$$

- After some algebria, one can verify that the Hessian condition (280) is satisfied provided if $c_3 < 0$ and
$$\frac{R^2E^2}{4LV_d^3} < c_1 < \frac{1}{CV_d}c_3$$
or if $c_3 > 0$, then
$$-\frac{1}{CV_d}c_3 < c_1 < -\frac{R^2E^2}{4LV_d^3}c_3$$

Even though the controller above ensures a stability property, it requires full state feedback and is sensitive to the load resistance R, which may be unknown or time-varying. We will now show that by changing the damping structure, these two issues can be avoided.. Towards this end, select the injected damping matrix as

$$R_a = \text{diag}\{R_a, -1/R\}$$

with $R_a > 0$. This yields the desired closed-loop damping $R_d = \text{diag}(R_a, 0)$, and so

$$K(x) = \frac{1}{\alpha(x_2)}\begin{bmatrix} -\frac{1}{RC}x_2 \\ -\frac{1}{L}R_ax_1 - E + \frac{R_a}{RC}\frac{x_2}{\alpha(x_2)} \end{bmatrix}$$

Since α is only a function of x_2, the integrability condition $\left(\frac{\partial k_2}{\partial x_1} = \frac{\partial k_1}{\partial x_2}\right)$ reduces to the ODE

$$\frac{d\alpha}{dx_2}(x_2) = \frac{1 - R_aRC/L}{x_2}\alpha(x_2) = \frac{\gamma}{x_2}\alpha(x_2)$$

This ODE can be solved by the separation of variables method to get $\alpha(x_2) = c_1x_2^\gamma$ where c_1 is a constant that we choose a $c_1 = \frac{u^*}{(x_2^*)^\gamma}$ to assign the equilibrium. To ensure that x^* is a minimum of $H_d(x)$, we examine the Hessian and verify that it is positive definite if and only if $-1 < \gamma < 1$. In summary, we've shown that the output feedback IDA-PBC

$$\alpha(x_2) = u^*\left(\frac{x_2}{x_2^*}\right)^\gamma, \text{for } 0 < \gamma < 1$$

asymptotically stabilizes x^* for all load resistance $R > 0$.

9. IDA-PBC Example: stabilization of food-chain

This section concludes the chapter with a system that is neither electrical or mechanical to demonstrate that these approaches may also be applied to biological systems [OABR00]. In particular, we consider a second order predator-prey system

$$\begin{aligned}\dot{x}_1 &= f(x) - x_1 \\ \dot{x}_2 &= -f(x) - x_2 + u\end{aligned}$$

The state variables represent the biomass of a two guilds in an aquatic ecosystem. The function $f(x)$ describes the predation response function. If one considers the Lotka-Volterra model [Lot25], then $f(x) = x_1 x_2$, if we consider a more realistic consumer-resource model [YI92] then $f(x) = \frac{x_1 x_2}{1+x_1}$ to show that predator consumption is bounded. In this section we'll use the Lotka-Volterra model. The control action u represents an inflow or outflow rate for the predator that could be associated with stocking or culling the existing population.

Note that the evolution of the system is restricted to the non-negative orthant, i.e. $x_i(t) \geq 0$ for all t. The control objective is to stabilize a specified positive equilibirum x^* with positive control (i.e. u ¿ 0 (stocking)). We define the total mass $H(x) = x_1 + x_2$ as the system's Hamiltonian and write the PCH equations with

$$J(x) = \begin{bmatrix} 0 & x_1 x_2 \\ -x_1 x_2 & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$$

with $g = [0, 1]^T$.

Since the system is fully damped, we can set $R_a(x) = 0$, (no additional damping is needed). In this case the vector function $K(x)$ reduces to

$$K(x) = \frac{-u(x)}{1 + x_1 x_2} \begin{bmatrix} 1 \\ \frac{1}{x_2} \end{bmatrix}$$

from which we can conclude that

$$x_2 k_2(x) = k_1(x) \tag{287}$$

The integrability condition in this two-dimensional case reduces to $\frac{\partial k_1}{\partial x_2}(x) = \frac{\partial k_2}{\partial x_1}(x)$, which combined with (287) yields the linear PDE

$$\frac{\partial k_1}{\partial x_1}(x) - x_2 \frac{\partial k_1}{\partial x_2}(x) = 0 \tag{288}$$

A family of solutions to this PDE is

$$k_1(x) = \Phi(\zeta(x)), \quad \zeta(x) = x_1 + \log x_2$$

for any differentiable function $\Phi(\cdot)$. From equation (287), we also obtain

$$k_2(x) = \frac{1}{x_2} \Phi(\zeta(x))$$

The equilibrium condition requires

$$\begin{bmatrix} k_1(x^*) \\ k_2(x^*) \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence $\Phi(\cdot)$ must be such that $\Phi(\zeta(x^*)) = -1$, where $\zeta(x^*) = x_1^* + \log x_2^* = x_1^*$. It is clear then that we cannot take $\Phi(\zeta) = \zeta$. Instead we use the function $\Phi(\zeta) = c_1 e^{c_2 \zeta}$ with c_1 and c_2 constants to be defined.

We now verify the Hessian condition. Some algebra shows that

$$\frac{\partial K}{\partial x}(x) = c_1 c_2 e^{c_2 \zeta} \begin{bmatrix} 1 & \frac{1}{x_2} \\ \frac{1}{x_2} & \frac{1}{x_2^2}\left(1 - \frac{1}{c_2}\right) \end{bmatrix}$$

which when evaluated at the equilibrium point yields,

$$\frac{\partial K}{\partial x}(x^*) = -c_2 \begin{bmatrix} 1 & 1 \\ 1 & \frac{c_2 - 1}{c_2} \end{bmatrix}$$

The determinant of this matrix is 1, and so it is positive definite if and only if $c_2 < 0$.

We now investigate the asymptotic stability properties. Note that the set

$$S = \left\{ x \in \mathbb{R}_+^2 \ : \ -x_1(1 + k_1(x))^2 - x_2(1 + k_2(x))^2 = 0 \right\}$$

consists only of the ponts $x = 0$ and $x = x^*$. It can be easily shown that $x = 0$ is unstable. So we have just shown that with the control

$$u(x) = (1 + x_1 x_2) x_2^c e^{c(x_1 - x_1^*)}$$

that all trajectories starting in $x(0)$ will converge to an exquilibrium $(x_1^*, 1)$.

10. Concluding Remarks

Passivity-based control is an important and useful method for constructing stabilizing controllers. The method redefines the output of the system and inputs of the system (though a feedback transformation) that renders the redefined system passive. Once passivated, it then becomes possible to stabilize the equilibrium of the system through a simple output feedback law. When that passivated system is zero-state detectable, then the equilibrium is asymptotically stable. The use of such feedback transformations, however, also limits the type of systems for which this feedback passivation method is applicable. In particular, it only works if the passivated system is minimum phase with a relative degree of one. When this is not the case, then more modern recursive methods such as backstepping, forwarding, or an interlacing of the two will need to be employed to circumvent these obstacles. Another potential issue with these methods is that they require knowledge of a storage function for the passivated system. Such storage functions may, in general, be difficult to obtain unless we can take advantage of first principle modeling of the system. This is possible for systems that can be realized as port-controlled Hamiltonian systems. In this case, the system's Hamiltonian and knowledge about its interconnection and dissipations structures can be used to provide a systematic method for constructing stabilizing controllers.

The discussion in this chapter was taken from two primary sources, the original book [**SJK12**] on constructive design methods and the original paper [**OVDSME02**].