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A WEIGHTED SEMILINEAR ELLIPTIC EQUATION INVOLVING CRITICAL SOBOLEV EXPONENTS

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Abstract. In this paper we prove the existence of a positive radial solution of the problem

$$-\Delta u = r^\sigma |u|^{p-1}u + \lambda r^\alpha u, \quad \text{in } B_R \subset \mathbf{R}^N \ (r = |x|)$$

for λ in a suitable (and almost optimal) range. Here $N \geq 3$, $\alpha, \sigma \geq -2$ and $p = (N + 2 + 2\sigma)/(N-2)$ corresponds to the critical Sobolev exponent $p + 1 = (2N + 2\sigma)/(N-2)$. Our result extends the previous one due to Brézis and Nirenberg when $\sigma = \alpha = 0$.

0. Introduction. In a previous paper [8] we considered the problem

$$\begin{cases} -\dfrac{1}{r^\gamma}(r^\gamma u')' = r^\sigma |u|^{q-1}u & \text{in } (0,1) \\ u(1) = 0, \quad \displaystyle\int_0^1 r^\gamma |u'|^2\, dr < \infty \\ u > 0. \end{cases} \tag{0.1}$$

We recall some of the results we obtained there.

"If $\gamma > 1$ then the problem has exactly one weak solution for $1 < q < \frac{\gamma+3+2\sigma}{\gamma-1}$ and no weak solution for $q > (\gamma + 3 + 2\sigma)/(\gamma - 1)$."

In this paper we shall deal exactly with the critical case, namely, $q = p = (\gamma + 3 + 2\sigma)/(\gamma - 1)$. Instead of (0.1) we consider the more general problem

$$\begin{cases} -\dfrac{1}{r^\gamma}(r^\gamma u')' = r^\sigma |u|^{p-1}u + \lambda r^\alpha u & \text{in } (0,\ 1) \\ u(1) = 0, \quad \displaystyle\int_0^1 r^\gamma |u'|^2\, dr < \infty \\ u > 0 \end{cases} \tag{0.2}$$

where $\gamma > 1$, $\sigma, \alpha > -2$.

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In order to describe our results we need to explain some notation. Let us consider the linear eigenvalue problems

$$-\frac{1}{r^\gamma}(r^\gamma u')' = \lambda r^\alpha u, \quad u(1)=0, \quad \int_0^1 r^\gamma |u'|^2\,dr < \infty \tag{0.3}$$

and

$$-\frac{1}{r^{2-\gamma}}(r^{2-\gamma} v')' = \mu r^\alpha v, \quad v(1)=0, \quad \int_0^1 r^{2-\gamma} |v'|^2\,dr < \infty \tag{0.4}$$

which should be understood in a suitable weak sense.

We denote by $\lambda_1(\alpha,\gamma)$ the least eigenvalue of (0.3) which exists for $\gamma > 1$, $\alpha > -2$ and by $\mu_1(\alpha,\gamma)$ the first eigenvalue of (0.4) which exists for $1 < \gamma < \alpha + 3$. We can now state our main results.

A. There exists no weak solution of (0.2) for $\lambda \le 0$ or $\lambda \ge \lambda_1(\alpha)$.

B. If $\gamma > \alpha + 3$ then (0.2) has at least a weak solution if and only if $\lambda \in \big(0, \lambda_1(\alpha)\big)$.

C. If $1 < \gamma < \alpha + 3$ then $\mu_1(\alpha,\gamma) < \lambda_1(\alpha,\gamma)$ and problem (0.2) has a weak solution for each $\lambda \in \big(\mu_1(\alpha,\gamma), \lambda_1(\alpha,\gamma)\big)$.

The proof of these results uses a variational method together with the techniques of Brézis-Nirenberg [3] in order to overcome the difficulties raised by the lack of compactness due to the critical Sobolev exponent $p+1 = (2\gamma+2+2\sigma)/(\gamma-1)$.

Our results can be directly applied to elliptic PDEs yielding a twofold generalization of some of the results of the Brézis-Nirenberg paper [3]. The paper is divided into three sections. Section 1 deals with weighted Sobolev spaces. In particular, here is proved the existence of a best Sobolev constant in a critical Sobolev imbedding and we compute it explicitly. Section 2 is the core of the paper. Here are stated and proved the existence results for (0.2). Section 3 is devoted to the proof of the inequality $\mu_1(\alpha,\gamma) < \lambda_1(\alpha,\gamma)$ for $1 < \gamma < \alpha + 3$.

1. Imbedding theorems for weighted Sobolev spaces. Existence of a best constant in the critical case. We first recall some known facts about weighted Sobolev spaces which we have stated and proved in [8] in a form suitable for our purposes. Let $R \in (0, +\infty]$, E_γ^R is the closure of the set

$$S = \{u \in C^1[0,R] : u \equiv 0 \text{ in a neighborhood of } R\}$$

in the norm $\|\cdot\|_{\gamma,R}$ defined by

$$\|u\|_{\gamma,R} = \Big(\int_0^R r^\gamma |u'|^2\,dr\Big)^{1/2}.$$

When $R = 1$ we write simply $\|u\|_\gamma$, $L_\theta^q(0,R)$ is the weighted $L^q(0,R; r^\theta\,dr)$.

We recall the following results (see [8] for a proof).

Radial Lemma. *There exists $C = C(\gamma) > 0$ such that for $u \in E_\gamma^R$*

$$|u(r)| \le \frac{C}{r^{(\gamma-1)/2}} \|u\|_{\gamma,R}, \quad \forall r \in (0,R).$$

Imbedding Lemma. *Let $\gamma > 1$, $R < \infty$ and $\theta > \max(-1, \gamma - 2)$. Then $E_\gamma^R \hookrightarrow L_\theta^q(0, R)$ continuously if and only if*

$$\frac{\theta+1}{q} \geq \frac{\gamma-1}{2}. \tag{1.1}$$

Compactness Lemma. *Let $\gamma > 1$, $R < \infty$ and $\theta > \max(-1, \gamma - 2)$. The imbedding $E_\gamma^R \hookrightarrow L_\theta^q(0, R)$ is compact if*

$$\frac{\theta+1}{q} > \frac{\gamma-1}{2}. \tag{1.1'}$$

We are interested mainly in the critical case; i.e., the situation $(\theta+1)/q = (\gamma-1)/2$ hence

$$q = \gamma^* = \frac{2(\theta+1)}{\gamma-1}. \tag{1.2}$$

The continuity of the critical imbedding is equivalent to the existence of a constant $K > 0$ such that

$$\|u\|_{L_\theta^{\gamma^*}(0,R)} \leq K\|u\|_{\gamma,R}. \tag{1.3}$$

In fact (1.3) holds also with $R = \infty$ (see Maz'ya [7], Sect. 1.3.1). Set

$$S(\gamma, \theta, R) = \inf\{\|u\|_{\gamma,R}^2 \,|\, u \in E_\gamma^R,\ \|u\|_{L_\theta^{\gamma^*}(0,R)}^2 = 1\}. \tag{1.4}$$

Following the ideas in Aubin [1] we can prove the following result.

Proposition 1.1. *Let $\theta > \gamma - 2 > -1$. Then in (1.4) with $R = \infty$ the infimum $S(\gamma, \theta, \infty)$ is achieved by the function*

$$\widetilde{U(r)} = \frac{C}{(1 + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}} = C \cdot U(r), \quad \sigma = \theta - \gamma \tag{1.5}$$

where C is a normalization constant; i.e., such that $\|\tilde{U}\|_{L_\theta^{\gamma^*}(0,\infty)} = 1$.

Proof: The proof will be carried out in two steps.

Step 1. If the infimum is achieved then it can also be achieved by a positive decreasing function. Let us assume the infimum is achieved. Obviously $\|u\|_{\gamma,\infty} = \||u|\|_{\gamma,\infty}$ and $\|u\|_{L_\theta^{\gamma^*}(0,\infty)} = \||u|\|_{L_\theta^{\gamma^*}(0,\infty)}$. (The former equality is a consequence of a variant of Stampacchia's lemma; see [6] for a proof.) Hence the infimum can also be reached by positive functions. The functions that realize the infimum satisfy the Euler-Lagrange equation

$$-\frac{1}{r^\gamma}(r^\gamma u')' = \lambda r^\sigma |u|^{\gamma^*-2}u, \tag{1.6}$$

where $\lambda \in \mathbf{R}^*$ is a Lagrange multiplier. We may assume $u > 0$. By the Radial Lemma we infer

$$\lim_{r\to\infty} u(r) = 0. \tag{1.7}$$

We make the change of variables $s = r^{-(\gamma-1)}$ and we denote $v(s) = u(r)$. Then v satisfies the following equation

$$v_{ss} + \frac{1}{(\gamma-1)^2}\frac{\lambda v^{\gamma^*-1}}{s^{2+(2+\sigma)/(\gamma-1)}} = 0 \quad \text{in } (0,\infty) \tag{1.8}$$

with $v(0) = 0$ and $v_s = dv/ds$. Hence the function λv is concave. One of the following two situations may occur.

A. $\lambda < 0$. Then v is convex and hence $v'(0)$ exists and $v'(0) \geq 0$. Therefore v is increasing and obviously u is decreasing.

B. $\lambda > 0$. Then v is concave and since $v > 0$ near ∞ we get $v_s(\infty) = 0$ and consequently v is increasing. Again we get that u is decreasing.

Step 2. The infimum is achieved by the function (1.5). It is easily seen that the function (1.5) satisfies (1.6) with a suitable λ. The statement of Step 2 follows from a sharp result due to G.A. Bliss [2]. We state a special case of it.

Lemma 1.2. *Let $q > 2$ and let $h(x) \geq 0$ be a measurable real-valued function such that $\int_0^\infty h^2(x)\,dx$ is finite. Set $g(x) = \int_0^x g(t)\,dt$. Then*

$$\Big(\int_0^\infty g^q(x)x^{\alpha-q}\,dx\Big) \leq K\Big(\int_0^\infty h^2(x)\,dx\Big)^{q/2} \tag{1.9}$$

where $q = 2\alpha - 2$ and $K = 1/(q-\alpha-1)\big[(\alpha\Gamma(q/\alpha))/(\Gamma(1/\alpha)\Gamma((q-1)/\alpha))\big]^\alpha$.

Here Γ is Euler's gamma function. The relation (1.9) holds with equality for every function $h(x)$ of the form

$$H_\alpha(x) = \frac{C}{(dx^\alpha+1)^{(\alpha+1)/\alpha}}. \tag{1.10}$$

We see that (1.9) can be restated as

$$\Big(\int_0^\infty g^q(x)x^{\alpha-q}\,dx\Big)^{1/q} \leq K^{1/q}\int_0^\infty |g'(x)|^2\,dx, \tag{1.9$'$}$$

for every increasing function g such that $g(0) = 0$ and $g' \in L^2(0,\infty)$. If in (1.9′) we make the change in variables $x = 1/r^\nu$, $u(r) = g(x)$ then we get

$$\nu^{1/q}\Big(\int_0^\infty u^q(r)r^{((\nu q)/2)-1}\,dr\Big)^{1/q} \leq K^{1/q}\nu^{-1}\Big(\int_0^\infty |u'(r)|^2 r^{\nu+1}\,dr\Big)^{1/2} \tag{1.11}$$

for every positive decreasing function u such that $u(r) \to 0$ as $r \to \infty$. If in (1.11) we further specialize ν and q such that $\nu + 1 = \gamma$, $\frac{\nu q}{2} - 1 = \theta$ we get $\nu = \gamma - 1$ and $q = (2(\theta+1))/(\gamma-1)$ and we obtain the critical Sobolev imbedding. Moreover (1.11) becomes equality when $d/dx\, u(r) = H_\alpha(x)$, where $x = r^{-(\gamma-1)}$. This happens when $u(r) = U(r)$ where U is given by (1.5).

Remark 1.3. The result above allows one to compute the exact value of $S(\gamma,\theta,\infty)$. In fact

$$S(\gamma,\theta,\infty) = \frac{\|U\|^2_{\gamma,\infty}}{\|U\|^2_{L^{\gamma^*}_\theta(0,\infty)}} \tag{1.12}$$

$$\|U\|^2_{\gamma,\infty} = \int_0^\infty r^\gamma |U'|^2\,dr, \quad U'(r) = (\gamma-1)\frac{r^{1+\sigma}}{(1+r^{2+\sigma})^{(\gamma+1+\sigma)/(2+\sigma)}}$$

$$\|U\|^2_{\gamma,\infty} = (\gamma-1)^2\int_0^\infty \frac{r^{\gamma+2+2\sigma}}{(1+r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,dr$$

$$= \frac{(\gamma-1)^2}{2+\sigma}\int_0^\infty \frac{s^{(\gamma+3+2\sigma)/(2+\sigma)-1}}{(1+s)^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,ds$$

so that

$$\|U\|^2_{\gamma,\infty} = \frac{(\gamma-1)^2}{2+\sigma}\frac{\Gamma\left(\frac{\gamma+3+2\sigma}{2+\sigma}\right)\Gamma\left(\frac{\gamma-1}{2+\sigma}\right)}{\Gamma\left(\frac{2(\gamma+1+\sigma)}{2+\sigma}\right)}, \tag{1.13}$$

where we have used the formula

$$\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}, \quad \forall m,n>0 \tag{1.14}$$

(cf. Dwight [4]). We now compute in a similar way

$$\|U\|^{\gamma^*}_{L_\theta^{\gamma^*}(0,\infty)} = \int_0^\infty \frac{r^\theta}{(1+r^{2+\sigma})^{(2(\theta+1))/(2+\sigma)}}\,dr \qquad (\theta=\gamma+\sigma)$$

$$= \frac{1}{2+\sigma}\int_0^\infty \frac{s^{(\gamma+\sigma+1)/(2+\sigma)-1}}{(1+s)^{(2(\gamma+\sigma+1))/(2+\sigma)}}\,ds = \frac{1}{2+\sigma}\frac{\left[\Gamma\left(\frac{\gamma+1+\sigma}{2+\sigma}\right)\right]^2}{\Gamma\left(\frac{2(\gamma+1+\sigma)}{2+\sigma}\right)} \tag{1.15}$$

where again we have used (1.14). It is a straightforward observation that $S(\gamma,\theta,R)$ is invariant under rescaling so that it does not really depend on R. We set

$$S(\gamma,\theta) := S(\gamma,\theta,R), \quad R\in(0,\infty).$$

2. Existence of a positive solution. In this section we consider the existence question for the following boundary value problem

$$\begin{cases} -\dfrac{1}{r^\gamma}(r^\gamma u')' = r^\sigma|u|^{p-1}u + \lambda r^\alpha u & \text{in } (0,1),\ \lambda\in R, \\ \alpha>-2,\ \sigma>-2,\ u\in E_\gamma,\ u>0 \end{cases} \tag{2.1}$$

where $p=(\gamma+3+2\sigma)/(\gamma-1)$, $\gamma>1$, $\sigma>-2$ so that the imbedding $E_\gamma \hookrightarrow L_\theta^{p+1}$, $\theta=\gamma+\sigma$ is noncompact.

We shall look for a weak solution of (2.1); i.e., a function $u\in E_\gamma$, $u>0$ such that

$$\int_0^1 r^\gamma u'\varphi'\,dr = \int_0^1 [r^\theta u^p\varphi + \lambda r^{\gamma+\alpha}u\varphi]\,dr, \quad \forall\varphi\in E_\gamma. \tag{2.2}$$

Remark 2.1. (2.1) has no weak solution for $\lambda\le 0$.

This will follow from a Pohozaev-type argument similar to that used in [8]. Assume the contrary; i.e., there exists a weak solution u of (2.1) with $\lambda \leq 0$. In (2.2) we set $\varphi = u$ and we get

$$\int_0^1 r|u'|^2\,dr = \int_0^1 [r^\theta u^{p+1} + \lambda r^{\gamma+\alpha}u^2]\,dr. \tag{2.3}$$

By standard elliptic regularity, $u \in C^2(0,1)$ so that u is a classical solution of

$$-(r^\gamma u')' = r^\theta u^p + \lambda r^{\gamma+\alpha}u, \quad \text{in } (0,1), \quad u(1) = 0. \tag{2.4}$$

We multiply (2.4) with ru' and we get

$$\begin{aligned} &r^{\gamma+1}u'u'' + \gamma r^\gamma |u'|^2 + r^{\theta+1}u^pu' + \lambda r^{\gamma+\alpha}uu' = 0\\ &= \frac{1}{2}r^{\gamma+1}\frac{d}{dr}|u'|^2 + r\gamma|u'|^2 + \frac{1}{p+1}r^{\theta+1}\frac{d}{dr}(u^{p+1}) + \frac{\lambda}{2}r^{\gamma+\alpha}\frac{d}{dr}(u^2) = 0. \end{aligned}$$

We integrate the last inequality by parts on $(\varepsilon,1)$. We obtain

$$\begin{aligned} &\tfrac{1}{2}|u'(1)|^2 - \tfrac{1}{2}e^{\gamma+1}|u'(\varepsilon)|^2 - \frac{1}{p+1}\varepsilon^{\theta+1}u^{p+1}(\varepsilon) - \frac{\lambda}{2}\varepsilon^{\gamma+\alpha+1}u^2(\varepsilon)\\ &+ \frac{\gamma-1}{2}\int_\varepsilon^1 r^\gamma|u'(r)|^2 dr - \frac{\theta+1}{p+1}\int_\varepsilon^1 r^\theta u^{p+1}(r)\,dr\\ &- \frac{\lambda(\gamma+\alpha+1)}{2}\int_\varepsilon^1 r^{\gamma+\alpha}u^2(r)\,dr = 0. \end{aligned} \tag{2.5}$$

Since $u \in E_\gamma \cap L_\theta^{p+1}$ we get that on a subsequence $\varepsilon_k \to 0$

$$\varepsilon_k^{\gamma+1}|u'(\varepsilon_k)|^2 + \varepsilon_k^{\theta+1}u^{p+1}(\varepsilon)_k \to 0 \quad \text{as } k \to \infty. \tag{2.6}$$

By the Radial Lemma $u^2(\varepsilon) \leq$ const. $\varepsilon^{-(\gamma-1)}$ so that

$$\varepsilon^{(\gamma+\alpha+1)}u^2(\varepsilon) \leq \text{ const. } \varepsilon^{\alpha+2} = O(1) \quad \text{as } \varepsilon \to 0, \text{ since } \alpha > -2. \tag{2.7}$$

If in (2.5) we let $\varepsilon = \varepsilon_k \to 0$ we infer by (2.6), (2.7)

$$\begin{aligned} &\tfrac{1}{2}|u'(1)|^2 + \frac{\gamma-1}{2}\int_0^1 r^\gamma|u'(r)|^2\,dr - \frac{\theta+1}{p+1}\int_0^1 r^\theta u^{p+1}(r)\,dr\\ &- \frac{\lambda(\gamma+\alpha+1)}{2}\int_0^1 r^{\gamma+\alpha}u^2(r)\,dr = 0. \end{aligned} \tag{2.8}$$

From (2.3) and (2.8) we infer

$$\tfrac{1}{2}|u'(r)|^2 + \Big(\frac{\gamma-1}{2} - \frac{\theta+1}{p+1}\Big)\int_0^1 r^\gamma|u'(r)|^2\,dr - \frac{\lambda(\alpha+2)}{2}\int_0^1 r^{\gamma+\alpha}u^2(r)\,dr = 0$$

and finally since $(\gamma-1)/2=(\theta+1)/(p+1)$

$$\tfrac{1}{2}|u'(1)|^2=\frac{\lambda(\alpha+2)}{2}\int_0^1 r^{\gamma+\alpha}u^2(r)\,dr\le 0. \tag{2.9}$$

We get $u'(1)=0$. We know that also $u(1)=0$. Therefore it follows—according to the uniqueness in a Lipschitzian Cauchy problem—that $u\equiv 0$.

A special part in our considerations will be played by the following generalized engenvalue problem

$$-\frac{1}{r^\gamma}(r^\gamma u')'=\lambda r^\alpha u,\quad \lambda\in\mathbf{R},\ \alpha>-2\quad u\in E_\gamma, \tag{2.10}$$

which is meant in the following generalized sense

$$\int_0^1 r^\gamma u'\varphi'\,dr=\lambda\int_0^1 r^{\gamma+\alpha}u\varphi\,dr,\quad \forall\varphi\in E_\gamma. \tag{2.11}$$

Since for $\alpha>-2$ the imbedding $E_\gamma\hookrightarrow L^2_{\gamma+\alpha}$ is compact one can prove in a standard manner the following facts:

(F1) The spectrum of (2.10) consists of an unbounded sequence of positive eigenvalues $0<\lambda_1(\alpha,\gamma)<\lambda_2(\alpha,\gamma)\le\lambda_3(\alpha,\gamma)<\cdots\to\infty$, each of them having finite multiplicity.

(F2) The eigenvalue $\lambda_1(\alpha,\gamma)$ is simple and the corresponding eigenspace is generated by a positive eigenfunction.

(F3) For every $\gamma>1$ the mapping $\alpha\to\lambda_1(\alpha,\gamma)$ is decreasing and continuous.

(For a proof of these by now classical statements we refer the reader to the work of D.J. de Figueiredo [5].)

When there is no possibility of confusion we shall write $\lambda_1(\alpha)$ instead of $\lambda_1(\alpha,\gamma)$.

Remark 2.2. Problem (2.1) has no weak solution for $\lambda\ge\lambda_1(\alpha,\gamma)$, $\gamma>1$, $\alpha>-2$. Indeed, we set in (2.2) $\varphi=\varphi_1$. We get

$$\int_0^1 r^\gamma u'\varphi_1'\,dr=\int_0^1[r^\theta u^p\varphi_1+\lambda r^{\gamma+\alpha}u\varphi_1]\,dr.$$

If in (2.11) we set $\varphi=u$ and $\lambda=\lambda_1(\alpha)$ we get

$$\int_0^1 r^\gamma u'\varphi_1'\,dr=\lambda_1\int_0^1 r^{\gamma+\alpha}u\varphi_1\,dr.$$

We infer

$$0\ge(\lambda-\lambda_1)\int_0^1 r^{\gamma+\alpha}u\varphi_1\,dr=\int_0^1 r^\theta u^p\varphi_1\,dr.$$

Due to the fact that $\varphi_1>0$ it follows $u\equiv 0$.

From the two remarks above we see that a necessary condition for the existence of a solution of (2.1) is that $\lambda\in\big(0,\lambda_1(\alpha)\big)$.

Following the ideas of Brézis-Nirenberg [3] we shall consider the minimization problem

$$S_\lambda = S_\lambda(\gamma, \theta) = \inf\{\|u\|_\gamma^2 - \lambda \|u\|^2_{L^2_{\gamma+\alpha}};\ \|u\|_{L^{p+1}_\theta} = 1\}. \tag{2.12}$$

If u is a solution of (2.12) we may assume $u \geq 0$ for otherwise we replace u by $|u|$. Then u satisfies

$$-\frac{1}{r^\gamma}(r^\gamma u')' - \lambda r^\alpha u = S_\lambda r^\sigma u^p \quad \text{in } (0,1), \quad u \in E_\gamma$$

in the weak sense (2.2). It follows that if $S_\lambda > 0$ then ku satisfies (2.2) for some appropriate constant $k > 0$ (namely $k = S_\lambda^{1/(p-1)}$). Thus in order to obtain a solution of (2.1) it is sufficient to check the following conditions.

$$\text{Problem (2.12) has a solution.} \tag{2.13}$$

$$S_\lambda > 0. \tag{2.14}$$

Condition (2.14) holds if and only if $\lambda < \lambda_1(\alpha)$ which is (according to Remark 2.2) a necessary condition for the existence of a solution of (2.1).

A sufficient condition so that (2.13) holds is supplied by the following result.

Proposition 2.3. *If $S_\lambda < S$ then the minimization problem* (2.12) *has a solution.*

The proof of this result follows the same lines as the proof of Lemma 1.2 in the paper of Brézis-Nirenberg [3] so we omit it.

Our task is to check when $S < S_\lambda$. We follow the arguments used in [3] Lemma 1.1 and Lemma 1.3. We treat separately two cases.

2.1. The case $\gamma \geq \alpha + 3$. The main result of this subsection is the following.

Proposition 2.4. *Let $\gamma \geq \alpha + 3$. Then $S_\lambda(\gamma, \theta) < S(\gamma, \theta)$ for every $\lambda > 0$.*

Proof: We shall estimate the ratio

$$Q_\lambda(u_\varepsilon) = \frac{\|u\|_\gamma^2 - \lambda\|u\|^2_{L^2_{\gamma+\alpha}}}{\|u_\varepsilon\|^2_{L^{p+1}_\theta}}, \quad \theta = \gamma + \sigma$$

with

$$u_\varepsilon(r) = \frac{\varphi(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}}, \quad \varepsilon > 0 \tag{2.15}$$

where $\varphi \in C^\infty[0,1)$ is a *fixed* function such that $\varphi(r) \equiv 1$ for r in some neighborhood of 0 and $\varphi(r) \equiv 0$ in some neighborhood of 1.

We claim that as $\varepsilon \to 0$ we have

$$\|u_\varepsilon\|_\gamma^2 = \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \tag{2.16}$$

$$\|u_\varepsilon\|^2_{L^{p+1}_\theta} = \frac{K_2}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \tag{2.17}$$

$$\|u_\varepsilon\|^2_{L^2_{\gamma+\alpha}} = \begin{cases} \dfrac{K_3}{\varepsilon^{(\gamma-3-\sigma)/(2+\sigma)}} + O(1), & \gamma > \alpha+3 \\ K_3|\log\varepsilon|, & \gamma = \alpha+3 \end{cases} \tag{2.18}$$

where K_1, K_2, K_3 are positive constants such that $K_1/K_2 = S$.

Verification of (2.16).

$$u'_\varepsilon(r) = \frac{\varphi'(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}} - (\gamma-1)\frac{r^{1+\sigma}\varphi(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma+1+\sigma)/(2+\sigma)}}$$

Since $\varphi \equiv 1$ near 0 it follows that

$$\begin{aligned} \|u_\varepsilon\|^2_\gamma &= \int_0^1 r^\gamma |u'_\varepsilon|^2\,dr = (\gamma-1)^2 \int_0^1 \frac{r^{\gamma+2+2\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,dr + O(1) \\ &= \frac{1}{\varepsilon^{(\gamma-1)/(2+\sigma)}}(\gamma-1)^2 \int_0^{\varepsilon^{-1/(2+\sigma)}} \frac{s^{\gamma+2+2\sigma}}{(1+s^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,ds \\ &= \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \end{aligned}$$

where

$$K_1 = (\gamma-1)^2 \int_0^\infty \frac{s^{\gamma+2+2\sigma}}{(1+s^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,ds = \|U\|^2_{\gamma,\infty}$$

and U is given by (1.5).

Verification of (2.17).

$$\begin{aligned} &\int_0^1 u_\varepsilon^{p+1} r^\theta\,dr = \int_0^1 \frac{r^{\gamma+\sigma}\varphi^{p+1}(r)}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,dr \\ &= \int_0^1 \frac{\varphi^{p+1}(r)-1}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,dr + \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,dr \\ &= \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,dr + O(1) = \frac{K'_2}{\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)}} + O(1) \end{aligned}$$

where

$$K'_2 = \int_0^\infty \frac{r^{\gamma+\sigma}}{(1+r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\,dr = \|U\|^{p+1}_{L^{p+1}_\theta(0,\infty)}\,.$$

Thus (2.17) follows with $K_2 = \|U\|^2_{L^{p+1}_\theta(0,\infty)}$ and $K_1/K_2 = S(\gamma,\theta)$.

Verification of (2.18). We have

$$\begin{aligned} \|u_\varepsilon\|^2_{L^2_{\gamma+\alpha}} &= \int_0^1 \frac{[\varphi^2(r)-1]r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\,dr + \int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\,dr \\ &= \int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\,dr + O(1). \end{aligned}$$

When $\gamma > \alpha + 3$ we have

$$\int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr = \frac{1}{\varepsilon^{(\gamma-3-\alpha)/(2+\sigma)}} \int_0^{\varepsilon^{-1/(2+\sigma)}} \frac{s^{\gamma+\alpha}}{(1+s^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} ds$$
$$= \frac{1}{\varepsilon^{(\gamma-3-\alpha)/(2+\sigma)}} \int_0^{\infty} \frac{s^{\gamma+\alpha}}{(1+s^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} ds + O(1)$$

and thus (2.18) follows with

$$K_3 = \int_0^{\infty} \frac{s^{\gamma+\alpha}}{(1+s^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} ds.$$

When $\gamma = \alpha + 3$ we have

$$\int_0^1 \frac{r^{2\alpha+3}}{(\varepsilon + r^{2+\sigma})^{(2(\alpha+2))/(2+\sigma)}} dr = \frac{1}{2+\sigma} \int_0^1 \left(\frac{s}{\varepsilon + s}\right)^{(2(\alpha+2))/(2+\sigma)} \frac{ds}{s}$$
$$= \frac{1}{2+\sigma} |\log \varepsilon| + O(1)$$

and thus (2.18) follows with $K_3 = 1/(2+\sigma)$.

Combining (2.16), (2.17) and (2.18) we get

$$Q_\lambda(u_\varepsilon) = \begin{cases} S - \lambda \frac{K_3}{K_2} \varepsilon^{(2+\alpha)/(2+\sigma)} + O(\varepsilon^{(\gamma-1)/2+\sigma)}), & \gamma > \alpha + 3 \\ S - \lambda \frac{K_3}{K_2} \varepsilon^{(2\alpha+2)/(2+\sigma)} |\log \varepsilon| + O(\varepsilon^{(2\alpha+2)/(2+\sigma)}), & \gamma = \alpha + 3. \end{cases}$$

In all cases we deduce that $Q_\lambda(u_\varepsilon) < S$ provided $\varepsilon > 0$ is small enough. ∎

We can state the following result

Theorem 2.5. *Let* $\gamma \geq \alpha + 3$. *There exists a solution of* (2.1) *if and only if* $0 < \lambda < \lambda_1(\alpha)$.

2.2. The case $\gamma < \alpha + 3$. Of special interest to us will be the following eigenvalue problem

$$-\frac{1}{r^{2-\gamma}} (r^{2-\gamma} u')' = \mu r^\alpha u, \quad u \in E_{2-\gamma} \tag{2.19}$$

which should be understood in the weak sense (2.11).

We shall consider the following more general eigenvalue problem

$$-\frac{1}{r^\beta} (r^\beta u') = \mu r^\alpha u, \quad u \in E_\beta, \quad \beta < 1 \tag{2.20}$$

which is understood in the E_β-weak sense

$$\int_0^1 r^\beta u' \varphi' \, dr = \mu \int_0^1 r^{\alpha+\beta} u \varphi \, dr, \quad \forall \varphi \in E_\beta. \tag{2.20'}$$

If one makes the change in variables $r = s^{1/(1-\beta)}$, $u(r) = v(s)$, then (2.20) reduces to

$$v'' = \frac{\mu}{(1-\beta)^2} s^{(\alpha+2)/(1-\beta)-2} v, \quad v \in E_0 \tag{2.21}$$

understood in E_0-weak sense.

The decisive remark is the following

Lemma 2.6. *The space E_0 is compactly imbedded in L^2_δ for every $\delta > -1$.*

Proof: Indeed E_0 is compactly imbedded in L^p_0 for every $p > 1$ according to Kondrachov's theorem. Now let $\delta > -1$.

$$\|u\|^2_{L^2_\delta} = \int_0^1 r^\delta u^2\, ds \le \Big(\int_0^1 r^{p\delta}\, dr\Big)^{1/p}\Big(\int_0^1 u^{2q}\, dr\Big)^{1/q}$$

for some $p, q > 1$ such that $p\delta > -1$ and $(1/p) + (1/q) = 1$. Hence $\|u\|_{L^2_\delta} \le$ const.$\|u\|_{L^{2q}}$ and therefore E_0 is compactly imbedded in L^2_δ. ∎

From the compactness result stated above it follows that if in (2.21) $(\alpha+2)/(\beta-1)-2 > -1$ (or simply $\alpha+\beta+1 > 0$) then the spectrum is discrete, positive, the first eigenvalue $\mu_1(\alpha,\beta)$ is simple and the corresponding eigenspace is generated by a positive function ψ_1.

Lemma 2.7. *Let ψ_1 be defined as above. Then $\psi_1(r) - \psi_1(0) = O(r^{\alpha+2})$ as $r \to 0$ and $\psi_1(0) > 0$.*

Proof: Since $\psi_1 \in E_\beta$ and $\beta < 1$ it follows that $\psi_1 \in L^\infty(0,1)$ and a simple argument shows that $\psi_1 \in C[0,1]$. Using the same arguments as in Proposition 2.4 of [8] it follows that

$$\psi_1(r) - \psi_1(0) = O(r^{\alpha+2}).$$

Suppose that $\psi_1(0) = 0$. If we make the change of variables $r = s^{1/(1-\beta)}$, $\omega(s) = \psi_1(r)$ then $\omega(s)$ satisfies

$$-\omega''(s) = \frac{\mu_1}{(1-\beta)^2} s^\delta \omega \quad \text{in } (0,1),\ \delta = \frac{\alpha+2}{1-\beta} - 2 \quad \omega \in E_0$$

in E_0-weak sense. By the strong maximum principle we infer $\omega'(0) > 0$ since $\omega(0) = 0$. Invoking once again the arguments of Proposition 2.4 in [8] we infer $\omega(s) = O(s^{\delta+2})$ as $s \to 0$. Since $\delta > -1$ it follows that $\omega'(0) = 0$. Contradiction! The lemma is proved. ∎

Thus the eigenvalue problem (2.19) leads to a positive discrete spectrum when $2-\gamma+\alpha+1 > 0$; i.e., $\gamma < \alpha+3$.

We denote the least eigenvalue with $\mu_1(\alpha,\gamma)$ and the corresponding eigenfunction with ψ_1 such that $\psi_1 > 0$

$$\psi_1(0) = 1, \quad \psi_1(r) - 1 = O(r^{\alpha+2}) \text{as } r \to 0. \tag{2.22}$$

Our next task is to estimate

$$Q_\lambda(u_\varepsilon) = \frac{\|u_\varepsilon\|^2_\gamma - \lambda\|u_\varepsilon\|^2_{L^2_{\gamma+\alpha}}}{\|u_\varepsilon\|^2_{L^{p+1}_{\gamma+\alpha}}}$$

where

$$u_\varepsilon(r) = \frac{\psi(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}}, \quad \psi(r) = \psi_1(r).$$

First we estimate the terms that enter $Q_\lambda(u_\varepsilon)$.

Estimation of $\|\mathbf{u}_\varepsilon\|_\gamma^2$.

$$u'(r) = \frac{\psi'(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}} - (\gamma-1)\frac{r^{1+\sigma}\psi(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma+1+\sigma)/(2+\sigma)}}$$

so that

$$\|u_\varepsilon\|_\gamma^2 = \int_0^1 \Big[\frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} - 2(\gamma-1)\frac{r^{\gamma+1+\sigma}\psi(r)\psi'(r)}{(\varepsilon + r^{2+\sigma})^{(2\gamma+\sigma)/(2+\sigma)}}$$
$$+ (\gamma-1)^2 \frac{r^{\gamma+2+2\sigma}\psi^2(r)}{(\varepsilon + r^{2+\sigma})^{(2\gamma+2+2\sigma)/(2+\sigma)}}\Big]\, dr.$$

We integrate by parts the second term above and we get

$$\|u_\varepsilon\|_\gamma^2 = \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\, dr$$
$$+ \varepsilon(\gamma-1)(\gamma+1+\sigma)\int_0^1 \psi^2(r)\frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\, dr$$

$$\int_0^1 \psi^2(r)\frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\, dr$$
$$= \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\, dr + \int_0^1 [\psi^2(r) - 1]\frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}$$
$$= I_1 + I_2$$

$$I_1 = \frac{K_1'}{\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)}} + O(1)$$

where

$$K_1' = \int_0^\infty \frac{r^{\gamma+\sigma}}{(1 + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\, dr. \tag{2.23}$$

By (2.22) we get

$$|I_2| \leq \text{ const. } \int_0^1 \frac{r^{\gamma+\sigma+\alpha+2}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} = O(f(\varepsilon))$$

where

$$f(\varepsilon) = \begin{cases} \varepsilon^{-(\gamma+\sigma-\alpha-1)/(2+\sigma)}, & \gamma+\sigma > \alpha+1 \\ |\log\varepsilon|, & \gamma+\sigma = \alpha+1 \\ 1, & \gamma+\sigma < \alpha+1. \end{cases}$$

We infer that

$$\|u_\varepsilon\|^2 = \int_0^1 \frac{|\psi^1(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\, dr + \frac{1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(\varepsilon) + O(\varepsilon(f(\varepsilon)),$$

$\varepsilon \to 0$.

A simple computation shows that $O(\varepsilon) + O(\varepsilon(f(\varepsilon)) = O(\varepsilon f(\varepsilon))$, $\varepsilon \searrow 0$ so that

$$\|u_\varepsilon\|_\gamma^2 = \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\, dr + \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(\varepsilon f(\varepsilon)) \tag{2.24}$$

as $\varepsilon \to 0$, where

$$K_1 = (\gamma - 1)(\gamma + 1 + \sigma)K_1' \tag{2.25}$$

and

$$\varepsilon f(\varepsilon) = \begin{cases} \varepsilon^{(\alpha+3-\gamma)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\ \varepsilon|\log \varepsilon| & \gamma + \sigma = \alpha + 1 \\ \varepsilon, & \gamma + \sigma < \alpha + 1. \end{cases}$$

Estimation of $\|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}_\theta^{p+1}}$.

$$\|u_\varepsilon\|^{p+1}_{L_\theta^{p+1}} = \int_0^1 \frac{\psi^{p+1}(r) r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} = \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\, dr$$
$$= \int_0^1 \frac{[\psi^{p+1}(r) - 1] r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}}\, dr = I_1 + I_2.$$

As above we infer

$$\|u_\varepsilon\|^{p+1}_{L_\theta^{p+1}} = \frac{K_1'}{\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)}} + O(g(\varepsilon))$$

where

$$g(\varepsilon) = \begin{cases} \varepsilon^{-(\gamma+\sigma-\alpha-1)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\ |\log \varepsilon|, & \gamma + \sigma = \alpha + 1 \\ 1, & \gamma + \sigma < \alpha + 1. \end{cases}$$

Hence

$$\|u_\varepsilon\|^2_{L_\theta^{p+1}} = \frac{K_2}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(g(\varepsilon)) \tag{2.26}$$

with

$$K_2 = [K_1']^{2/(p+1)}. \tag{2.27}$$

Estimation of $\|\mathbf{u}_\varepsilon\|_\gamma^2 - \lambda\|\mathbf{u}_\varepsilon\|^2_{\mathbf{L}^2_{\gamma+\alpha}}$. By (2.24) we get

$$\|u_\varepsilon\|_\gamma^2 - \lambda\|u_\varepsilon\|^2_{L^2_{\gamma+\alpha}} = \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\, dr$$
$$- \lambda \int_0^1 \frac{\psi^2(r) r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} + O(\varepsilon f(\varepsilon))$$

$$\int_0^1 \frac{|\psi'(r)|^2 r}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\,dr - \lambda \int_0^1 \frac{\psi^2(r) r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\,dr$$
$$= \int_0^1 [|\psi'(r)|^2 r^{2-\sigma} - \lambda r^{\alpha+2-\gamma}\psi^2(r)] \frac{r^{2\gamma-2}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}}\,dr$$
$$= \int_0^1 R(\psi) N(\varepsilon, r)\,dr = \int_0^1 R(\psi)\,dr + \int_0^1 R(\psi)[N(\varepsilon, r) - 1]\,dr$$
$$= \int_0^1 R(\psi)\,dr + O(\varepsilon).$$

We infer

$$\|u_\varepsilon\|_\gamma^2 - \lambda \|u_\varepsilon\|_{L^2_{\gamma+\alpha}}^2 = \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + \int_0^1 R(\psi)\,dr + O(\varepsilon f(\varepsilon)) \quad \text{as } \varepsilon \to 0 \tag{2.28}$$

where again we have used the fact that $O(\varepsilon) + O(\varepsilon(f(\varepsilon)) = O(\varepsilon(f(\varepsilon))$ as $\varepsilon \searrow 0$. By (2.26) and (2.28) we infer

$$Q_\lambda(u_\varepsilon) = \frac{K_1}{K_2} + \varepsilon^{(\gamma-1)/2} \int_0^1 R(\psi)\,dr + O\big(\varepsilon^{(\gamma-1)/2} \varepsilon f(\varepsilon)\big).$$

From $\gamma < \alpha + 3$ we get $\varepsilon f(\varepsilon) = o(1)$ and therefore

$$= \frac{K_1}{K_2} + \varepsilon^{(\gamma-1)/(2+\sigma)} \int_0^1 R(\psi)\,dr + o(\varepsilon^{(\gamma-1)/(2+\sigma)}) \quad \text{as } \varepsilon \searrow 0. \tag{2.29}$$

If we recall our definition of ψ we get

$$\int_0^1 R(\psi)\,dr = \int_0^1 [r^{2-\gamma}|\psi'(r)|^2 - \lambda r^{2+\alpha-\gamma}\psi^2(r)]\,dr$$
$$= (\mu_1(\alpha) - \lambda) \int_0^1 r^{\alpha+2-\gamma}\psi^2(r)\,dr = (\mu_1(\alpha) - \lambda) K_3,$$

where K_3 is a positive constant. This yields

$$Q_\lambda(u_\varepsilon) = \frac{K_1}{K_2} + K_3(\mu_1(\alpha) - \lambda)\varepsilon^{(\gamma-1)/(2+\sigma)} + o(\varepsilon^{(\gamma-1)/(2+\sigma)}) \quad \text{as } \varepsilon \searrow 0. \tag{2.30}$$

Now we claim that $K_1/K_2 = S$. Indeed $S = \frac{\|U\|_{\gamma,\infty}^2}{\|U\|_{L_\theta^{p+1}}^2}$. By (2.23), (2.27) we see that $K_2 = \|U\|_{L_\theta^{p+1}}^2$. We have to check that $K_1 = \|U\|_{\gamma,\infty}^2$. From (1.13) we get

$$\|U\|_{\gamma,\infty}^2 = \frac{(\gamma-1)^2}{2+\sigma} \frac{\Gamma\big(\frac{\gamma+3+2\sigma}{2+\sigma}\big)\Gamma\big(\frac{\gamma-1}{2+\sigma}\big)}{\Gamma\big(\frac{2(\gamma+1+\sigma)}{2+\sigma}\big)}.$$

K_1 can be easily computed using (1.14) and we get

$$K_1 = \frac{(\gamma-1)(\gamma+1-\sigma)}{(2+\sigma)^2} \frac{\big[\Gamma\big(\frac{\gamma+\sigma+1}{2+\sigma}\big)\big]^2}{\Gamma\big(\frac{2(\gamma+1+\sigma)}{2+\sigma}\big)}.$$

So that we actually have to check

$$(\gamma+1+\sigma)\Big[\Gamma\Big(\frac{\gamma+1+\sigma}{2+\sigma}\Big)\Big]^2 = (\gamma-1)\Gamma\Big(\frac{\gamma+3+2\sigma}{2+\sigma}\Big)\Gamma\Big(\frac{\gamma-1}{2+\sigma}\Big). \tag{2.31}$$

From the well-known relation $\Gamma(x+1) = x\Gamma(x)$ we infer

$$\Gamma\Big(\frac{\gamma+1+\sigma}{2+\sigma}\Big) = \frac{\gamma-1}{2+\sigma}\Gamma\Big(\frac{\gamma-1}{2+\sigma}\Big)$$

and

$$\Gamma\Big(\frac{\gamma+3+2\sigma}{2+\sigma}\Big) = \frac{\gamma+1+\sigma}{2+\sigma}\Gamma\Big(\frac{\gamma+1+\sigma}{2+\sigma}\Big).$$

(2.31) follows from the equalities above. Hence

$$\frac{K_1}{K_2} = S. \tag{2.32}$$

By (2.30) and (2.32) we get

$$\text{If } \lambda > \mu_1(\alpha) \text{ then } S_\lambda < S. \tag{2.33}$$

Now we can state

Proposition 2.8. *If $\mu_1(\alpha) < \lambda_1(\alpha)$ $(\alpha > -2,\ 1 < \gamma < \alpha+3)$ then the problem possesses at least a solution for each $\lambda \in (\mu_1(\alpha), \lambda_1(\alpha))$.*

There is still a question we must answer, namely when does the following spectral equality hold?

$$\mu_1(\alpha) < \lambda_1(\alpha). \tag{2.34}$$

We recall that $\mu_1(\alpha) = \mu_1(\alpha,\gamma)$ and $\lambda_1(\alpha) = \lambda_1(\alpha,\gamma)$, are the least eigenvalues of the following eigenvalue problems

$$-\frac{1}{r^{2-\gamma}}(r^{2-\gamma}\psi')' = \mu r^\alpha \psi, \quad \psi \in E_{2-\gamma}, \tag{2.35}$$

and respectively

$$-\frac{1}{r^\gamma}(r^\gamma \psi')' = \lambda r^\alpha \varphi, \quad \varphi \in E_\gamma \tag{2.36}$$

where $\alpha > -2$, $1 < \gamma < \alpha+3$ and these eigenvalue problems are understood in the weak sense (2.20′) and (2.11).

The aim of our next section is to answer this question.

3. Proof of the spectral inequality. The main result of this section is the following.

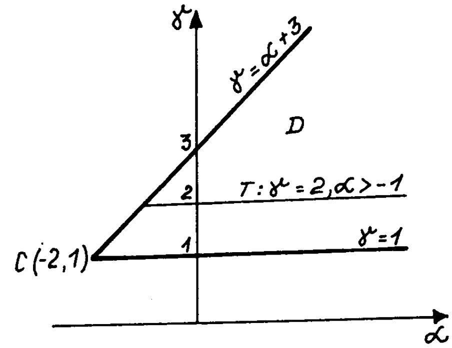

Figure 1.

Proposition 3.1. *$\mu_1(\alpha,\gamma) < \lambda_1(\alpha,\gamma)$ for all $\alpha > -2$, $1 < \gamma < \alpha + 3$.*

The proof will be carried out in several steps. Let us first denote by D the set

$$D = \{(\alpha,\gamma) \,|\, 1 < \gamma < \alpha + 3\}.$$

In the (α,γ)-plane D looks like Figure 1.

Let us denote by S the set

$$S = \{(\alpha,\gamma) \in D \,|\, \mu_1(\alpha,\gamma) < \lambda_1(\alpha,\gamma)\}.$$

We also consider the following semigroup of transformations of the (α,γ)-plane $(H_\beta)_{\beta>0}$ where the H_β are given by the law

$$(\alpha,\gamma) \xrightarrow{H_\beta} (\alpha_\beta,\gamma_\beta) = \beta(\alpha+2,\gamma-1) + (-2,1). \tag{3.1}$$

$(H_\beta)_{\beta>0}$ is a semigroup of homoteties of pole $C(-2,1)$. Clearly, $H_\beta(D) \subset D\, \forall \beta > 0$.

Proof of Proposition 3.1.

Step 1. $H_\beta(S) \subset S$, $\forall \beta > 0$. We have to prove that if $\mu_1(\alpha,\gamma) < \lambda_1(\alpha,\gamma)$ then also

$$\mu_1(\alpha_\beta,\gamma_\beta) < \lambda_1(\alpha_\beta,\gamma_\beta).$$

Indeed, let us make the change of variables $r = s^\beta$ in the equations (2.35) and (2.36). We get

$$-\frac{1}{s^{2-\gamma_\beta}}(s^{2-\gamma_\beta}\psi')' = \beta^2\mu s^{\alpha_\beta}\psi, \quad \psi \in E_{2-\gamma_\beta}, \tag{3.2}$$

and respectively

$$-\frac{1}{s^{\gamma_\beta}}(s^{\gamma_\beta}\varphi')' = \beta^2\lambda s^{\alpha_\beta}\varphi, \quad \varphi \subset E_{\gamma_\beta} \tag{3.3}$$

where α_β and γ_β are given by (3.1). This yields $\mu_1(\alpha_\beta,\gamma_\beta) = \beta^2\mu_1(\alpha,\beta)$ and $\lambda_1(\alpha_\beta,\gamma_\beta) = \beta^2\lambda_1(\alpha,\gamma)$ and thus $\mu_1(\alpha,\gamma) < \lambda_1(\alpha,\gamma)$ if and only if $\mu_1(\alpha_\beta,\gamma_\beta) < \lambda_1(\alpha_\beta,\gamma_\beta)$.

Step 2. $\mu_1(\alpha,2) < \lambda_1(\alpha,2)$, for all $\alpha > -1$. We denote by ψ and respectively φ eigenfunctions corresponding to μ_1 and λ_1 such that $\psi,\varphi > 0$ in $(0,1)$. Then ψ and φ satisfy

$$-\psi'' = \mu_1 r^\alpha \psi, \quad \varphi \in E_0 \tag{3.4}$$

and respectively

$$-\frac{1}{r^2}(r^2\varphi')' = \lambda_1 r^\alpha \varphi, \quad \psi \in E_2. \tag{3.5}$$

As in Lemma 2.7 we can prove that

$$\varphi(0) > 0 \tag{3.6}$$

$$\varphi(r) - \varphi(0) = O(r^{\alpha+2}) \quad \text{as } r \searrow 0. \tag{3.7}$$

Hence $\varphi'(0) = 0$ and $\varphi \in E_0$. From (3.5) we infer

$$-\varphi'' = \lambda_1 r^\alpha + \frac{2}{r}\varphi'. \tag{3.8}$$

Multiplying (3.8) with φ and then integrating on $(0,1)$ we get

$$\int_0^1 |\varphi'|^2\,dr = \lambda_1 \int_0^1 r^\alpha \varphi^2\,dr + \int_0^1 \frac{2\varphi\varphi'}{r}\,dr.$$

Since

$$\int_0^1 |\varphi'|^2\,dr \geq \mu_1 \int_0^1 r^\alpha \varphi^2\,dr$$

it follows

$$\mu_1 \int_0^1 r^\alpha \varphi^2\,dr \leq \lambda_1 \int_0^1 r^\alpha \varphi^2\,dr + \int_0^1 \frac{2\varphi\varphi'}{r}\,dr.$$

It is a routine exercise using (3.5) to prove that $\varphi' < 0$. Thus

$$\mu_1 \int_0^1 r^\alpha \varphi^2\,dr < \lambda_1 \int_0^1 r^\alpha \varphi^2\,dr$$

hence $\mu_1 < \lambda_1$.

Step 3. $S = D$. We have proved so far that the half line $(T) : \alpha > -1, \gamma = 2$ lies in S (see Figure 1). Hence by Step 1 we get $\bigcup_{\beta>0} H_\beta(T) \subset S$. It is an easy geometric remark that

$$\bigcup_{\beta>0} H_\beta(T) = D \quad \text{(see Figure 1).}$$

Proposition 3.1 is proved. ∎

Now we can state the following result.

Theorem 3.2. *a) For $\gamma \geq \alpha + 3$, $\alpha > -2$ problem (2.1) has a solution if and only if $\lambda \in (0, \lambda_1(\alpha,\gamma))$.*

b) For $1 < \gamma < \alpha + 3$, $\alpha > -2$ we have $\mu_1(\alpha,\gamma) < \lambda_1(\alpha,\gamma)$ and problem (2.1) has a solution if $\lambda \in (\mu_1(\alpha,\beta), \lambda_1(\alpha,\gamma))$ and no solution if $\lambda \leq 0$ or $\lambda \geq \lambda_1$.

Remark 3.3. In the paper [3] of Brézis-Nirenberg it was proved that if $\gamma = 2$, $\alpha = 0$ then problem (2.1) has a solution if and only if $\lambda \in (\pi^2/4, \pi^2)$. An easy computation shows that $\mu_1(0,2) = \pi^2/4$ and $\lambda_1(0,2) = \pi^2$ and thus our result seems optimal. It is therefore natural to ask the following question:

Is it true that in the case $1 < \gamma < \alpha + 3$ problem (2.1) has no solution for $\lambda < \mu_1(\alpha, \gamma)$?

Remark 3.4. We consider the problem

$$\begin{aligned} -\Delta u &= r^\sigma |u|^{p-1} u + \lambda r^\alpha u \quad \text{in } B_R \subset \mathbf{R}^N \ (r = |x|) \\ u &= 0 \quad \text{on } \partial B_R \\ u &> 0 \end{aligned} \tag{3.9}$$

$\alpha, \sigma \geq -2$, $p + 1 = (2N + 2\sigma)/(N-2)$.

By a weak solution of (3.8) we mean a function $u \in H_0^1(B_R)$ $u > 0$ satisfying

$$\int_{B_R} \nabla u \nabla \varphi = \int_{B_R} r^\sigma |u|^{p-1} u\varphi \, dx + \lambda \int_{B_R} r^\alpha u \varphi \, dx, \quad \forall \varphi \in H_0^1(B_R). \tag{3.10}$$

It is easily seen that at least at a formal level the radial solutions of (3.9) satisfy

$$-\frac{1}{r^{N-1}}(r^{N-1}u') = r^\sigma |u|^{p-1} u + \lambda r^\alpha u \quad \text{in } (0, R),\ u' = du/dr,\ u > 0,\ u \in E_{N-1}. \tag{3.11}$$

We claim that weak solutions of (3.11) in the sense of (2.2) satisfy (3.10). Indeed, if u satisfies (3.11), then u satisfies

$$-\Delta u = r^\sigma |u|^{p-1} u + \lambda r^\alpha u \quad \text{in } B_R \backslash \{0\} \tag{3.12}$$

by standard elliptic regularity. Let $\varphi \in H_0^1(\Omega)$. We multiply (3.12) with φ and then we integrated on $D_\varepsilon = \{x \in \mathbf{R}^N : \varepsilon \leq |x| \leq R\}$ using Green's formula

$$\int_{D_\varepsilon} \nabla u \nabla \varphi \, dx - \int_{D_\varepsilon} r^\sigma |u|^{p-1} u \varphi \, dx - \lambda \int_{D_\varepsilon} r^\alpha u \varphi \, dx + \int_{|x|=\varepsilon} u'(\varepsilon)\varphi(x) \, dS = 0.$$

Here ds is the surface element on $|x| = \varepsilon$, $dS = \sum_{N-1} \varepsilon^{N-1} \, d\theta$; $d\theta$ is the surface element on $[|x| = 1]$ and $\sum_{N-1}$ is the $(N-1)$-dimensional measure of the sphere $|x| = 1$. We get

$$\begin{aligned} &\int_{D_\varepsilon} \nabla u \nabla \varphi \, dx - \int_{D_\varepsilon} r^\sigma |u|^{p-1} u \varphi \, dx - \lambda \int_{D_\varepsilon} r u \varphi \, dx \\ &\quad + \sum_{N-1} \int_{|x|=\varepsilon} \varepsilon^{N-1} u'(\varepsilon) \varphi(\varepsilon, \theta) \, d\theta = 0. \end{aligned} \tag{3.13}$$

Since $u \in E_{N-1}$ we deduce that $\varepsilon^N u'(\varepsilon)^2 \to 0$ on a subsequence $\varepsilon = \varepsilon_k \to 0$. Hence $\varepsilon_k^{N-1} u'(\varepsilon_k) \to 0$ since $N > 1$. In (3.13) we let $\varepsilon = \varepsilon_k \to 0$. This yields

$$\int_{B_R} \nabla u \cdot \nabla \varphi \, dx - \int_{B_R} r^\sigma |u|^{p-1} u \varphi \, dx - \lambda \int_{B_R} r^\alpha u \varphi \, dx = 0;$$

i.e., exactly (3.10). Now it is easy to formulate existence results for the problem (3.9). Let $\lambda_1(\alpha)$ be the first eigenvalue of the following eigenvalue problem

$$-\Delta u = \lambda r^\alpha u \quad \text{in } B_R, \quad u = 0 \quad \text{on } \partial B_R. \tag{3.14}$$

As in Figueiredo [5] we infer that $\lambda_1(\alpha)$ is simple and the corresponding eigenfunction may be chosen positive. We deduce that in fact $\lambda_1(\alpha) = \lambda_1(\alpha, N-1)$ where $\lambda_1(\alpha, N-1)$ was defined at (2.11). Let $\mu_1(\alpha)$ be the first eigenvalue of the eigenvalue problem (2.35) with $\gamma = N - 1$.

We can now state

Theorem 3.5. *a) For $N \geq \alpha+4$, $\alpha \geq -2$ problem* (3.9) *has a radial weak solution if and only if $\lambda \in (0, \lambda_1(\alpha))$.*

b) For $2 < N < \alpha+4$, $\alpha > -2$, we have $\mu_1(\alpha) < \lambda_1(\alpha)$ and if $\lambda \in (\mu_1(\alpha), \lambda_1(\alpha))$ problem (3.9) *has a radial weak solution. It has no radial weak solution if $\lambda \leq 0$, $\lambda \geq \lambda_1$.*

Concerning the regularity of radial weak solutions it can be easily proved that these are classical solutions of (3.9). Indeed $u \in E_{N-1}$. Let $\theta = (N-1)+(\sigma+1)/2$, $q = (p-1)N/2$. Then, a simple computation shows that $\frac{(N-1)-1}{2} = \frac{\theta+1}{q}$ so that by the imbedding lemma we get that $u \in L^q_\theta(0,R)$; i.e., $r^{(N-1)} r^{(\sigma+1)/2} |u|^{(p-1)N/2} \in L^1(0,R)$ which can be restated as $r^\sigma |u|^{p-1} \in L^{N/2}(B_R)$. Hence if we set $a(r) = r^\sigma |u|^{p-1}$ then u is a weak solution of

$$-\Delta u = a(r)u + \lambda r^\alpha u \quad \text{in } H^1_0(B_R)$$

with $a \in L^{N/2}(B_R)$.

We can now proceed as in Brézis-Nirenberg [3] to infer that $u \in L^\infty(B_R)$ and hence by standard elliptic regularity $u \in C^2(B_R)$.

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