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Geometric connections and geometric Dirac operators on contact manifolds ☆

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Abstract

We construct some natural metric connections on metric contact manifolds compatible with the contact structure and characterized by the Dirac operators they determine. In the case of CR manifolds these are invariants of a fixed pseudo-hermitian structure, and one of them coincides with the Tanaka–Webster connection.

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Introduction

This work has its origin in our attempt to better understand the nature of Seiberg–Witten monopoles on contact 3-manifolds. The main character of this story is a metric contact manifold (M, g, η, J), where g is a Riemann metric, and η is a contact form and J is an almost complex structure on $V := \ker \eta$ such that

$$g(X, Y) = d\eta(X, JY), \quad \forall X, Y \in C^\infty(V).$$

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We are interested in the local differential geometry of such a manifold and in particular, we seek "natural" connection in the tangent bundle TM.

Gauge theory suggests that a "natural" connection ought to be compatible with g and J. We will refer to these as *metric contact connections*. These requirements alone still leave open a wide range of choices. On the other hand such manifolds are also equipped with some natural elliptic partial differential operators. For the simplicity of the exposition assume M is equipped with a *spin* structure with associated complex spinor bundle $\mathbb{S}$. Every metric connection ∇ on M induces a Dirac type operator

$$\mathfrak{D}(\nabla): C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S}).$$

A metric connection ∇ is called *balanced* if $\mathfrak{D}(\nabla)$ is symmetric. Two connections ∇^i, $i = 0, 1$, will be called *Dirac equivalent* if $\mathfrak{D}(\nabla^0) = \mathfrak{D}(\nabla^1)$. The first question we address in this paper is the existence of a metric contact connection Dirac equivalent with the Levi-Civita connection.

On the other hand, a metric contact manifold is equipped with a natural elliptic, first order operator $\mathcal{H}$ resembling very much the Hodge–Dolbeault operator on a complex manifold (see Section 3.3 for more details). This operator acts on the sections of the complex spinor bundle $\mathbb{S}_c$ associated to the canonical *spin*^{c} structure determined by the contact structure. A metric contact connection ∇ induces a (geometric) Dirac operator $\mathfrak{D}_c(\nabla)$ on $C^\infty(\mathbb{S}_c)$.

The second question we address in this paper concerns the existence of a metric contact connection ∇ such that $\mathfrak{D}_c(\nabla) = \mathcal{H}$. We say that such a connection is *adapted to* $\mathcal{H}$.

To address these questions we rely on the work P. Gauduchon (see [4] or Section 2.1), concerning hermitian connections on almost-hermitian manifolds. More precisely, to implement Gauduchon's results we will regard M as boundary of certain (possible non-complete) almost hermitian manifolds. We will concentrate only on two cases frequently arising in gauge theory.

- The symplectization $\tilde{M} = \mathbb{R}_+ \times M$ with symplectic form $\omega = \hat{d}(t\eta)$, metric $\tilde{g} = dt^2 + \eta^{\otimes 2} + tg|_V$, and almost complex structure $\tilde{J}$.
- The cylinder $\hat{M} = \mathbb{R} \times M$ with metric $\hat{g} = dt^2 + g$ and almost complex structure $\hat{J}$ defined by $\hat{J}\partial_t = \xi$, $\hat{J}|_V = J$.

To answer the second question we use the cylinder case and a certain natural perturbation of the first canonical connection on $(T\hat{M}, \hat{g}, \hat{J})$. This new connection on $T\hat{M}$ preserves the splitting $T\hat{M} = \mathbb{R}\partial_t \oplus TM$ and induces a connection on TM with the required properties (see Section 3.1). Moreover, when M is a CR manifold this connection coincides with the Tanaka–Webster connection, [10,13].

To answer the first question we use the symplectization $\tilde{M}$ and a natural perturbation of the Chern connection on $T\tilde{M}$. We obtain a new connection on $\tilde{M}$ whose restriction to $\{1\} \times M$ is a contact connection (see Section 3.4). When M is CR this contact connection is also CR, but it never coincides with the Tanaka–Webster connection. We are not aware whether this contact connection has been studied before.

Theorem. (a) *On any metric contact manifold there exists a* balanced contact connection *adapted to* $\mathcal{H}$ *and a* balanced contact connection *Dirac equivalent to the Levi-Civita connection. If the manifold is CR these connections are also CR.*

(b) *On a CR manifold each Dirac equivalence class of balanced connections contains at most one CR connection. Moreover, the Tanaka–Webster connection is the unique balanced CR connection adapted to* $\mathcal{H}$.

Finally, we present several Weitzenböck formulæ involving the operator $\mathcal{H}$ (see Section 3.3). We expect these facts will allows us to extend the computations in [9] to more general links of isolated surface singularities.

1. General properties geometric Dirac operators

1.1. Dirac operators compatible with a metric connection

Suppose (M, g) is an oriented, n-dimensional Riemannian manifold. We will denote a generic local, oriented, orthonormal synchronous frame of TM by (e_i). Its dual coframe is denoted by (e^i). We will denote the natural duality between a vector space and its dual by $\langle \bullet, \bullet \rangle$.

A *metric connection* on TM is a connection ∇ on TM such that $\nabla g = 0$. The *torsion* of a metric connection ∇ is the TM-valued 2-form $T = T(\nabla)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The Levi-Civita connection, denoted by D in the sequel, is the metric connection uniquely determined by the condition $T(D) = 0$. Any metric connection ∇ can be uniquely written as $D + A$, where $A \in \Omega^1(\mathrm{End}_-(TM))$, where End_- denotes the space of skew-symmetric endomorphisms. A is called the *the potential of* ∇. There are natural *isomorphisms*

$$\Omega^2(TM) \to \Omega^2(T^*M), \quad T \mapsto T^\dagger, \quad \Omega^1\big(\mathrm{End}_-(TM)\big) \to \Omega^2(T^*M), \quad A \mapsto A^\dagger$$

defined by

$$\Omega^2(TM) \ni T \mapsto T^\dagger, \quad \big\langle X, T^\dagger(Y, Z)\big\rangle = g\big(X, T(Y, Z)\big)$$

and

$$\Omega^1\big(\mathrm{End}_-(TM)\big) \ni A \mapsto A^\dagger, \quad \big\langle X, A^\dagger(Y, Z)\big\rangle = g(A_X Y, Z) =: A^\dagger(X; Y, Z),$$

$\forall X, Y, Z \in \mathrm{Vect}(M)$. In local coordinates, if

$$T(e_j, e_k) = \sum_i T^i_{jk} e_i, \quad A_{e_i} e_j = \sum_k A^k_{ij} e_k$$

then

$$T^\dagger(e_j, e_k) = \sum_i T^i_{jk} e^i, \quad A^\dagger(e_j, e_k) = \sum_i A^k_{ij} e^i,$$

or equivalently, $T^\dagger_{ijk} = T^i_{jk}$, $A^\dagger_{ijk} = A^k_{ij}$. To simplify the exposition, when no confusion is possible, we will drop the $\dagger$ from notations and when working in local coordinates, we will write A_{ijk} instead of $A^\dagger_{ijk}$ etc. Define

$$\mathrm{tr} : \Omega^2(T^*M) \to \Omega^1(M), \quad \Omega^2(T^*M) \ni (B_{ijk}) \mapsto (\mathrm{tr}\, B) = \sum_{i,k} B_{iik} e^k$$

and the *Bianchi projector*

$$\mathfrak{b} : \Omega^2(T^*M) \to \Omega^3(M),$$

$$\Omega^2(T^*M) \ni (B_{ijk}) \mapsto \mathfrak{b}B = \sum_{i<j<k} (B_{ijk} + B_{kij} + B_{jki}) e^i \wedge e^j \wedge e^k.$$

Note that if $B \in \Omega^3(M) \subset \Omega^2(T^*M)$ then $B = \frac{1}{3}\mathfrak{b}B$.

For any $A \in \mathrm{End}(TM)$ and $\alpha \in \Omega^1(M)$ we define $A \wedge \alpha \in \Omega^2(T^*M)$ by the equality

$$\begin{aligned}(A \wedge \alpha)(X; Y, Z) &= \big((AX)_\flat \wedge \alpha\big)(Y, Z) \\ &= g(AX, Y)\alpha(Z) - g(AX, Z)\alpha(Y), \quad \forall X, Y, Z \in \mathrm{Vect}(M),\end{aligned}$$

where $\bullet_\flat$ (resp. $\bullet^\flat$) denotes the g-dual of a vector (resp. covector) $\bullet$. The proof of the following result is left to the reader.

Lemma 1.1. *Let $A \in \mathrm{End}(TM)$, $\alpha \in \Omega^1(M)$ and set*

$$A_+ = \frac{1}{2}(A + A^*), \quad A_- = \frac{1}{2}(A - A^*).$$

Then $\mathrm{tr}(A \wedge \alpha) = (\mathrm{tr}\, A)\alpha - A^t\alpha$, $\quad \mathfrak{b}(A \wedge \alpha) = 2\omega_{A_-} \wedge \alpha$, *where*

$$\omega_{A_-}(X, Y) = g(A_-X, Y), \quad \forall X, Y \in \mathrm{Vect}(M).$$

Using the above operations we can orthogonally decompose $\Omega^2(T^*M)$ as

$$\Omega^2(T^*M) = \Omega^1(M) \oplus \Omega^3(M) \oplus \Omega^2_0(T^*M),$$

where

$$\Omega^2_0(T^*M) := \big\{A \in \Omega^2(T^*M);\ \ \mathfrak{b}A = \mathrm{tr}\, A = 0\big\},$$

and $\Omega^1(M)$ embeds in $\Omega^2(T^*M)$ via the map

$$\Omega^1(M) \to \Omega^2(T^*M), \quad \alpha \mapsto \tilde{\alpha} := \frac{1}{n-1}(\mathbb{1}_{TM} \wedge \alpha).$$

Using this orthogonal splitting we can decompose any $A \in \Omega^2(T^*M)$ as

$$A = \widetilde{\mathrm{tr}\, A} + \frac{1}{3}\mathfrak{b}A + P_0A, \quad P_0A := A - \widetilde{\mathrm{tr}\, A} - \frac{1}{3}\mathfrak{b}A \in \Omega^2_0(T^*M).$$

The next result, whose proof can be found in [4], states that a metric connection is determined by its torsion in a very explicit way.

Proposition 1.2. *Suppose that ∇ is a metric connection with potential A and torsion T. Then*

$$T^\dagger = -A^\dagger + \mathfrak{b}A^\dagger, \tag{1.1}$$

$$A^\dagger = -T^\dagger + \frac{1}{2}\mathfrak{b}T^\dagger. \tag{1.2}$$

In particular $\mathfrak{b}A^\dagger = \frac{1}{2}\mathfrak{b}T^\dagger$, $\mathrm{tr}\, A^\dagger = -\,\mathrm{tr}\, T^\dagger$.

Since all the computations we are about to perform are local we can assume that M is equipped with a *spin* structure and we denote by $\mathbb{S}$ the associated complex spinor bundle.[1] We have a *Clifford multiplication map*

$$\mathbf{c}:\Omega^*(M)\to \mathrm{End}(\mathbb{S}).$$

A hermitian connection $\tilde{\nabla}$ on $\mathbb{S}$ is said to be *compatible with the Clifford multiplication and the metric connection* ∇ on TM if

$$\tilde{\nabla}_X\big(\mathbf{c}(\alpha)\psi\big)=\mathbf{c}(\nabla_X\alpha)\psi+\mathbf{c}(\alpha)\tilde{\nabla}_X\psi,\quad \forall X\in \mathrm{Vect}(M),\ \alpha\in\Omega^1(M),\ \psi\in C^\infty(\mathbb{S}).$$

We denote by $\mathfrak{A}_\nabla=\mathfrak{A}_\nabla(\mathbb{S})$ the space of hermitian connections on $\mathbb{S}$ compatible with the Clifford multiplication and ∇.

Proposition 1.3. *The space $\mathfrak{A}_\nabla(\mathbb{S})$ is an affine space modelled by the space $\mathbf{i}\Omega^1(M)$ of imaginary 1-forms on M.*

Proof. Suppose $\tilde{\nabla}^0,\tilde{\nabla}^1\in\mathfrak{A}_\nabla$. Set $C:=\tilde{\nabla}^1-\tilde{\nabla}^0\in\Omega^1(\mathrm{End}(\mathbb{S}))$. Since both $\tilde{\nabla}^i$, $i=0,1$, are compatible with the Clifford multiplication and ∇ we deduce that for every $X\in\mathrm{Vect}(M)$ the endomorphism $C(X):=X\lrcorner C$ commutes with the Clifford multiplication. Since the fibers of $\mathbb{S}$ are irreducible Clifford modules we deduce from Schur's Lemma that $C(X)$ is a constant in each fiber, i.e., $C\in\Omega^1(M)\otimes\mathbb{C}$. Since both $\tilde{\nabla}^i$ are hermitian connections we conclude that C must be purely imaginary 1-form. □

Definition 1.4. A *geometric Dirac operator* on $\mathbb{S}$ is a first order partial differential operator $\mathfrak{D}$ of the form

$$\mathfrak{D}=\mathfrak{D}(\tilde{\nabla}):C^\infty(\mathbb{S})\xrightarrow{\tilde{\nabla}}C^\infty(T^*M\otimes\mathbb{S})\xrightarrow{\mathbf{c}}C^\infty(\mathbb{S}),$$

where $\tilde{\nabla}\in\mathfrak{A}_\nabla(\mathbb{S})$ for some metric connection ∇ on TM.

Locally, a geometric Dirac operator has the form $\mathfrak{D}(\tilde{\nabla})=\sum_i\mathbf{c}(e^i)\tilde{\nabla}_{e_i}$. Every metric connection ∇ canonically determines a connection $\hat{\nabla}\in\mathfrak{A}_\nabla(\mathbb{S})$ locally described as follows. If the $\underline{so}(n)$-valued 1-form ω associated by the frame (e_i) to the connection ∇ is defined by

$$\nabla e_j=\sum_{i,k}e^k\otimes\omega^i_{kj}e_i,\quad \omega^i_{kj}+\omega^j_{ki}=0,$$

then the induced connection on $\mathbb{S}$ is given by the $\mathrm{End}_-(\mathbb{S})$-valued 1-form (see [8])

$$\hat{\omega}=-\frac{1}{4}\sum_{i,j,k}e^k\otimes\omega^i_{kj}\mathbf{c}(e^i)\mathbf{c}(e^j).\tag{1.3}$$

We set $\mathfrak{D}(\nabla):=\mathfrak{D}(\hat{\nabla})$ and $\mathfrak{D}_0:=\mathfrak{D}(\hat{D})$. $\mathfrak{D}_0$ is the usual *spin* Dirac operator. We see that every geometric operator has the form

$$\mathfrak{D}=\mathfrak{D}(\nabla)+\mathbf{c}(\mathbf{i}a),$$

[1] $\mathbb{S}$ is $\mathbb{Z}_2$-graded if $n=\dim M$ is even and it is ungraded if n is odd.

where ∇ is a metric connection on M and $a \in \Omega^1(M)$. We have the following elementary identity

$$\mathfrak{D}(D + A) = \mathfrak{D}(D) - \frac{1}{2}\mathbf{c}(\operatorname{tr} A) + \frac{1}{2}\mathbf{c}\big(\mathfrak{b}(A)\big). \tag{1.4}$$

Definition 1.5. The connection ∇ is called *balanced* if $\mathfrak{D}(\nabla)$ is symmetric. We denote by $\mathfrak{A}_{bal}(M)$ the space of balanced connections on M.

The identity (1.4) implies immediately the following result.

Proposition 1.6. (a) *The connection ∇ with torsion T is balanced if and only if* $\operatorname{tr} T = 0$.
(b) *Suppose that $\nabla = D + A$ is a balanced connection on TM. Then*

$$\mathfrak{D}(\nabla) = \mathfrak{D}_0 + \frac{1}{2}\mathbf{c}(\mathfrak{b}A) = \mathfrak{D}_0 + \frac{1}{4}\mathbf{c}(\mathfrak{b}T).$$

Corollary 1.7. *Suppose $\mathfrak{D} = \mathfrak{D}_0 + \mathbf{c}(\varpi)$, $\varpi \in \Omega^3(M)$. Then $\mathfrak{D} = \mathfrak{D}(\nabla)$, where*

$$\nabla = D + A, \quad A^\dagger = \frac{2}{3}\varpi.$$

The above result can also be rephrased in the language of superconnections described, e.g., in [1]. Suppose $\varpi \in \Omega^3(M)$. The operator $d + \mathbf{c}(\varpi)$ is a superconnection on the trivial line bundle $\underline{\mathbb{C}}$. Taking the tensor product it with the connection $\hat{D}$ on $\mathbb{S}$ we obtain a superconnection on $\mathbb{S} = \mathbb{C} \otimes \mathbb{S}$

$$\mathbb{A}_\varpi := \varpi \otimes \mathbb{1} + \mathbb{1} \otimes \hat{D} : C^\infty(\mathbb{S}) \to \Omega^*(\mathbb{S}).$$

The Dirac operator determined by this superconnection is

$$\mathbf{c} \circ \mathbb{A}_\varpi = \mathfrak{D}_0 + \mathbf{c}(\omega).$$

Definition 1.8. Two connections $\nabla^0, \nabla^1 \in \mathfrak{A}_{bal}(M)$ will be called *Dirac equivalent* if

$$\mathfrak{D}(\hat{\nabla}^0) = \mathfrak{D}(\hat{\nabla}^1).$$

The above results show that two balanced connections ∇^0 and ∇^1 are Dirac equivalent if and only if

$$\mathbf{c}\big(\mathfrak{b}T(\nabla^0)\big) = \mathbf{c}\big(\mathfrak{b}T(\nabla^1)\big) \iff \mathfrak{b}T(\nabla^1) = \mathfrak{b}T(\nabla^0). \tag{1.5}$$

Two metric connections on TM, not necessarily balanced, are will be called *quasi-equivalent* if they satisfy the condition (1.5).

1.2. Weitzenböck formulæ

Suppose (E, h) is a hermitian vector bundle over M. A *generalized Laplacian* is a *formally self-adjoint*, second order partial differential operator $L : C^\infty(E) \to C^\infty(E)$ whose principal symbol satisfies

$$\sigma_L(\xi) = -|\xi|_g^2 \mathbf{1}_E.$$

The next classical result shows that the class of generalized Laplacians is quite narrow.

Proposition 1.9 [1, Section 2.1], [5, Section 4.1.2]. *Suppose L is a generalized Laplacian on E. Then there exists a* unique hermitian *connection $\tilde{\nabla}$ on E and a unique selfadjoint endomorphism $\mathcal{R}$ of E such that*

$$L = \tilde{\nabla}^*\tilde{\nabla} + \mathcal{R}. \tag{1.6}$$

We will refer to this presentation of a generalized Laplacian as the Weitzenböck presentation of L.

If $\mathfrak{D}$ is a geometric Dirac operator on $\mathbb{S}$ then both $\mathfrak{D}^*\mathfrak{D}$ and $\mathfrak{D}\mathfrak{D}^*$ are generalized Laplacians. Suppose now that ∇ is a balanced connection on our spin manifold (M, g). It determines a symmetric Dirac operator $\mathfrak{D}(\nabla)$. We denote by $\nabla^{\mathfrak{w}}$ and respectively $\mathcal{R}_\nabla$ the Weitzenböck connection and respectively remainder of the generalized Laplacian $\mathfrak{D}(\nabla)^2$. A classical result of Lichnerowicz states that if ∇ is the Levi-Civita connection then $\nabla^{\mathfrak{w}} = \hat{\nabla}$ and $\mathcal{R} = s/4$, where s is the scalar curvature of the Riemann metric g. When ∇ is not symmetric the situation is more complicated but we can still produce explicit descriptions of $\nabla^{\mathfrak{w}}$ and $\mathcal{R}$.

More precisely we know from Proposition 1.6 that

$$\mathfrak{D}(\nabla) = \mathfrak{D}_0 + \frac{1}{4}\mathbf{c}(\mathfrak{b}T^\dagger).$$

We set $\varpi := \frac{1}{4}\mathfrak{b}T^\dagger$. As explained at the end of Section 1.1, $\mathfrak{D}(\nabla)$ is the Dirac operator associated to the superconnection $\hat{D} + \varpi$. Using [2, Theorem 1.3] we deduce the following result.

Theorem 1.10. *Denote by $\mathfrak{D}_{spin}$ the spin-Dirac operator induced by the Levi-Civita connection D, $\mathfrak{D}_{spin} = \mathfrak{D}(\hat{D})$. Any geometric Dirac operator $\mathfrak{D}$ can be written as*

$$\mathfrak{D} = \mathfrak{D}_{spin} + \mathbf{c}(\varpi) + \mathbf{c}(\mathbf{i}a), \quad a \in \Omega^1(M),\ \varpi \in \Omega^3(M).$$

Additionally, if $\nabla = D + \frac{2}{3}\varpi + U$, where $U \in \Omega^2(T^*M)$ is such that $\operatorname{tr} U = 0 = \mathfrak{b}U = 0$ *then*

$$\mathfrak{D} = \mathfrak{D}(\hat{\nabla}) + \mathbf{c}(\mathbf{i}a), \quad \mathfrak{D}(\hat{\nabla})^2 = (\nabla^{\mathfrak{w}})^*\nabla^{\mathfrak{w}} + \mathcal{R}_\nabla + \mathbf{c}(\mathbf{i}da),$$

where

$$\nabla^{\mathfrak{w}} = \hat{\nabla} + \frac{1}{4}\sum_{i,j,k} e^i \otimes T_{ijk}\mathbf{c}(e^j)\mathbf{c}(e^k), \quad \mathcal{R}_\nabla = \frac{1}{4}s(g) + \big(\mathbf{c}(d\varpi) - 2\|\varpi\|^2\big). \tag{1.7}$$

The last theorem has an obvious extension where we replace $\mathbb{S}$ by the complex spinor bundle $\mathbb{S}_\sigma$ determined by a *spin*^{c}-structure σ on M. This case requires the choice of a hermitian connection on the line bundle $\det \mathbb{S}_\sigma$. In the spin case $\det \mathbb{S} \cong \mathbb{C}$ and the additional hermitian connection on the trivial line bundle is encoded by the imaginary 1-from $\mathbf{i}a$ appearing in the statement of Theorem 1.10.

2. Dirac operators on almost-hermitian manifolds

2.1. Basic differential geometric objects on an almost-hermitian manifolds

In this subsection we survey a few differential geometric[2] facts concerning almost complex manifolds. For more details we refer to [4,6,7].

Consider an almost-hermitian manifold (M^{2n}, g, J). Recall that this means that (M, g) is a Riemann manifold and J is a skew-symmetric endomorphism of TM such that $J^2 = -1$. Fix $x_0 \in M$ and consider a local, oriented orthonormal frame of TM, $(e_1, f_1, \dots, e_n, f_n)$. We also assume it is adapted to J that is,

$$f_j = Je_j, \quad \forall j = 1, \dots, n.$$

We denote by $(e^1, f^1, \dots, e^n, f^n)$ the dual coframe. Let $\mathbf{i} := \sqrt{-1}$. We split $TM \otimes \mathbb{C}$ into $\pm\mathbf{i}$-eigen-subbundles of J, $TM^{1,0}$ and $T^{0,1}$. These are naturally equipped with hermitian metrics induced by g and have natural local unitary frames near x_0

$$TM^{1,0}\colon \quad \varepsilon_k := \frac{1}{\sqrt{2}}(e_k - \mathbf{i} f_k), \quad k = 1, \dots, n,$$

$$TM^{0,1}\colon \quad \bar{\varepsilon}_k := \frac{1}{\sqrt{2}}(e_k + \mathbf{i} f_k), \quad k = 1, \dots, n.$$

Form by duality $T^*M^{1,0}$ and $T^*M^{0,1}$ with local unitary frames given by

$$\varepsilon^k := \frac{1}{\sqrt{2}}(e^k + \mathbf{i} f^k) \quad \text{and respectively,} \quad \bar{\varepsilon}^k := \frac{1}{\sqrt{2}}(e^k - \mathbf{i} f^k), \quad k = 1, \dots, n.$$

For $m = 0, \dots, 2n$ we have unitary decompositions

$$\Lambda^m T^*M \otimes \mathbb{C} = \bigoplus_{p+q=m} \Lambda^{p,q} T^*M, \quad \Lambda^{p,q} T^*M := \Lambda^p T^*M^{1,0} \otimes \Lambda^q T^*M^{0,1}.$$

Set $K_M := \Lambda^{n,0} T^*M$. We denote by $P^{p,q}$ the unitary projection onto $\Lambda^{p,q}$ and define

$$\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M), \quad \bar{\partial} := P^{p,q+1} \circ d$$

and

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \partial := P^{p+1,q} \circ d.$$

The space $\Omega^3(M) \otimes \mathbb{C}$ splits unitarily as

$$\Omega^3 \otimes \mathbb{C} = \Omega^+ \oplus \Omega^-, \quad \Omega^+ := \Omega^{2,1} \oplus \Omega^{1,2}, \quad \Omega^- := \Omega^{3,0} \oplus \Omega^{0,3}.$$

Finally, introduce the involution $\mathfrak{M}$ on $\Omega^2(T^*M)$ defined by

$$\mathfrak{M}B(X; Y, Z) = B(X; JY, JZ).$$

[2] Our conventions for the wedge product and exterior derivative differ from those used in [3] or [6, I.§3]. They agree with those in [4,8]. This explains some discrepancies between formulæ in [3,6] and the present paper.

Observe that

$$\psi^+ = \mathfrak{b}\mathfrak{M}\psi^+, \quad \forall \psi^+ \in \Omega^+.$$

We denote by $\Omega^{1,1}(T^*M)$ the 1-eigenspace of $\mathfrak{M}$ and by $\Omega_s^{1,1}(T^*M)$ the intersection of $\ker \mathfrak{b}$ with $\Omega^{1,1}(T^*M)$. Thus

$$A \in \Omega_s^{1,1}(T^*M) \quad \Longleftrightarrow \quad A = \mathfrak{M}A, \quad \mathfrak{b}A = 0.$$

The Nijenhuis tensor $N \in \Omega^2(TM)$ is defined by

$$N(X,Y) := \frac{1}{4}\big([JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\big), \quad \forall X, Y \in \mathrm{Vect}(M).$$

Notice that $N(JX,Y) = N(X,JY) = -JN(X,Y)$. This implies immediately that $\operatorname{tr} N^\dagger = 0$.

We denote by D the Levi-Civita connection determined by the metric g and by ω the fundamental 2-form defined by

$$\omega(X,Y) = g(JX,Y), \quad \forall X, Y \in \mathrm{Vect}(M).$$

Locally we have $\omega = \mathbf{i}\sum_j \varepsilon^j \wedge \bar{\varepsilon}^j$. Define $d^c\omega \in \Omega^3(M)$ by

$$d^c\omega(X,Y,Z) = -d\omega(JX,JY,JZ).$$

The *Lee form* θ determined by (g,J) is defined by

$$\theta = \Lambda(d\omega) = -J\Lambda\big((d^c\omega)^+\big),$$

where Λ denotes the contraction by ω, $\Lambda = (\omega\wedge\ )^*$, and J acts on the 1-form α by

$$J\alpha(X) = -\alpha(JX), \quad \forall X \in \mathrm{Vect}(M).$$

We have the following identity (see [6, §IX.4] where the authors use slightly different conventions)

$$g\big((D_XJ)Y,Z\big) = -\frac{1}{2}d\omega(X,JY,JZ) + \frac{1}{2}d\omega(X,Y,Z) + 2g\big(N(Y,Z),JX\big). \tag{2.1}$$

The form ω determines the skew-symmetric part of $N^\dagger$ via the identity

$$\mathfrak{b}N^\dagger = (d^c\omega)^-.$$

The almost complex structure defines a Cauchy–Riemann operator

$$\bar{\partial}_J : C^\infty(TM^{1,0}) \to \Omega^{0,1}(TM^{1,0})$$

defined by $X \lrcorner \bar{\partial}_J Y = [X,Y]^{1,0}$, $\forall X \in C^\infty(TM^{0,1})$, $Y \in C^\infty(TM^{1,0})$.

A *hermitian connection* on TM is a metric connection ∇ such that $\nabla J = 0$. A hermitian connection ∇ is completely determined by $\psi^+ := \frac{1}{3}(\mathfrak{b}T)^+$ and $B := (T^\dagger)_s^{1,1}$ via the equality (see [4, Section 2.3])

$$T(\nabla)^\dagger = N^\dagger + \frac{1}{8}(d^c\omega)^+ - \frac{3}{8}\mathfrak{M}(d^c\omega^+) + \frac{9}{8}\psi^+ - \frac{3}{8}\mathfrak{M}\psi^+ + B.$$

We will denote the above connection by $\nabla(\psi^+, B)$. When $B = 0$ we write $\nabla(\psi^+)$ instead of $\nabla(\psi^+, B)$. Observe that if T is the torsion of $\nabla(\psi^+, B)$ then

$$\mathfrak{b}T^\dagger = \mathfrak{b}N^\dagger + 3\psi^+ = (d^c\omega)^- + 3\psi^+.$$

Using the formulæ [4, (1.3.5), (1.4.9)] and the equality $\psi^+ = \mathfrak{b}\mathfrak{M}\psi^+$, $\forall \psi^+ \in \Omega^+$ we deduce that

$$\operatorname{tr}\mathfrak{M}\psi^+ = -2J\Lambda\psi^+, \quad \forall \psi^+ \in \Omega^+(M).$$

Since $\operatorname{tr} N^\dagger = 0$ we deduce that the trace of the torsion of $\nabla(\psi^+, B)$

$$\operatorname{tr} T\big(\nabla(\psi^+, B)\big) = \operatorname{tr} B + \frac{3}{4}J\Lambda\big((d^c\omega)^+ + \psi^+\big) = \operatorname{tr} B - \frac{3}{4}\theta + \frac{3}{4}J\Lambda\psi^+.$$

Example 2.1. The *first canonical connection* (see [4, Section 2.5] or [7]) is the hermitian connection ∇^0 defined by $B = 0$ and $\mathfrak{b}T_0^\dagger = (d^c\omega)^- - (d^c\omega^+)$ so that $\psi^+ = -\frac{1}{3}(d^c\omega)^+$. Its torsion is

$$T_0^\dagger = N^\dagger - \frac{1}{4}\big((d^c\omega)^+ + \mathfrak{M}(d^c\omega)^+\big).$$

In general, it is not a balanced connection since $\operatorname{tr} T_0^\dagger = -\frac{1}{2}\theta$.

Example 2.2. The *Chern connection* or the *second fundamental connection*, [4,7], is the *unique* hermitian connection ∇ on TM such that $\nabla^{0,1} = \bar{\partial}_J$. We will denote it by ∇^c. Alternatively (see [4, Section 2.5]), it is the hermitian connection defined by $B = 0$ and $\mathfrak{b}T^\dagger = (d^c\omega)^- + (d^c\omega)^+$, i.e., it is determined by $\psi^+ = \frac{1}{3}(d^c\omega)^+$. Its torsion is given by

$$T_c^\dagger = N^\dagger + \frac{1}{2}\big((d^c\omega)^+ - \mathfrak{M}(d^c\omega)^+\big).$$

In general, it is not a balanced connection since $\operatorname{tr} T_c^\dagger = -\theta$.

Theorem 2.3. *For every $B \in \Omega_s^{1,1}(T^*M)$ such that $\operatorname{tr} B = \frac{1}{2}\theta$ there exists a* hermitian *connection $\nabla^b = \nabla^b(B)$ uniquely determined by the following conditions.*

(i) *∇^b is balanced.*
(ii) *$(T^\dagger)_s^{1,1} = B$.*
(iii) *∇^b is quasi-equivalent to ∇^0 (see (1.5)).*

Proof. We seek ∇^b of the form $\nabla^b = \nabla(\psi^+, B)$. The condition (iii) implies that its torsion satisfies $\mathfrak{b}T_b = (d^c\omega)^- - (d^c\omega)^+$. Thus we need to choose $\psi^+ = -\frac{1}{3}(d^c\omega)^+$. Now observe that $0 = \operatorname{tr} T_b^\dagger = \operatorname{tr} B - \frac{1}{2}\theta$. □

Definition 2.4. We will refer to any of the connections ∇^b constructed in Theorem 2.3 as a *basic connection* determined by an almost hermitian structure.

The torsion of a basic connection $\nabla^b(B)$ is

$$T_b^\dagger = N^\dagger - \frac{1}{4}\big((d^c\omega)^+ + \mathfrak{M}(d^c\omega)^+\big) + B. \tag{2.2}$$

Observe also that on an almost Kähler manifold the first and second fundamental connection coincide. The resulting connection is basic with $B \equiv 0$. They are precisely the connections used by Taubes [12], to analyze the Seiberg–Witten monopoles on a symplectic manifold.

For any basic connection ∇^b we have the following identities ([4, Section 3.5])

$$(\bar\partial\phi)(Z_0, Z_1, \dots, Z_p) = \sum_{j=0}^{p} (-1)^j \nabla^b_{Z_j}\phi(Z_0, \dots, \hat{Z}_j, \dots, Z_p), \tag{2.3a}$$

$$\bar\partial^*\phi(Z_1, \dots, Z_{p-1}) = -\sum_{i=1}^{n} \bigl(e_i \lrcorner \nabla^b_{e_i}\phi + f_j \lrcorner \nabla^b_{f_i}\phi\bigr)(Z_1, \dots, Z_{p-1}),$$
$$\forall Z_0, \dots, Z_p \in C^\infty(T^{0,1}M),\ \phi \in \Omega^{0,p}(M). \tag{2.3b}$$

2.2. *Hodge–Dolbeault operators*

An almost hermitian manifold M is equipped with a canonical *spin*^{c} structure and the associated complex spinor bundle is

$$\mathbb{S}_c := \Lambda^{0,*}T^*M = \bigoplus_{p\geqslant 0} \Lambda^{0,p}T^*M.$$

Note that $\det \mathbb{S}_c = K_M^{-1}$. The Chern connection induces a hermitian connection $\det \nabla^c$ on K_M^{-1} and we denote by $\mathfrak{D}_c$ the geometric Dirac operator induced by the Levi-Civita connection D on TM and the connection $\det\nabla^c$ on K_M^{-1}.

If M is spinable, then a choice of spin structure is equivalent to a choice of a square root of K_M and in this case $\mathbb{S}_c := \mathbb{S} \otimes K_M^{-1/2}$. The bundle $\mathbb{S}_c$ has a natural Dirac type operator, the Hodge–Dolbeault operator

$$\mathcal{H}_J := \sqrt{2}(\bar\partial + \bar\partial^*) : C^\infty(\mathbb{S}_J) \to C^\infty(\mathbb{S}_J).$$

We have the following result [2, Theorem 2.2] and [4, Section 3.6].

$$\mathcal{H}_J = \mathfrak{D}_c - \frac{1}{4}\bigl\{\mathbf{c}\bigl((d^c\omega)^+\bigr) - \mathbf{c}\bigl((d^c\omega)^-\bigr)\bigr\}.$$

Using Theorem 1.10 we deduce that $\mathcal{H}_J$ is a geometric Dirac operator, more precisely $\mathcal{H}_J$ is induced by $\widehat{\nabla} \otimes \mathbb{1} + \mathbb{1} \otimes \det\nabla^c$, where ∇ is the connection

$$\nabla = D - \frac{1}{6}\bigl((d^c\omega)^+ - (d^c\omega)^-\bigr) \quad \text{with torsion } T^\dagger = \frac{1}{3}\bigl(d^c(\omega)^- - (d^c\omega)^+\bigr).$$

3. Dirac operators on contact manifolds

3.1. *Differential objects on metric contact manifolds*

We review a few basic geometric facts concerning metric contact manifolds. For more details we refer to [3,11].

A *metric contact manifold* (m.c. manifold for brevity) is an oriented manifold of odd dimension $2n+1$ equipped with a Riemann metric g and a 1-form η such that

- $|\eta(x)|_g = 1$, $\forall x \in M$. Denote by $\xi \in \mathrm{Vect}(M)$ the metric dual of η and set $V := \ker \eta \subset TM$. V is a hyperplane sub-bundle of TM and we denote by P_V the orthogonal projection onto V.
- There exists $J : TM \to TM$ such that

$$d\eta(X, Y) = g(JX, Y), \quad J^2 X = -X + \eta(X)\xi, \quad \forall X, Y \in \mathrm{Vect}(M).$$

Definition 3.1. A *contact metric connection* on (M^{2n+1}, η, J, g) is a metric connection such that $\nabla J = 0 = \nabla \xi$.

The manifold M is called *positively oriented* if the orientation induced by the nowhere vanishing $(2n+1)$-form $\eta \wedge (d\eta)^n$ coincides with the given orientation of M. In this case $dv_g = \frac{1}{n!}\eta \wedge (d\eta)^n$. Set $\omega := d\eta$. We have decompositions

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}, \quad V^* \otimes \mathbb{C} = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

and we set $K_M := \det(V^*)^{1,0}$, $\Phi := L_\xi J$. The operator Φ is a traceless, symmetric endomorphism of V (see [3]). Since $L_\xi(J^2) = 0$ we deduce

$$J\Phi + \Phi J = 0 \quad \Longrightarrow \quad (J\Phi)^* = (J\Phi). \tag{3.1}$$

Define the Nijenhuis tensor $N \in \Omega^2(TM)$ by[3]

$$N(X, Y) = \frac{1}{2}\big\{J^2[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y]\big\}.$$

Notice that $N(\xi, X) = -\frac{1}{2}J\Phi X$, $\forall X \in \mathrm{Vect}(M)$. (M, g, η) is a *Cauchy–Riemann manifold* (CR for brevity) if and only if $JN(X, Y) = 0$, $\forall X, Y \in C^\infty(V)$. Equivalently, this means, and

$$N(X, Y) + \frac{1}{2}\omega(X, Y)\xi = -J^2 N(X, Y) = 0, \quad \forall X, Y \in C^\infty(V).$$

In this case, the Nijenhuis tensor can be given the more compact description

$$N^\dagger = \frac{1}{2}(J\Phi \wedge \eta - \eta \otimes d\eta). \tag{3.2}$$

In particular, M is a CR manifold when $\dim M = 3$.

3.2. *The generalized Tanaka–Webster connection*

To each metric contact manifold M we can associate an almost hermitian manifold $(\hat{M}, \hat{g}, \hat{J})$ defined as follows.

$$\hat{M} = \mathbb{R} \times M, \quad \hat{g} = dt^2 + g, \quad \hat{J}\partial_t = \xi.$$

We will denote by $\hat{d}$ the exterior differentiation on $\hat{M}$. If we set

$$\hat{\omega}(X, Y) = \hat{g}(\hat{J}X, Y), \quad \forall X, Y \in \mathrm{Vect}(\hat{M})$$

[3] We used the factor $\frac{1}{2}$ rather than the $\frac{1}{4}$ used in the almost complex case only to keep up with the conventions in [3].

then $\hat{\omega} = dt \wedge \eta + \omega$ and $\hat{d}\hat{\omega} = -dt \wedge \omega$. We deduce that the Lee form $\theta = \Lambda(-dt \wedge d\eta)$ is $-n\,dt$. We will work with local, oriented orthonormal frames $(e_0, f_0, e_1, \dots, e_n, f_n)$ adapted to $\hat{J}$ such that

$$e_0 = \partial_t, \quad f_0 = \xi, \quad e^0 = dt, \quad f^0 = \eta,$$

$$\hat{\omega} = \mathbf{i}\varepsilon^0 \wedge \bar{\varepsilon}^0 + \mathbf{i}\sum_{k=1}^n \varepsilon^k \wedge \bar{\varepsilon}^k, \quad \hat{d}\hat{\omega} = -\frac{\mathbf{i}}{\sqrt{2}}(\varepsilon^0 + \bar{\varepsilon}^0) \wedge \sum_{k=1}^n \varepsilon^k \wedge \bar{\varepsilon}^k.$$

Hence

$$\hat{d}^c\hat{\omega} = -\frac{1}{\sqrt{2}}(\varepsilon^0 - \bar{\varepsilon}^0) \wedge \sum_{k=1}^n \varepsilon^k \wedge \bar{\varepsilon}^k = -\eta \wedge d\eta$$

so that $(\mathfrak{b}\hat{N}^\dagger) = (\hat{d}^c\hat{\omega})^- = 0$. We have the following identity, [3].

$$\hat{N}(X, Y) = \frac{1}{2}\left(N(X, Y) + \frac{1}{2}\omega(X, Y)\xi\right), \quad \hat{N}(\partial_t, X) = \frac{1}{4}\Phi X, \quad \forall X, Y \in \operatorname{Vect}(M). \tag{3.3}$$

Observe that $\hat{N}^\dagger|_M = \frac{1}{2}N^\dagger + \frac{1}{4}\eta \otimes d\eta$ so that

$$0 = \mathfrak{b}\hat{N}^\dagger|_M = \frac{1}{2}\mathfrak{b}N^\dagger + \frac{1}{4}\mathfrak{b}(\eta \otimes d\eta) = \frac{1}{2}\mathfrak{b}N^\dagger + \frac{1}{4}\eta \wedge d\eta.$$

Hence

$$\mathfrak{b}N^\dagger = -\frac{1}{2}\eta \wedge d\eta.$$

We want to find $B \in \Omega_s^{1,1}(T^*\hat{M})$ such that $\operatorname{tr} B = -\frac{n}{2}dt$ and the basic connection it induces on $T^*\hat{M}$ is compatible with the splitting $\partial_t \oplus TM$. From (2.2) we deduce that the torsion of such a connection is

$$\hat{T}_b^\dagger = \hat{N}^\dagger - \frac{1}{4}\big((\hat{d}^c\hat{\omega})^+ + \mathfrak{M}(\hat{d}^c\hat{\omega})^+\big) + B = \hat{N}^\dagger + \frac{1}{4}\big(\eta \wedge \omega + \mathfrak{M}(\eta \wedge \omega)\big) + B. \tag{3.4}$$

Thus $\mathfrak{b}T_b^\dagger = \eta \wedge d\eta$. Using Proposition 1.2 we deduce that $\nabla^b = D + A$ where

$$A_b^\dagger = \frac{1}{2}\mathfrak{b}T_b^\dagger - T_b^\dagger = \frac{1}{4}\big(\eta \wedge d\eta - \mathfrak{M}(\eta \wedge d\eta)\big) - \hat{N}^\dagger - B.$$

Thus, for all $X, Y \in \operatorname{Vect}(M)$ which are t-independent we have

$$\hat{g}(\nabla_t^b X, Y) = A_b^\dagger(\partial_t; X, Y).$$

Since

$$B(\partial_t; \bullet, \bullet) = 0 \quad \text{and} \quad \hat{g}(\hat{N}(X, Y), \partial_t) = 0, \quad \forall X, Y \in \operatorname{Vect}(M).$$

we deduce

$$\hat{g}(\nabla_t^b X, Y) = -\frac{1}{4}\mathfrak{M}(\eta \wedge d\eta)(\partial_t; X, Y) = 0.$$

Similarly, we deduce

$$\hat{g}(\nabla_t^b X, \partial_t) = A_b^\dagger(\partial_t; X, \partial_t) = 0 \quad \Longrightarrow \quad \nabla_t^b Z = 0, \quad \forall Z \in \operatorname{Vect}(M).$$

Since ∇^b is a metric connection we deduce $\hat{g}(\nabla^b_\bullet \partial_t, \partial_t) = 0$. On the other hand, for any vector fields X, Y on M we have

$$\hat{g}(\nabla^b_X \partial_t, Y) = A_b^\dagger(X; \partial_t, Y) = -\frac{1}{4}\mathfrak{M}\eta \wedge d\eta(X, \partial_t, Y) - \hat{g}\big(\hat{N}(\partial_t, Y), X\big) - B(X; \partial_t, Y)$$
$$= \frac{1}{4} g(X_V, Y_V) - \frac{1}{4} g(\Phi Y, X) - B(X; \partial_t, Y),$$

where $X_V = P_V X$, $Y = P_V Y$. Next, $\forall X, Y \in \mathrm{Vect}(M)$, we have

$$\hat{g}(\nabla^b_X Y, \partial_t) = A_b^\dagger(X; Y, \partial_t) = -\frac{1}{4}\mathfrak{M}\eta \wedge d\eta(X; Y, \partial_t) - \hat{g}\big(\hat{N}(Y, \partial_t), X\big) - B(X; Y, \partial_t)$$
$$= -\frac{1}{4} g(X_V, Y_V) + \frac{1}{4} g(\Phi Y, X) - B(X; Y, \partial_t).$$

Lemma 3.2. *There exists $B_0 \in \Omega_s^{1,1}(T^*\hat{M})$ such that* $\operatorname{tr} B = -\frac{n}{2}\, dt$ *and*

$$B(\partial_t; \bullet, \bullet) = 0, \tag{3.5a}$$

$$B(X; Y, \partial_t) = \frac{1}{4} g(X, \Phi Y) - \frac{1}{4} g(X_V, Y_V), \quad \forall X, Y \in \mathrm{Vect}(V). \tag{3.5b}$$

Proof. Define

$$B = \frac{1}{4}(\Phi \wedge dt + J\Phi \wedge \eta) - \frac{1}{4}(P_V \wedge dt + J P_V \wedge \eta) + \frac{1}{2}\eta \otimes d\eta \tag{3.6}$$

and we set

$$B_0 = \frac{1}{4}(\Phi \wedge dt + J\Phi \wedge \eta), \quad B_1 = -\frac{1}{4}(P_V \wedge dt + J P_V \wedge \eta).$$

We need to show that this definition is correct, i.e., the above B satisfies all the required conditions (3.5a), (3.5b) and

$$\operatorname{tr} B = -\frac{n}{2}\, dt, \quad \mathfrak{b} B = 0, \quad B \in \Omega^{1,1}(T^*M).$$

Here the elementary properties in Lemma 1.1 will come in handy. Since Φ and $J\Phi$ are symmetric and traceless we deduce that $\operatorname{tr} B_0 = 0$, $\mathfrak{b} B_0 = 0$. The condition $B_0 \in \Omega^{1,1}$ follows from the identity $\phi J = -J\Phi$. Now observe that $B_1 \in \Omega^{1,1}$ and

$$\mathfrak{b} B_1 = -\frac{1}{2}\eta \wedge d\eta, \quad \operatorname{tr} B_1 = -\frac{n}{2}\, dt.$$

Finally $\eta \otimes d\eta \in \Omega^{1,1}$, it is traceless and $\mathfrak{b}(\eta \otimes d\eta) = \eta \wedge d\eta$. The condition (3.5b) follows by direct computation. The lemma follows putting together the above facts. $\square$

If we choose B as in Lemma 3.2 we deduce

$$\hat{g}(\nabla^b_\bullet X, \partial_t) = 0, \quad \forall X \in \mathrm{Vect}(M).$$

The above computations show that the basic connection ∇^b of $(\hat{M}, \hat{g}, \hat{J})$ determined by B_0 preserves the orthogonal splitting $T\hat{M} = \langle \partial_t \rangle \oplus TM$ and thus induces a *balanced* contact metric connection ∇^{TW}

on TM. To compute its torsion we use the identity (3.4). Observe that

$$\hat{N}^\dagger|_M = \frac{1}{2}\left\{N^\dagger + \frac{1}{2}\eta \otimes d\eta\right\},$$

and $\mathfrak{M}(\eta \wedge d\eta)|_M = \eta \otimes d\eta$. Finally

$$B|_M = \frac{1}{4}(J\Phi) \wedge \eta - \frac{1}{4}JP_V \wedge \eta + \frac{1}{2}\eta \otimes d\eta.$$

Since on M we have the equality $JP_V = J$, the torsion T_{TW} of ∇^{TW} given by

$$T_{\mathrm{TW}}^\dagger = \frac{1}{2}N^\dagger + \eta \otimes d\eta + \frac{1}{4}\eta \wedge d\eta + \frac{1}{4}(J\Phi - J) \wedge \eta. \tag{3.7}$$

Moreover, $\mathfrak{b}T_{\mathrm{TW}} = \eta \wedge d\eta$.

Definition 3.3. We will call the above connection ∇^{TW} the *generalized Tanaka–Webster* connection of M. It is the unique metric connection with torsion given by (3.7).

To explain the terminology in the above definition suppose now that M is a CR-manifold. Using (3.2) and (3.7) we deduce

$$T_{\mathrm{TW}}^\dagger = \frac{3}{4}\eta \otimes d\eta + \frac{1}{4}\eta \wedge d\eta - \frac{1}{4}(J \wedge \eta) + \frac{1}{2}J\Phi \wedge \eta.$$

In particular,

$$T_{\mathrm{TW}}(X, Y) = d\eta(X, Y)\xi, \quad \forall X, Y \in \mathrm{Vect}(V).$$

Because the distribution $V^{1,0}$ is integrable we deduce

$$T_{\mathrm{TW}}(X, Y) = 0, \quad \forall X, Y \in C^\infty(V^{1,0}).$$

A contact metric connection with the above property will be called a *CR metric connection*. Next observe that for $X, Y \in C^\infty(V)$ we have

$$g\big(X, T_{\mathrm{TW}}(\xi, Y)\big) = T_{\mathrm{TW}}^\dagger(X; \xi, Y) = -\frac{1}{4}d\eta(X, Y) + \frac{1}{4}g(JX, Y) + \frac{1}{2}g(J\Phi X, Y).$$

Hence $T_{\mathrm{TW}}(\xi, Y) = \frac{1}{2}J\Phi Y$. Since $\Phi J = -J\Phi$ we deduce $JT_{\mathrm{TW}}(\xi, X) = -T_{\mathrm{TW}}(\xi, JX)$.

Remark 3.4. Using [11, Proposition 3.1], we deduce that when M is a Cauchy–Riemann manifold, the connection ∇^{TW} on (V, J) is the Tanaka–Webster connection determined by the CR structure (see [10,11,13] for more details). The generalized Tanaka–Webster connection we have constructed does not agree with the generalized Tanaka–Webster connection constructed by S. Tanno in [11] because that connection is not compatible with J if M is not a CR-manifold.

Finally, let us point out that when M is a CR manifold then

$$g(\nabla_\xi^{\mathrm{TW}}X, Y) = g(D_\xi X, Y) + \frac{1}{2}\mathfrak{b}T_{\mathrm{TW}}^\dagger(\xi, X, Y) - T_{\mathrm{TW}}^\dagger(\xi; X, Y) = g\left(D_\xi X - \frac{1}{2}JX, Y\right)$$

so that $\nabla_\xi^{\mathrm{TW}} = D_\xi^V := P_V D_\xi - \frac{1}{2}J$.

Example 3.5. Let us consider in greater detail the special case of a metric, contact, *spin* 3-manifold M. M is automatically a CR-manifold so that the torsion of the (generalized) Tanaka–Webster connection satisfies

$$T_{\mathrm{TW}}(X,Y)=d\eta(X,Y)\xi,\quad T_{\mathrm{TW}}(\xi,X)=\frac{1}{2}J\Phi X,\quad \forall X,Y\in C^{\infty}(V),\quad \mathfrak{b}T^{\dagger}_{\mathrm{TW}}=\eta\wedge d\eta.$$

The *spin* Dirac operator $\mathfrak{D}_0$ on M is related to the Dirac operator $\mathfrak{D}(\nabla^{\mathrm{TW}})$ by the equality

$$\mathfrak{D}(\nabla^{\mathrm{TW}})=\mathfrak{D}_0+\frac{1}{4}\mathbf{c}(\mathfrak{b}T^{\dagger}_{\mathrm{TW}})=\mathfrak{D}_0+\frac{1}{4}\mathbf{c}(\eta\wedge d\eta)=\mathfrak{D}_0-\frac{1}{4}.$$

When M is Sasakian, i.e., $\Phi=0$, the above equality shows that $\mathfrak{D}(\nabla^{\mathrm{TW}})$ coincides with the adiabatic Dirac operator introduced in [9] (see in particular [9, Eq. (2.20)] with $\lambda=\frac{1}{2}$, $\delta=1$).

Later on we will need to compare the connections $\det\nabla^c$ and $\det\nabla^b$ induced by the Chern connection ∇^c and respectively ∇^b on $K^{-1}_{\hat{M}}$.

Proposition 3.6. $\det\nabla^c=\det\nabla^b+\frac{n\mathbf{i}}{2}\eta$.

Proof. Denote by ∇^0 the first fundamental connection of $(\hat{M},\hat{J})$. We have $\nabla^b=\nabla^0-B$, where B is described in Lemma 3.2. Set $\delta:=\varepsilon_0\wedge\varepsilon_1\wedge\cdots\wedge\varepsilon_n$. Then for every vector field X on $\hat{M}$ we have

$$\det\nabla^b_X\delta=\det\nabla^0_X\delta-B_X\delta.$$

Observe that $B_X\varepsilon^k=\sum_{j=0}^n C^j_k\varepsilon_j$ so that $B_X\delta=(\sum_{j=0}^n C^k_k)\delta$. On the other hand, $C^k_k=g_c(B_X\varepsilon_k,\bar{\varepsilon}_k)$, where g_c denotes the complex bilinear extension of g. Hence

$$C^k_k=\frac{1}{2}g_c\big(B_X(e_k-\mathbf{i}f_k),e_k+\mathbf{i}f_k\big)=\mathbf{i}g(B_Xe_k,Je_k)+\mathbf{i}g(B_Xf_k,Jf_k).$$

Thus

$$\sum_k C^k_k=-\mathbf{i}\sum_{k=0}^n\big(g(JB_Xe_k,e_k)+g(JB_Xf_k,f_k)\big)=-\mathbf{i}\,\mathrm{tr}\,JB_X. \tag{3.8}$$

Using equality (3.6) we deduce

$$\hat{g}(B_XY,JY)=\frac{1}{4}\big\{\hat{g}(\Phi X,Y)\,dt(JY)-\hat{g}(J\Phi X,Y)\eta(JY)\big\}$$
$$+\frac{1}{4}\big\{\hat{g}(P_VX,Y)\,dt(JY)-\hat{g}(JP_VX,Y)\eta(JY)\big\}+\frac{1}{2}\eta(X)\,d\eta(Y,JY).$$

We see that $\mathrm{tr}\,JB_X\neq0$ only if $X=\xi$ in which case shows that the sum (3.8) is n. Hence

$$\nabla^b\delta=\nabla^0\delta-\mathbf{i}n\eta.$$

On the other hand we have the identity, [4, Eq. (2.7.6)],

$$\det\nabla^c=\det\nabla^0+\frac{\mathbf{i}}{2}J\theta=\det\nabla^0-\frac{n\mathbf{i}}{2}J\,dt=\det\nabla^b+\frac{n\mathbf{i}}{2}\eta.\qquad\Box$$

Corollary 3.7. $F(\det\nabla^c)=F(\det\nabla^b)+\frac{n\mathbf{i}}{2}d\eta$.

3.3. Geometric Dirac operators on contact manifolds

Consider the Hodge–Dolbeault operator $\hat{\mathcal{H}}$ on $\hat{M}$

$$\hat{\mathcal{H}} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*) : \Omega^{0,*}(\hat{M}) \to \Omega^{0,*}(\hat{M}).$$

It is a geometric Dirac operator and it satisfies

$$\hat{\mathcal{H}} = \sqrt{2}\sum_{k=0}^{n}\big(\hat{\mathbf{c}}(\varepsilon^k)\hat{\nabla}_{\varepsilon_k} + \hat{\mathbf{c}}(\bar{\varepsilon})\hat{\nabla}_{\bar{\varepsilon}_k}\big),$$

where $\hat{\mathbf{c}}$ denotes the Clifford multiplication on $\hat{\mathbb{S}}_c \cong \Lambda^{0,*}T^*\hat{M}$, $\hat{\nabla} = \hat{\nabla}^b \otimes \mathbb{1} + \mathbb{1} \otimes \det \nabla^c$, and $\det \nabla^c$ denotes the hermitian connection on $K_{\hat{M}}^{-1}$ induced by the Chern connection on $T\hat{M}$. More precisely

$$\hat{\mathbf{c}}(\bar{\varepsilon}^k) = \sqrt{2}\bar{\varepsilon}^k \wedge \bullet, \quad \hat{\mathbf{c}}(\varepsilon^k) = -\sqrt{2}\varepsilon^k \lrcorner \bullet\,.$$

Above, $\varepsilon^k \lrcorner \bullet$ denotes the odd derivation of $\Omega^{0,*}(\hat{M})$ uniquely determined by the requirements

$$\varepsilon^k \lrcorner\, \bar{\varepsilon}^j = \delta_{kj}, \quad \forall j, k = 0, \ldots, n.$$

We want to point out that $(\bar{\varepsilon}^k \wedge)^* = \varepsilon^k \lrcorner$. We set

$$\mathcal{J} := \hat{\mathbf{c}}(dt) = \frac{1}{\sqrt{2}}\big(\hat{\mathbf{c}}(\varepsilon^0) + \hat{\mathbf{c}}(\bar{\varepsilon}^0)\big), \quad \mathbb{S}_c := \hat{\mathbb{S}}_c^+\big|_{0\times M}.$$

Note that $\hat{\mathbb{S}}_c|_M \cong \mathbb{S}_c \oplus \mathcal{J}\mathbb{S}_c$.

The metric contact structure on M produces a $U(n)$-reduction of the tangent bundle TM. This $U(n)$-reduction induces a *spin*^{c} structure on M and $\mathbb{S}_c$ is the associated bundle of complex spinors and $\det \mathbb{S}_c \cong K_M^{-1}$.

The Clifford multiplication on $\mathbb{S}_c$ is defined by the equality

$$\mathbf{c}(\alpha) = \mathcal{J}\hat{\mathbf{c}}(\alpha), \quad \forall \alpha \in \Omega^1(M).$$

Along M we can identify $\hat{\mathbb{S}}_c^-$ with $\mathcal{J}\mathbb{S}_c^+$ and as such $\mathcal{J}$ we can write.

$$\mathcal{J} = \begin{bmatrix} 0 & -G^* \\ G & 0 \end{bmatrix}, \quad GG^* = G^*G = \mathbb{1}_{\mathbb{S}_c}.$$

We can view the Hodge–Dolbeault operator as an operator on $\mathbb{S}_c \oplus \mathbb{S}_c$

$$\hat{\mathcal{H}} = \mathcal{J}\left(\hat{\nabla}_t^b - \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -G\mathcal{H}G^* \end{bmatrix}\right), \quad \mathcal{H}^* = \mathcal{H}.$$

$\mathcal{H}$ is the geometric Dirac operator induced by $\hat{\nabla}^{\mathrm{TW}} \otimes \mathbb{1} + \mathbb{1} \otimes \det \nabla^c$. We want to provide a more explicit description of the operator $\mathcal{H}$. Observe that

$$C^\infty(\hat{\mathbb{S}}_c^+) = \Omega^{0,even}(\hat{M}) = \Omega^{0,even}(V^*) \oplus \bar{\varepsilon}^0 \wedge \Omega^{0,odd}(V^*)$$

where $\Omega^{0,p}(V^*) := C^\infty(\Lambda^p(V^*)^{0,1})$. We can represent $\psi \in C^\infty(\hat{\mathbb{S}}_c^+)$ as a sum

$$\psi = \psi_+ \oplus \bar{\varepsilon}^0 \wedge \psi_-, \quad \psi_+ \in \Omega^{0,even}(V^*),\ \psi_- \in \Omega^{0,odd}(V^*).$$

The above decomposition can be alternatively described as follows. The operator

$$\mathbf{c}(\eta) = \mathcal{J}\hat{\mathbf{c}}(\eta) : C^\infty(\hat{\mathbb{S}}_c^+) \to C^\infty(\hat{\mathbb{S}}_c^+)$$

satisfies $\mathbf{c}(\eta)^2 = -1$ and thus $\mathbf{c}(\mathbf{i}\eta)$ is an involution of $C^\infty(\hat{\mathbb{S}}_c^+)$. More explicitly

$$\mathbf{c}(\eta) = \frac{\mathbf{i}}{2}\big(\hat{\mathbf{c}}(\bar{\varepsilon}^0) + \hat{\mathbf{c}}(\varepsilon^0)\big)\big(\hat{\mathbf{c}}(\bar{\varepsilon}^0) - \hat{\mathbf{c}}(\varepsilon^0)\big) = \mathbf{i}(\bar{\varepsilon}^0 \wedge -\varepsilon^0 \lrcorner)(\bar{\varepsilon}^0 \wedge +\varepsilon^0 \lrcorner).$$

Thus, for every $\phi \in \Omega^{0,*}(V^*)$ we have

$$\mathbf{c}(\mathbf{i}\eta)(\bar{\varepsilon}^0 \wedge \phi) = -\bar{\varepsilon}^0 \wedge \phi, \quad \mathbf{c}(-\mathbf{i}\eta)\phi = \phi.$$

This shows that the above decomposition is defined by the ± 1 eigenspaces of the involution $\mathbf{c}(\eta)$. The restriction of the operator $\bar{\partial} : \Omega^{0,*}(\hat{M}) \to \Omega^{0,p}(\hat{M})$ to $\Omega^{0,*}(V^*)$ decomposes into two parts. More precisely, if $\phi \in \Omega^{0,*}(V^*)$ then

$$\bar{\partial}\phi = \bar{\varepsilon}^0 \wedge \bar{\partial}_0\phi + \bar{\partial}_V\phi := \frac{1}{2}\big(1 + \mathbf{c}(\mathbf{i}\eta)\big)\bar{\partial} + \frac{1}{2}\big(1 - \mathbf{c}(\mathbf{i}\eta)\big)\bar{\partial}.$$

Note that

$$\bar{\partial}_0\phi := \varepsilon^0 \lrcorner \bar{\partial}\phi \in \Omega^{0,p}(V^*), \quad \bar{\partial}_V \in \Omega^{0,p+1}(V^*).$$

We will regard $\bar{\partial}_0$ and $\bar{\partial}_V$ as operators

$$\bar{\partial}_0 : \Omega^{0,*}(V^*) \to \Omega^{0,*}(V^*), \quad \bar{\partial}_V : \Omega^{0,*}(V^*) \to \Omega^{0,*+1}(V^*).$$

Pick a t-independent section $\psi = C^\infty(\hat{\mathbb{S}}_c^+)$. It decomposes as

$$\psi = \psi_+ + \bar{\varepsilon}^0 \wedge \psi_-, \quad \psi_\pm \in \Omega^{0,even/odd}(V^*).$$

We have the equality

$$\hat{\mathcal{H}}\begin{bmatrix} \psi \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 & -G^* \\ G & 0 \end{bmatrix}\begin{bmatrix} \mathcal{H} & 0 \\ 0 & -G\mathcal{H}G^* \end{bmatrix}\begin{bmatrix} \psi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{H}G^* \\ G\mathcal{H} & 0 \end{bmatrix}\begin{bmatrix} \psi \\ 0 \end{bmatrix}.$$

Thus

$$\sqrt{2}(\bar{\partial} + \bar{\partial}^*)\psi = G\mathcal{H}\psi = \hat{\mathbf{c}}(dt)\mathcal{H}\psi \quad \Longrightarrow \quad \mathcal{H}\psi = -\sqrt{2}\mathcal{J}(\bar{\partial} + \bar{\partial}^*)\psi.$$

We compute

$$\begin{aligned}
(\bar{\partial} + \bar{\partial}^*)(\psi_+ + \bar{\varepsilon}^0 \wedge \psi_-) &= \bar{\partial}\psi_+ + (\bar{\partial}\bar{\varepsilon}^0) \wedge \psi_- - \bar{\varepsilon}^0 \wedge \bar{\partial}\psi_- + \bar{\partial}^*\psi_+ + \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \psi_-) \quad (\bar{\partial}\bar{\varepsilon}^0 = 0) \\
&= \bar{\varepsilon}^0 \wedge \bar{\partial}_0\psi_+ + \bar{\partial}_V\psi_+ - \bar{\varepsilon}^0 \wedge \bar{\partial}_V\psi_- + (\bar{\varepsilon}^0 \wedge \bar{\partial}_0 + \bar{\partial}_V)^*\psi_+ + \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \psi_-) \\
&= \bar{\varepsilon}^0 \wedge (\bar{\partial}_0\psi_+ - \bar{\partial}_V\psi_-) + \bar{\partial}_V\psi_+ + \bar{\partial}_V^*\psi_+ + \bar{\partial}_0^*(\varepsilon^0 \lrcorner \psi_+) + \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \psi_-) \\
&= \bar{\varepsilon}^0 \wedge (\bar{\partial}_0\psi_+ - \bar{\partial}_V\psi_-) + \bar{\partial}_V\psi_+ + \bar{\partial}_V^*\psi_+ + \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \psi_-).
\end{aligned}$$

To proceed further we need to provide a more explicit description for $\bar{\partial}^*(\varepsilon^0 \lrcorner)^*\psi_-$. We denote by $\langle \bullet, \bullet \rangle_M$ the L^2-inner product on M. For every t-independent compactly supported $\alpha \in \Omega^{0,odd}(\hat{M})$ we have $\alpha = \alpha_- + \bar{\varepsilon}^0 \wedge \alpha_+$, $\alpha_\pm \in \Omega^{0,odd/even}(V^*)$, and

$$\begin{aligned}
\big\langle \alpha, \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \phi_-)\big\rangle_M &= \langle \bar{\partial}\alpha, \bar{\varepsilon}^0 \wedge \phi_- \rangle_M = \langle \bar{\varepsilon}^0 \wedge \bar{\partial}_0\alpha_-, \bar{\varepsilon}^0 \wedge \phi_- \rangle_M - \langle \bar{\varepsilon}^0 \wedge \bar{\partial}_V\alpha_+, \bar{\varepsilon}^0 \wedge \phi_- \rangle_M \\
&= \langle \bar{\partial}_0\alpha_-, \phi_- \rangle_M - \langle \bar{\partial}_V\alpha_+, \phi_- \rangle_M = \langle \alpha_-, \bar{\partial}_0^*\phi_- \rangle_M - \langle \alpha_+, \bar{\partial}_V^*\phi_- \rangle_M.
\end{aligned}$$

We conclude

$$\bar\partial^*(\bar\varepsilon^0\wedge\phi_-)=\bar\partial_0^*\phi_- - \bar\varepsilon^0\wedge\bar\partial_V^*\phi_-,$$

and

$$(\bar\partial+\bar\partial^*)(\psi_+ + \bar\varepsilon^0\wedge\psi_-)=\bar\varepsilon^0\wedge(\bar\partial_0\psi_+ - \bar\partial_V\psi_- - \bar\partial_V^*\phi_-)+\bar\partial_V\psi_+ + \bar\partial_V^*\psi_+ + \bar\partial_0^*\phi_-.$$

Now observe that

$$\hat{\mathbf{c}}(dt)\bullet=\frac{1}{\sqrt2}\big(\hat{\mathbf{c}}(\bar\varepsilon^0)+\hat{\mathbf{c}}(\varepsilon^0)\big)\bullet=(\bar\varepsilon^0\wedge\bullet-\varepsilon^0\lrcorner\bullet)$$

so that

$$\begin{aligned}\mathcal{H}\psi&=-\sqrt2(\varepsilon^0\lrcorner\ -\bar\varepsilon^0\wedge)\big\{\bar\varepsilon^0\wedge(\bar\partial_0\psi_+ - \bar\partial_V\psi_- - \bar\partial_V^*\phi_-)+\bar\partial_V^*\psi_+ + \bar\partial_V\psi_+ + \bar\partial_0^*\psi_-\big\}\\ &=-\sqrt2\big\{(\bar\partial_0\psi_+ - \bar\partial_V\psi_- - \bar\partial_V^*\phi_-)-\bar\varepsilon^0\wedge(\bar\partial_V^*\psi_+ + \bar\partial_V\psi_+ + \bar\partial_0^*\psi_-)\big\}.\end{aligned}$$

In block form

$$\mathcal{H}\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}=\sqrt2\begin{bmatrix}-\bar\partial_0 & (\bar\partial_V^*+\bar\partial_V)\\ (\bar\partial_V^*+\bar\partial_V) & \bar\partial_0^*\end{bmatrix}\cdot\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}.$$

The above equality can be further simplified as follows. If $\phi\in\Omega^{0,p}(V^*)\subset\Omega^*(M)\otimes\mathbb{C}$ then

$$d\phi\in\eta\wedge\big(\Omega^{0,p}(V^*)+\Omega^{1,p-1}(V^*)\big)\oplus\Omega^{0,p+1}(V^*)\oplus\Omega^{1,p}(V^*)\oplus\Omega^{2,p-1}(V^*).$$

and

$$-\sqrt2\bar\partial_0\phi=-\mathbf{i}(\xi\lrcorner d\phi)^{0,p}=:-\mathbf{i}L_\xi^V\phi.$$

On the other hand, the identity (2.3a) implies

$$\bar\partial_0\phi=\nabla^b_{\bar\varepsilon_0}\phi=\frac{\mathbf{i}}{\sqrt2}\nabla^{\mathrm{TW}}_\xi\phi.$$

Since $\mathbf{div}_g\xi=0$ the operator $\mathbf{i}\nabla^{\mathrm{TW}}_\xi$ is symmetric and so must by $\mathbf{i}L_\xi^V$. Hence $\bar\partial_0^*\phi=\mathbf{i}L_\xi^V$ and

$$\mathcal{H}\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}=\begin{bmatrix}-\mathbf{i}L_\xi^V & \sqrt2(\bar\partial_V^*+\bar\partial_V)\\ \sqrt2(\bar\partial_V+\bar\partial_V^*) & \mathbf{i}L_\xi^V\end{bmatrix}\cdot\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix}$$

or equivalently,

$$\mathcal{H}=\mathbf{c}(\mathbf{i}\eta)L_\xi^V+\begin{bmatrix}0 & \sqrt2(\bar\partial_V+\bar\partial_V^*)\\ \sqrt2(\bar\partial_V^*+\bar\partial_V^*) & 0\end{bmatrix}. \tag{3.9}$$

We will refer to $\mathcal{H}$ as the *contact Hodge–Dolbeault operator*. The next result summarizes the results we have proved so far.

Theorem 3.8. *Suppose (M^{2n+1},g,η) is a metric contact manifold, $V:=\ker\eta$. Denote by $\mathbb{S}_c$ the bundle of complex spinors associated to the spin^c structure determined by the contact structure. Denote the corresponding Clifford multiplication by* $\mathbf{c}$.

(i) $\mathbb{S}_c \cong \Lambda^{0,*}V^*$, $\mathbf{c}(\mathbf{i}\eta)\phi = (-1)^p\phi$, $\forall\phi \in \Omega^{0,p}(V^*)$. *We decompose*

$$\mathbb{S}_c = \mathbb{S}_c^+ \oplus \mathbb{S}_c^-, \quad \mathbb{S}_c^\pm = \Lambda^{0,even/odd}(V^*).$$

(ii) *The operator* $\mathcal{H}: C^\infty(\mathbb{S}_c) \to C^\infty(\mathbb{S}_c)$ *defined by* (3.9) *is a geometric Dirac operator induced by the connection* ∇^{TW} *on* TM *and* $\det\nabla^c$ *on* $\det\mathbb{S}_c$.

(iii) *If we denote by* $\mathfrak{D}_c$ *the Dirac operator on* $\mathbb{S}_c$ *induced by the Levi-Civita connection on* TM *and* $\det\nabla^c$ *on* $\det\mathbb{S}^c$ *then*

$$\mathcal{H} = \mathfrak{D}_c + \frac{1}{4}\mathbf{c}(\eta \wedge d\eta).$$

(iv) *Using the identity* $F(\det\nabla^c) = F(\det\nabla^{\mathrm{TW}}) + \frac{n\mathbf{i}}{2}d\eta$, *we deduce that* $\mathcal{H}$ *satisfies a Weitzenböck formula*

$$\mathcal{H}^2 = (\nabla^{\mathfrak{w}})^*(\nabla^{\mathfrak{w}}) + \frac{s(g)}{4} + \frac{1}{16}\big(4\mathbf{c}(d\eta \wedge d\eta) - 2n\big) + \frac{1}{2}\mathbf{c}\big(F(\det\nabla^{\mathrm{TW}})\big) + \frac{n\mathbf{i}}{4}\mathbf{c}(\omega),$$

where $\nabla^{\mathfrak{w}}$ *is the Weitzeböck connection defined in* (1.7). *In particular, if* $\dim M = 3$ (*so that* $n = 1$ *and* $\mathbf{c}(\eta \wedge d\eta) = -1$) *we have*

$$\mathfrak{D}_c = \mathcal{H} + \frac{1}{4}, \quad \mathcal{H}^2 = (\nabla^{\mathfrak{w}})^*(\nabla^{\mathfrak{w}}) + \frac{s}{4} - \frac{1}{8} + \frac{1}{2}\mathbf{c}\big(F(\det\nabla^{\mathrm{TW}})\big) + \frac{\mathbf{i}}{4}\mathbf{c}(d\eta).$$

3.4. Connections induced by symplectizations

The *symplectization* of the positively oriented metric contact manifold (M^{2n+1}, η, g, J) is the manifold $\tilde{M} = \mathbb{R}_+ \times M$ equipped with the symplectic form

$$\tilde{\omega} = dt \wedge \eta + t\,d\eta = dt \wedge \eta + t\omega.$$

If we denote by $\tilde{d}$ the exterior derivative on $\tilde{M}$ then we can write $\tilde{\omega} = \tilde{d}(t\eta)$. $\tilde{M}$ is equipped with a compatible metric $\tilde{g} = dt^2 + \eta^2 + t\omega(\bullet, J\bullet)$.

We denote by $\tilde{J}$ the associated almost complex structure. We will identify M with the slice $\{1\} \times M$ of $\tilde{M}$. If we fix as before a local, oriented, orthonormal frame $\xi, e_1, f_1, \dots, e_n, f_n$ compatible with the metric contact structure on M then we get a symplectic frame

$$\tilde{e}_0 = \partial_t, \quad \tilde{f}_0 = \xi, \quad \tilde{e}_k = t^{-1/2}e_k, \quad \tilde{f}_k = t^{-1/2}f_k, \quad k = 1, \dots, n.$$

The dual coframe is

$$\tilde{e}^0 = dt, \quad \tilde{f}^0 = \eta, \quad \tilde{e}^k = t^{1/2}e^k, \quad \tilde{f}^k = t^{1/2}f^k, \quad k = 1, \dots, n.$$

We denote by $\tilde{N}$ the Nijenhuis tensor of $\tilde{J}$ and by $\hat{N}$ the Nijenhuis tensor of the almost complex manifold $(\hat{M}, \hat{J})$ used in Section 3.1. The Chern connection $\tilde{\nabla}^c$ of $(\tilde{M}, \tilde{g}, \tilde{J})$ is the metric connection with torsion $\tilde{T} = \tilde{N}$. In this case $\theta = 0$, $\mathfrak{b}\tilde{T} = 0$. Observe that $\tilde{J} = \hat{J}$. We deduce that for $j, k = 1, \dots, n$, we have

$$\tilde{N}(\tilde{e}_j, \tilde{e}_k) = \frac{1}{t}\hat{N}(e_j, e_k), \quad \tilde{N}(\tilde{e}_j, \tilde{f}_k) = \frac{1}{t}\hat{N}(e_j, f_k), \quad \tilde{N}(\tilde{f}_j, \tilde{f}_k) = \frac{1}{t}\hat{N}(f_j, f_k),$$

$$\tilde{N}(\partial_t, \tilde{e}_j) = \frac{1}{\sqrt{t}}\hat{N}(\partial_t, e_j), \quad \tilde{N}(\partial_t, \tilde{f}_k) = \frac{1}{\sqrt{t}}\hat{N}(\partial_t, f_k),$$

$$\tilde{N}(\xi, \tilde{e}_j) = \frac{1}{\sqrt{t}}\hat{N}(\partial_t, e_j), \quad \tilde{N}(\xi_t, \tilde{f}_k) = \frac{1}{\sqrt{t}}\hat{N}(\partial_t, f_k).$$

Denote by $\tilde{D}$ the Levi-Civita connection determined by $\tilde{g}$. It determined by (see [6,8])

$$\begin{aligned} 2\tilde{g}(\tilde{D}_X Y, Z) &= X\tilde{g}(Y, Z) + Y\tilde{g}(X, Z) - Z\tilde{g}(X, Y) \\ &\quad + \tilde{g}\big([X, Y], Z\big) + \tilde{g}\big([Z, X], Y\big) + \tilde{g}\big(X, [Z, Y]\big). \end{aligned}$$

We deduce from the above identity that if X, Y are t-independent vectors tangent along M

$$2\tilde{g}(\tilde{D}_t X, Y) = g(X_V, Y_V) = \omega(X, JY),$$

where $X_V := P_V X$. Hence

$$2\tilde{g}(\tilde{D}_X Y, \partial_t) = -\partial_t \tilde{g}(X, Y) = -g(X_V, Y_V) = \omega(JX, Y).$$

As in Section 3.1 we want to alter $\tilde{\nabla}^c$ by $B \in \Omega_s^{1,1}(T^*\tilde{M})$ such that $\operatorname{tr} B = 0$ so that the new basic hermitian connection $\tilde{\nabla}^b$ with torsion $\tilde{T}_b^\dagger := \tilde{N}^\dagger + B$ satisfies

$$\tilde{\nabla}_X^b \xi = 0, \quad \tilde{g}(\tilde{\nabla}_X^b Y, \partial_t) = 0, \tag{3.10}$$

for all t-independent tangent vectors X, Y along M. We have $\tilde{\nabla} = \tilde{D} + A$, where $A^\dagger = -\tilde{T}_b^\dagger$. Thus we need

$$\begin{aligned} 0 = \tilde{g}(\tilde{\nabla}_X Y, \partial_t) &= \tilde{g}\big(\tilde{D}_X(Y, \partial_t)\big) - \tilde{g}\big(X, \tilde{N}(Y, \partial_t)\big) - B(X; Y, \partial_t) \\ &= -\frac{1}{2}\omega(JX, Y) + \tilde{g}\big(X, \tilde{N}(\partial_t, Y)\big) - B(X; Y, \partial_t). \end{aligned}$$

If $Y = \xi$ we deduce

$$B(X; \xi, \partial_t) = 0.$$

If $Y \in C^\infty(V)$ then we deduce

$$\begin{aligned} 0 &= -\frac{1}{2}\omega(JX, Y) + \frac{1}{\sqrt{t}}\tilde{g}\big(X, \hat{N}(\partial_t, Y)\big) + B(X; \partial_t, Y) \\ &= \frac{1}{2}g(X, Y) + \frac{1}{4\sqrt{t}}\tilde{g}(X, \Phi Y) + B(X; \partial_t, Y) \\ &= \frac{1}{2}g(X, Y) + \frac{\sqrt{t}}{4}g(X, \Phi Y) + B(X; \partial_t, Y). \end{aligned}$$

We conclude that B must satisfy the additional conditions

$$B(\xi; \partial_t, Y) = 0, \quad Y \in C^\infty(V),$$

$$B(X; \partial_t, Y) = -\frac{1}{2\sqrt{t}}\left(\frac{1}{\sqrt{t}}\tilde{g}(X, Y) + \frac{1}{2}\tilde{g}(X, \Phi Y)\right).$$

We write $B = B_0 + B_1$ where B_0 is defined as in Lemma 3.2 by the equality

$$B_0 = \frac{1}{4\sqrt{t}}\{\Phi \wedge dt + J\Phi \wedge \eta\}.$$

B_1 must satisfy the equalities $\operatorname{tr} B_1 = 0$,

$$B_1(X; \partial_t, Y) = -\frac{1}{2t}\tilde{g}(X, Y), \quad \forall X, Y \in C^\infty(V), \tag{3.11a}$$

$$B_1(X; \xi, \partial_t) = B_1(\xi; \partial_t, Y) = 0, \quad \forall X \in \operatorname{Vect}(M),\ Y \in C^\infty(V). \tag{3.11b}$$

We try B_1 of the form

$$B_1 = x\,dt \otimes dt \wedge \eta + y\eta \otimes d\eta + U + V$$

where

$$U = \frac{1}{2t} P_V \wedge dt, \quad V = \frac{1}{2t} J P_v \wedge \eta.$$

Clearly $B_1 \in \Omega^{1,1}(T^*\tilde{M})$. Next observe that

$$\mathfrak{b}B_1 = y\eta \wedge d\eta + \mathfrak{b}V = \left(y + \frac{1}{t}\right)\eta \wedge d\eta, \quad \operatorname{tr} B_1 = \left(x + \frac{n}{t}\right)dt.$$

Thus, set $x = -\frac{n}{t}$, $y = \frac{1}{t}$. These choices guarantee that $B_1 \in \Omega_s^{1,1}(T^*\tilde{M})$ and $\operatorname{tr} B_1 = 0$. The conditions (3.11a) and (3.11b) can now be verified by direct computation. We can now conclude that if

$$B = \frac{1}{4\sqrt{t}}(\Phi \wedge dt + J\Phi \wedge \eta) - \frac{n}{t} dt \otimes dt \wedge \eta - \frac{1}{t}\eta \otimes d\eta + \frac{1}{2t}(P_V \wedge dt + J P_V \wedge \eta)$$

then the connection $\tilde{\nabla}^b$ with torsion $\tilde{N}^\dagger + B$ satisfies the conditions (3.10). These conditions show that $\tilde{\nabla}^b$ induces by restriction to the slice $\{t\} \times M$ a connection ∇^t on TM. The torsion of $\nabla^1 = \nabla^{t=1}$ is given by

$$\begin{aligned}
(T_1)^\dagger &= \tilde{N}^\dagger|_{t=1} + B|_{t=1} = \hat{N}^\dagger|_M + \frac{1}{4}(J\Phi \wedge \eta) - \eta \otimes d\eta + \frac{1}{2}(J P_V \wedge \eta) \\
&\overset{(3.3)}{=} \frac{1}{2}N^\dagger - \frac{3}{4}\eta \otimes d\eta + \frac{1}{2}(J P_V \wedge \eta) + \frac{1}{4}(J\Phi \wedge \eta).
\end{aligned}$$

This connection *never* coincides with the generalized Tanaka–Webster connection constructed in Section 3.1, because in this case we have $\mathfrak{b}T_1^\dagger = 0$. This shows ∇^1 is Dirac equivalent to the Levi-Civita connection. We have thus proved the following result.

Theorem 3.9. *On every metric contact manifold (M, g, J) there exists a canonical balanced contact metric connection ∇^1 induced by a basic hermitian connection on the symplectization of M. This contact connection is Dirac equivalent to the Levi-Civita connection and its torsion is given by*

$$T_1^\dagger = \frac{1}{2}N^\dagger - \frac{3}{4}\eta \otimes d\eta + \frac{1}{2}(J P_V \wedge \eta) + \frac{1}{4}(J\Phi \wedge \eta).$$

When M is a CR manifold we deduce from (3.2)

$$T_1^\dagger = -\eta \otimes d\eta + \frac{1}{2}J \wedge \eta + \frac{1}{2}J\Phi \wedge \eta.$$

In particular

$$T_1(X,Y) = -d\eta(X,Y)\xi, \quad \forall X,Y \in C^\infty(V).$$

Let us observe that in this case for every $X,Y \in C^\infty(V)$ we have

$$g(\nabla^1_\xi X,Y) = g(D_\xi X,Y) - g\big(\xi, T_1(X,Y)\big) = g(D_\xi X,Y) + \omega(X,Y)$$

so that

$$\nabla^1_\xi = D^V_\xi + J = P_V D_\xi + J = \nabla^{\mathrm{TW}}_\xi + \frac{3}{2}J.$$

Remark 3.10. Let us point out a difference between contact and hermitian connections. We have shown that there always exist contact connections with torsion T satisfying $\mathfrak{b}T^\dagger = 0$.

On the other hand, if ∇ is a hermitian connection on an almost complex hermitian manifold (M,g,J) with Nijenhuis tensor N then its torsion satisfies (see [4])

$$(\mathfrak{b}T)^- = (\mathfrak{b}N^\dagger) = (d^c\omega)^-.$$

If $\dim M = 4$ then always $(d^c\omega)^- = 0$ and in this case it is possible to find hermitian connections Dirac equivalent to the Levi-Civita connection. However, in higher dimensions this is possible if and only if $(d^c\omega)^- = 0$.

3.5. A uniqueness result

The constructions we performed in the previous subsection may seem a bit ad-hoc but as we will show in this section they produce, at least for CR manifolds, connections uniquely determined by a few natural requirements.

Proposition 3.11. *Suppose (M,η,g,J) is a CR manifold. Then each Dirac equivalence class of connections contains at most one balanced CR connection.*

Proof. Suppose ∇ is a balanced CR connection with torsion T. Set $\Omega := \mathfrak{b}T$. We get a hermitian connection $\hat\nabla = dt \wedge \partial_t + \nabla$ on $(T\hat M, \hat J)$ with the property $\mathfrak{b}T(\hat\nabla)^\dagger = \Omega$, $\operatorname{tr} T(\hat\nabla)^\dagger = 0$. Denote by ∇^b the basic hermitian connection on $(T\hat M, \hat J)$ we have constructed in Section 3.1. The results in Section 2.1 imply that

$$T(\hat\nabla)^\dagger = T_b^\dagger + \frac{9}{8}\psi^+ - \frac{3}{8}\mathfrak{M}\psi^+ + B =: T_b^\dagger + S,$$

where

$$\begin{aligned} &\psi^+ \in \Omega^{3,+}(\hat M), \quad B \in \Omega^{1,1}_s(T^*\hat M), \quad \Omega = \mathfrak{b}T_b^\dagger + 3\psi^+ = 3\psi^+ + \eta\wedge d\eta,\\ &B(\partial_t;\bullet,\bullet) = 0 = B(\bullet;\bullet,\partial_t) = 0, \quad \operatorname{tr} B = 0. \end{aligned} \tag{$*$}$$

Thus ψ^+ is uniquely determined. Moreover, since ∇ is a CR connection we deduce that

$$g\big(X, T(Y,Z)\big) = 0, \quad \forall X,Y,Z \in C^\infty(V).$$

Since the restriction of ∇^b to M is also a CR connection we deduce

$$S(X; Y, Z) = 0, \quad \forall X, Y, Z \in C^\infty(V).$$

Thus the restriction of B to V is uniquely determined. The condition $B \in \Omega_s^{1,1}(T^*\hat{M})$ coupled with $(*)$ show that the restriction of B to $\mathbb{R}\partial_t \oplus \mathbb{R}\xi \subset T\hat{M}$ is also uniquely determined. This concludes the proof of Proposition 3.11. $\square$

Corollary 3.12. *The Tanaka–Webster connection on a CR manifold is the unique balanced CR connection adapted to $\mathcal{H}$. Moreover, the connection ∇^1 of Section 3.4 is the unique balanced CR connection with torsion satisfying $\mathfrak{b}T^\dagger = 0$, i.e., Dirac equivalent with the Levi-Civita connection.*

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