# THE EULER CHARACTERISTIC

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Abstract. I will describe a few basic properties of the Euler characteristic and then I use them to prove special case of a cute formula due to Bernstein-Khovanskii-Koushnirenko.

## 1. Basic properties of the Euler characteristic

The Euler characteristic is a function $\chi$ which associates to each *reasonable*[1] topological space $X$ an integer $\chi(X)$. For us a reasonable space would be a space which admits a finite simplicial decomposition (a.k.a. triangulation.) For example, all algebraic varieties are reasonable. In the sequel we will tacitly assume that all spaces are reasonable and so we will drop this attribute from our discourse.

More explicitly, the Euler characteristic of $X$ is defined as the alternating sum

$$\chi(X) = \sum_{k\geq 0}(-1)^k \dim_{\mathbb{R}} H_c^k(X,\mathbb{R}),$$

where $H_c^\bullet(X,\mathbb{R})$ denotes the cohomology with compact supports and real coefficients of the space $X$.

The Euler characteristic is uniquely determined by the following properties.

- **Normalization.**

$$\chi(\{\text{point}\}) = 1.$$

- **Topological invariance.**

$$\chi(X) = \chi(Y) \;\text{ if } X \text{ is homeomorphic to } Y.$$

- **Proper homotopy invariance**

$$\chi(X) = \chi(Y) \;\text{ for any homotopic compact spaces } X \text{ and } Y.$$

- **Excision.**

$$\chi(X) = \chi(C) + \chi(X\setminus C), \;\text{ for every closed subset } C\subset X.$$

The excision property has a dual form

$$\chi(X) = \chi(U) + \chi(X\setminus U) \;\text{ for every open subset } U\subset X.$$

- **Multiplicativity.**

$$\chi(X\times Y) = \chi(X)\chi(Y).$$

The excision property is frequently used under the guise of the *inclusion-exclusion formula*. More precisely, if $X$ is a union of two closed sets $X = S_1\cup S_2$ then

$$\chi(X) = \chi(S_1) + \chi(S_2) - \chi(S_1\cap S_2).$$

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[1]For example, every space defined by polynomial equalities and inequalities is reasonable. The technical term would be *subanalytic*.

Indeed

$$\chi(X) = \chi(S_1) + \chi(X \setminus S_1) = \chi(S_1) + \chi(S_2 \setminus S_1 \cap S_2) = \chi(S_1) + \chi(S_2) - \chi(S_1 \cap S_2).$$

We have a similar formula with *closed* replaced by *locally closed*[2].

Let us compute the Euler characteristic of a few reasonable spaces. Note first that the Euler characteristic of a finite set (equipped with the discrete topology) is equal to the cardinality of that set.

Denote by $\Delta_n$ the closed $n$-dimensional simplex. Thus $\Delta_0$ is a point, $\Delta_1$ is a segment, $\Delta_2$ is a triangle, $\Delta_3$ is a tetrahedron etc. Every simplex $\Delta_n$ is homotopic to a point and thus

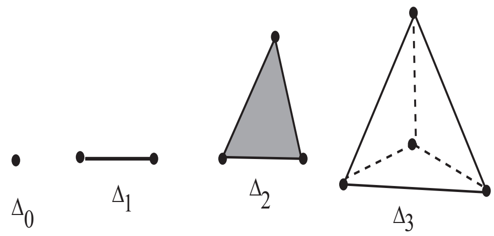


FIGURE 1. *Simplices.*

$$\chi(\Delta_n) = 1, \ \ \forall n \geq 0.$$

Observe that $\partial\Delta_n$ is homeomorphic to the $(n-1)$-sphere $S^{n-1}$. Since $S^0$ is a union of two points we deduce $\chi(S^0) = 2$. In general, the $n$-dimensional sphere is a union of two closed hemispheres intersecting along the Equator which is a $(n-1)$ sphere. Hence,

$$\chi(S^n) = 2\chi(\Delta_n) - \chi(S^{n-1}) = 2 - \chi(S^{n-1}).$$

We deduce inductively

$$2 = \chi(S^n) + \chi(S^{n-1}) = \chi(S^{n-1}) + \chi(S^{n-2}) = \cdots = \chi(S^1) + \chi(S^0)$$

so that

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & \text{if} \quad n \in 2\mathbb{Z} \\ 0 & \text{if} \quad n \in 2\mathbb{Z}+1 \end{cases}.$$

Now observe that the interior of $\Delta_n$ is homeomorphic to $\mathbb{R}^n$ so that

$$\chi(\mathbb{R}^n) = \chi(\Delta_n) - \chi(\partial\Delta_n) = 1 - \chi(S^{n-1}) = 1 - (1 + (-1)^{n-1}) = (-1)^n.$$

The excision property implies the following useful formula. Suppose

$$\emptyset \subset X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(N)} = X$$

is an increasing filtration of $X$ by closed subsets. Then

$$\chi(X) = \chi(X^{(0)}) + \chi(X^{(1)} \setminus X^{(0)}) + \cdots + \chi(X^{(N)} \setminus X^{(N-1)}).$$

Let's apply this in the case when $X$ is a simplicial complex. We denote by $X^{(k)}$ the union of the simplices of dimension $\leq k$. Then $X^{(k)} \setminus X^{(k-1)}$ is the union of the interiors of the $k$-dimensional simplices. We denote by $f_k(X)$ the number of such simplices. Each of them is homeomorphic

[2]A subset of a topological space is called *locally closed* if is the intersection of an open subset with a closed subset

to $\mathbb{R}^k$ and thus its Euler characteristic is equal to $(-1)^k$. Hence $\chi(X^{(k)} \setminus X^{(k-1)}) = (-1)^k f_k(X)$ so that

$$\chi(X) = \sum_{k \geq 0} (-1)^k f_k(X).$$

Suppose $\pi : X \to Y$ is a $d : 1$ covering map. Then we can find an increasing filtration of $Y$ by closed subsets

$$Y^{(0)} \subset \cdots \subset Y^{(k)} \subset Y^{(k+1)} \subset \cdots \subset Y$$

such that over $Y^{(k)} \setminus Y^{(k-1)}$ the projection $\pi$ is a trivial covering map. If we set

$$X^{(k)} = \pi^{-1}(Y^{(k)})$$

then $X^{(k)} \setminus X^{(k-1)}$ is homeomorphic to a product $F \times (Y^{(k)} \setminus Y^{(k-1)})$, where $F$ is a finite set of cardinality $d$. Hence $\chi(X^{(k)} \setminus X^{(k-1)}) = d\chi(Y^{(k)} \setminus Y^{(k-1)})$ and we deduce

$$\chi(X) = d\chi(Y).$$

More generally, suppose $f : X \to Y$ is a reasonable[3], proper, continuous map with finite fibers. Denote by $Y^{(k)}$ the subset of $Y$ consisting of points $y$ such that the fiber $f$ has cardinality $k$. We set

$$X^{(k)} := f^{-1}\big(Y^{(k)}\big).$$

The map $f : X^{(k)} \to Y^{(k)}$ is a $k : 1$ cover and we deduce

$$\chi(X^{(k)}) = k \cdot \chi(Y^{(k)}).$$

Summing over $k$ we deduce the *slicing formula*

$$\chi(X) = \sum_{k \geq 0} k\chi(Y^{(k)}) = \sum_{k \geq 0} k \cdot \chi\big(\{y \in Y;\ \ |f^{-1}(y)| = k\}\big). \tag{1.1}$$

## 2. A Bézout type formula

To put things into perspective we start with a classic elementary fact. Suppose we are given a polynomial with complex coefficients in one complex variable

$$A(z) = a_k z^k + \cdots + a_n z^n,\ \ k < n,\ \ a_k \cdot a_n \neq 0.$$

Then for generic choices of coefficients the number of nonzero roots of $P$ is equal to $n - k$. We want to rephrase this in a more sophisticated way.

For a polynomial $P$ in $\nu$ complex variables $\vec{z} = (z_1, \cdots, z_\nu)$

$$\mathcal{Z}_P^* := \{\vec{z} \in (\mathbb{C}^*)^\nu;\ \ P(\vec{z}) = 0\}.$$

In our special case the number of nonzero roots of $A$ is the Euler characteristic of $\mathcal{Z}_A^*$.

To every nonzero monomial $a_\alpha \vec{z}^\alpha = a_\alpha z_1^{\alpha_1} \cdots z_n \nu^{\alpha_\nu}$ of $P$ we associate the point

$$\alpha = (\alpha_1, \cdots, \alpha_\nu) \in \mathbb{Z}^\nu \subset \mathbb{R}^\nu.$$

We obtain in this fashion a finite set of lattice points denoted by $\operatorname{supp} P \subset \mathbb{R}^\nu$. The *Newton polytope* of $P$ is by definition the convex hull of $\operatorname{supp} P$. We denote it by $\Delta(P)$. For example,

$$\operatorname{supp} A(z) = \{k, k+1, \cdots, n\} \subset \mathbb{R}^1,$$

and $\Delta(A)$ is the line segment connecting the points $k, n \in \mathbb{R}$. The fact that a generic polynomial $A$ with Newton polytope $[k, n]$ has $(n - k)$ nonzero roots can be rewritten in the following sophisticated fashion

$$\chi(\mathcal{Z}_A^*) = \operatorname{vol}_1\big(\Delta(A)\big).$$

[3]A reasonable map would be for example a real analytic map.

Above, $\mathrm{vol}_1$ denotes the 1-dimensional volume of the segment $[k,n]$, i.e. its length. The above equality is a special case of the following general result.

**Theorem 2.1** (Bernstein-Khovanskii-Koushnirenko)**.** *Fix a convex polytope in $\mathbb{R}^\nu$ with vertices in $(\mathbb{Z}_{\geq 0})^\nu$ then for a generic polynomial $P$ such that $\Delta(P)=\Delta$ we have*

$$\chi(\mathcal{Z}_P^*) = (-1)^{\nu-1}\nu!\cdot \mathrm{vol}_\nu(\Delta),$$

*where* $\mathrm{vol}_\nu$ *denotes the $\nu$-dimensional Euclidean volume in $\mathbb{R}^\nu$.* □

Let us prove this theorem in a special case when $\nu=2$ and

$$P = ax^4 + bx^3y^4 + cy^5 + d.$$

The Newton polytope of $P$ is depicted in Figure 2.

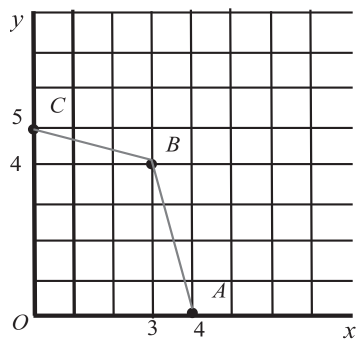


FIGURE 2. *The Newton polygon of $ax^4+bx^3y^4+cy^5+d$.*

To compute the area of this polygon we decompose it into two triangles $OAB$ and $OBC$ and we have

$$2\mathrm{Area}\,(OABC) = 2\mathrm{Area}\,(OAB) + 2\mathrm{Area}\,(OBC) = \begin{vmatrix} 4 & 0 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 4 & 5 \end{vmatrix} = 16+15 = 31.$$

Consider the curve

$$\mathcal{Z}_P = \{(x,y)\in\mathbb{C}^2;\;\; P(x,y)=0,\;\; y\neq 0\}.$$

and set $\mathcal{Z}_P^0 = \{(0,y)\in\mathcal{Z}_P\} = \{(0,y);\;\; cy^5+d=0\}$. For generic $(c,d)$ we have $\chi(\mathcal{Z}_P)=5$ so that

$$\chi(\mathcal{Z}_P^*) = \chi(\mathcal{Z}_P) - \chi(\mathcal{Z}_P^0) = \chi(\mathcal{Z}_P) - 5.$$

To compute the Euler characteristic of $\mathcal{Z}_P$ we use the slicing formula (1.1) applied to the map

$$f:\mathcal{Z}_P\to\mathbb{C}^*,\;\; \mathcal{Z}_P\ni(x,y)\mapsto y.$$

Note that for every $y_0\in\mathbb{C}$ the fiber $f^{-1}(y_0)$ is the intersection of the horizontal line $y=y_0$ with the curve $\mathcal{Z}_P$. The points on this intersection are found by solving the polynomial equation in $x$

$$ax^4 + bx^3y_0^4 + cy_0^5 + d = 0.$$

For all but finitely many $y_0$'s this equation has exactly 4 distinct solutions. The intersection contains less than 4 points precisely when the horizontal line $y=y_0$ is tangent to the curve $\mathcal{Z}_P$, i.e. $y_0$ is a critical value of the map $\pi$ (see Figure 3). Denote by $S^*$ the set of $y_0\in\mathbb{C}^*$ such that the line $y=y_0$ intersects the curve $\mathcal{Z}_P$ transversally. The *discriminant set* is $D=\mathbb{C}^*\setminus S^*$.

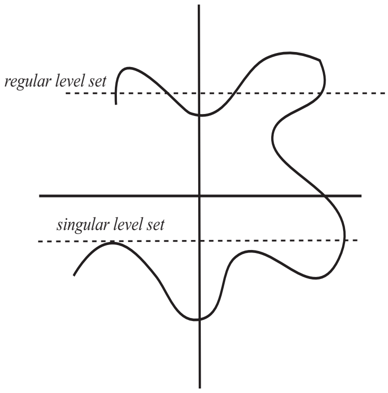


FIGURE 3. *Slicing the curve* $\mathcal{Z}_P^*$.

For generic coefficients $a, b, c, d$ the curve $\mathcal{Z}_P$ can be represented near a horizontal tangency point as the graph of the implicit function $y = y(x)$. The horizontal tangency condition can be rewritten as

$$\frac{dy}{dx} = 0.$$

Derivating the equation $P = 0$ with respect to $x$ we deduce

$$y^3(4bx^3 + 5cy)\frac{dy}{dx} = -x^2(4ax + 3by^4), \;\; xy \neq 0.$$

The points with horizontal tangents on $\mathcal{Z}_P$ are obtained by solving the system

$$ax^4 + bx^3y^4 + cy^5 + d = 0, \;\; x^2(4ax + 3by^4) = 0, \;\; xy \neq 0. \tag{2.1}$$

We distinguish two cases, $x = 0$ and $x \neq 0$.

• If $x = 0$ then $cy^5 + d = 0$. For general $c, d$ this equation has five distinct roots. If $\hat{y}$ is one of them then the intersection of the line $y = \hat{y}$ with the curve $\mathcal{Z}_P$ consists of two points $(x, \hat{y})$ found by solving for $x$ the equation

$$x^3(ax + b\hat{y}^4) = 0.$$

• If $x \neq 0$ then (2.1) implies $x = -\frac{3by^4}{4a}$ and using this in the equation $P(x, y) = 0$ we obtain a degree 16 equation in $y$

$$R(y) = P(-\frac{3by^4}{4a}, y) = 0.$$

If the coefficients $(a, b, c, d)$ are generic then the equation $R(y) = 6$ has 16 different roots and for every solution $\hat{y}$ of $R(y) = 0$ the polynomial $x \mapsto P(x, \hat{y})$ has a double root at

$$\hat{x} = -\frac{3b\hat{y}^4}{4a}$$

Thus for every root $\hat{y}$ of $R(y) = 0$ the equation $P(x, \hat{y}) = 0$ has exactly three solutions.

We have thus decomposed the discriminant set $D$ into two parts

$$D = D_1 \cup D_2 = \{R(y) = 0\} \cup \{cy^5 + d = 0\}.$$

These two parts are disjoint for general coefficients. Note that

$$\chi(D_1) = 16, \;\; \chi(D_2) = 5$$

$$\chi(f^{-1}(y)) = 3, \;\; \forall y \in D_1, \;\; \chi(f^{-1}(y)) = 2, \;\; \forall y \in D_2.$$

Since $\chi(f^{-1}(y)) = 4$ for $y \in S^*$ we deduce from the slicing formula (1.1) that

$$\begin{aligned}\chi(\mathcal{Z}_P) &= 4\chi(S^*) + 3\chi(D_1) + 2\chi(D_2)\\ &= 4\chi(\mathbb{C}^* \setminus D) + 3\cdot 16 + 2\cdot 5\\ &= 4\cdot(0 - 16 - 5) + 48 + 10 = -26.\end{aligned}$$

Hence

$$\chi(\mathcal{Z}_P^*) = \chi(\mathcal{Z}_P) - 5 = -26 - 5 = -31 = -2\mathrm{Area}\,(OABC)$$

as predicted by Theorem 2.1.

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