TWO PROOFS OF THE DE RHAM THEOREM

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Contents

Introduction

The main goal of this paper is to state and prove the De Rham Theorem in two different ways. We will work exclusively in the realm of smooth manifolds, and we will discuss various different ways of associating cohomology groups to a smooth manifold. Of primary concern for us will be the language of differential forms. At this point, we wish to give the reader some geometric intuition for the De Rham Theorem, since the treatment in the paper will be largely technical.

In the discussion that follows, we will consider the question of when a planar vector field on $\mathbb{R}^2$ is a gradient vector field. That is, we wish to decide when, for a vector field $\vec{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$ defined on a region $D \subset \mathbb{R}^2$, there exists a function $f(x, y)$ on D such that

$$\nabla f := \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} = P\boldsymbol{i} + Q\boldsymbol{j}$$

If this happens, we will say that $\vec{F}$ is *exact*. We see that by the equality of mixed partial derivatives that if $\vec{F} = P\boldsymbol{i} + Q\boldsymbol{j}$ is exact, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In general, we say that if this condition is satisfied, then the vector field is *closed*. We wish to know if a closed vector field is exact. It turns out that if we have a closed vector field $\vec{F}$ on a convex subset $D \subset \mathbb{R}^2$, then the vector field is exact on D. Also, in this case, we have that

$$\int_C \vec{F} \cdot ddr = \int_C Pdx + Qdy = 0, \quad \vec{r} = x\boldsymbol{i} + y\boldsymbol{j},$$

for any closed path C in D. Now, consider the vector field $\vec{F}(x, y)$ given by

$$\vec{F}(x, y) = \frac{-y}{x^2 + y^2}\boldsymbol{i} + \frac{x}{x^2 + y^2}\boldsymbol{j}$$

With this definition, it is easily checked that, on $\mathbb{R}^2 - (0, 0)$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

so that $\vec{F}$ is closed. However, it can be shown that $\vec{F}$ is not exact. Furthermore, if we let S^1 denote the unit circle, then we have

$$\int_{S^1} \vec{F} d\vec{r} = 2\pi$$

The difference between these two cases is that in the first case we had a convex subset, and in the other case we have the set $\mathbb{R}^2 - (0, 0)$. We see that this particular vector field is not a gradient because it has strange behavior near zero. We'd like to say the problem is that $(0, 0)$ is missing from the space, or that it is in some sense a "hole" in our space. Of course, there are a priori many different ways to define holes as subsets of $\mathbb{R}^2$.

One way is to look at integrals over closed paths of closed vector fields. If we happen to find a closed vector field with integral not equal to 0 over a closed path, then we can say that this path surrounds a hole in our space. Similarly, we could say that if we have a closed path which we cannot homotopically shrink to a point, then we say this path surrounds a hole. There are of course other possible definitions.

Intuitively, in more abstract settings, the various different ways of defining holes in a space are described by the various definitions of the cohomology of a manifold, and the De Rham theorem says that all of these methods of measuring holes are the same. In effect, the De Rham theorem tells us that it does not matter how we compute how many holes there are in our manifold, we will always get the same answer.

To prove this striking theorem, we will first discuss in great detail the language of differential forms, stating many important theorems, defining the De Rham cohomology, and proving the Poincaré Lemma.

In section 2 we will discuss the notion of simplicial complexes, and will develop the theory of simplicial homology and cohomology, defining many key concepts that will be used in the proof of the De Rham Theorem.

In section 3, we will define the notion of a smoothly triangulated manifold, and we will also define a period map which will go from differential forms on M to simplicial cochains on the triangulation of M. Finally, we will prove that this map is an isomorphism in cohomology, which is the first proof of the De Rham Theorem.

In section 4, we will develop the notion of singular homology and cohomology, and we will discuss the generalized Mayer-Vietoris exact sequence, proving its exactness.

Finally, in section 5, we will build up the notion of double complexes, which we will then use to define another period map which goes from differential forms on M to singular cochains on M. We will then prove that this map induces isomorphisms in cohomology, which will be the second proof of the De Rham theorem.

We offer two proofs of the De Rham theorem in this paper because the two proofs represent two widely different views of the subject. The first proof is given in a very classical setting, and represents the classical point of view, whereas the second proof uses very modern machinery and represents a more modern point of view. Both of these points of view have merit, and so we demonstrate them both.

At this point, I would like to thank all the people who made this thesis possible. I would like to thank all the faculty of the University for their help in building up my mathematical career and teaching me all that I know about math. Most of all, I would like to thank my advisor, Professor Liviu Nicolaescu, without whom none of this would have been possible. He has been there since the beginning of this project, helping me put together the paper you see here.

1. The Basics

1.1. **The calculus of differential forms.** We want survey without proofs the basic facts concerning the calculus of differential forms on a smooth manifold. For details we refer to [4].

For any smooth manifold M we denote by $\mathrm{Vect}(M)$ the vector space of smooth vector fields on M, and by $\Omega^k(M)$ the vector space of differential forms of degree k, i.e., maps

$$\omega : \underbrace{\mathrm{Vect}(M) \times \cdots \times \mathrm{Vect}(M)}_{k} \to C^\infty(M),$$

such that, for every $X_1, \dots, X_k \in \mathrm{Vect}(M)$, $f_1, \dots, f_k \in C^\infty(M)$ and any permutation σ of $\{1, \dots, k\}$ we have

$$\omega\big(X_{\sigma(1)}, \dots, X_{\sigma(k)} \big) = \epsilon(\sigma)\omega(X_1, \dots, X_k),$$
$$\omega(f_1 X_1, \dots, f_k X_k) = (f_1 \cdots f_k)\omega(X_1, \dots, X_k),$$

where $\epsilon(\sigma) \in \{\pm 1\}$ denotes the signature of the permutation σ. By definition, $\Omega^0(M)$ is the space $C^\infty(M)$ of smooth real valued functions on M.

We form the graded vector space

$$\Omega^\bullet(M) := \bigoplus_{k=0}^{\dim M} \Omega^k(M).$$

We say that $\omega \in \Omega^\bullet(M)$ is *homogeneous* if it belongs to one of the summands $\Omega^k(M)$. For a homogeneous element ω we denote by $|\omega|$ its degree. A linear map

$$L : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

is called *homogeneous of degree* q if it maps any homogeneous form ω to a homogeneous form $L\omega$ and

$$|L\omega| = |\omega| + q.$$

The space $\Omega^\bullet(M)$ is an associative $\mathbb{R}$-algebra with respect to the *wedge* or *exterior product*

$$\wedge : \Omega^\bullet(M) \times \Omega^\bullet(M) \to \Omega^\bullet(M).$$

The product of any two homogeneous forms ω, η is a homogeneous form $\omega \wedge \eta$ and

$$|\omega \wedge \eta| = |\omega| + |\eta|, \ \ \omega \wedge \eta = (-1)^{|\omega|\cdot|\eta|}\eta \wedge \omega.$$

A differential form of degree k can also be interpreted as a smooth section of the vector bundle $\Lambda^k T^*M$, the k-th exterior power of the cotangent bundle T^*M,

$$\omega : M \to \Lambda^k T^*M, \ \ M \ni x \mapsto \omega_x \in \Lambda^k T_x^*M.$$

The *support* of ω is defined as the closed set

$$\operatorname{supp}\omega := \text{closure}\Big\{ x \in M; \ \ \omega_x \neq 0 \in \Lambda^k T_x^*M \Big\}.$$

We denote by $\Omega_c^k(M)$ the space of smooth differential forms of degree k with compact support.

The *exterior derivative on* M is the homogeneous linear operator

$$d : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

of degree 1 *uniquely* determined by the following conditions.

For every pair of homogeneous forms ω, η we have

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge (d\eta). \tag{$\boldsymbol{P}_-$}$$

$$d^2 = 0. \tag{1.1}$$

$$\forall f \in C^\infty(M) = \Omega^0(M), \ \ df \text{ is the differential of } f.$$

Any vector field $X \in \operatorname{Vect}(M)$ determines a homogeneous linear operator of degree -1

$$i_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

called the *contraction with* X and defined by

$$i_X\omega(X_2, \dots, X_k) = \omega(X, X_2, \dots, X_k), \ \ \forall \omega \in \Omega^k(M), \ \ X_2, \dots, X_k \in \operatorname{Vect}(M).$$

The operator i_X satisfies the odd product rule ($\boldsymbol{P}_-$).

The *Lie derivative* along a vector field $X \in \operatorname{Vect}(M)$ is the homogeneous linear operator of degree 0

$$L_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

uniquely determined by the following conditions.

$$\forall \omega, \eta \in \Omega^\bullet(M), \ \ L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge (L_X\omega), \tag{$\boldsymbol{P}_+$}$$

$$\forall f \in C^\infty(M), \ \ L_X f = df(X), \tag{1.2}$$

$$\forall \alpha \in \Omega^1(M), \ \ Y \in \operatorname{Vect}(M), \ \ (L_X\alpha)(Y) = L_X\big(\alpha(Y)\big) - \alpha([X, Y]), \tag{1.3}$$

where $[X, Y]$ denotes the Lie bracket of the vector fields X, Y.

Example 1.1. Suppose that $x^1, \dots, x^n$, $n = \dim M$, are local coordinates on an open subset $U \subset M$. For every ordered multi-index

$$I = (i_1 < \cdots < i_k), \ \ 1 \leq i_j \leq n$$

we

$$dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(U).$$

Any $\omega \in \Omega^k(U)$ is a linear combination

$$\omega = \sum_{|I|=k} \omega_I dx^I,$$

where $\omega_I \in C^\infty(U)$, and the summation is carried over all ordered multi-indices I such that $|I| = k$. Moreover

$$d\omega = \sum_I (d\omega_I) \wedge dx^I. \qquad \square$$

Proposition 1.2. *For any $X, Y \in \mathrm{Vect}(M)$, and any $\omega \in \Omega^\bullet(M)$ we have the following commutation formulæ*

$$(L_X d - dL_X)\omega = 0. \tag{1.4a}$$

$$\big(L_X L_Y - L_Y L_X \big)\omega) = L_{[X,Y]}\omega, \ \ (i_X i_Y + i_Y i_X)\omega = 0. \tag{1.4b}$$

$$(L_X i_Y - i_Y L_X)\omega = i_{[X,Y]}\omega. \tag{1.4c}$$

$$L_X\omega = (i_X d + di_X)\omega. \tag{1.4d}$$

The equality (1.4d) is known as Cartan's homotopy formula. $\square$

Any smooth map $F : M \to N$ induces a homogeneous linear operator of degree 0

$$F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$$

called the *pullback by F* and defined by

$$F^*\omega_x(v_1, \dots, v_k) := \omega_{F(x)}\big(F_*(v_1), \dots, F_*(v_k) \big), \ \ \forall x \in M, \ \ v_1, \dots, v_k \in T_xM,$$

where $F_* : T_xM \to T_{F(x)}N$ denotes the differential of F.

Example 1.3. Suppose $F : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map. We denote by (x^i) the coordinates in $\mathbb{R}^m$ and by (u^j) the coordinates in $\mathbb{R}^n$. Then the map F can be interpreted as a collection of functions

$$u^j = u^j(x^1, \dots, x^m), \ \ 1 \leq j \leq n.$$

A differential form ω of degree k on the target space $\mathbb{R}^n$ has the form

$$\omega = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \omega_{i_1 \dots i_k}(u^1, \dots, u^n) du^{j_1} \wedge \cdots \wedge du^{j_k}, \ \ \omega_{i_1 \dots i_k} \in C^\infty(\mathbb{R}^n).$$

Then $F^*\omega$ is the differential form obtained from the above expression by thinking of the qunatities u^j as function of x^i so that

$$du^j = \sum_{i=1}^m \frac{\partial u^j}{\partial x^i} dx^i \in \Omega^1(\mathbb{R}^m)$$

and

$$\omega_{j_1, \dots, j_k}(\dots, u^j, \dots) = \omega_{j_1, \dots, j_k}\big(\dots, u^j(x^1, \dots, x^m), \dots\big) \in C^\infty(\mathbb{R}^m). \qquad \square$$

Proposition 1.4. *The pullback is a morphism of $\mathbb{R}$-algebras which commutes with the exterior derivatives commute with pullbacks. More precisely, for any smooth map $F : M \to N$, and any $\omega, \eta \in \Omega^\bullet(N)$ we have*

$$(F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta), \ \ d_M F^*\omega = F^* d_N\omega.$$

Moreover,

$$\mathbb{1}_M^* = \mathbb{1}_{\Omega^\bullet(M)}$$

and if $G : N \to P$ is another smooth map, then

$$(G \circ F)^* = F^* \circ G^*. \qquad \square$$

Recall that an orientation on a finite dimensional vector space V is an equivalence class of *ordered bases* of V where two bases $\underline{\boldsymbol{e}} := (\boldsymbol{e}_i)_{1\leq i\leq \dim V}$ and $\underline{\boldsymbol{f}} := (\boldsymbol{f}_j)_{1\leq j\leq \dim V}$ are called equivalent if the transition matrix $T_{\underline{\boldsymbol{f}},\underline{\boldsymbol{e}}} = (T^i_j)_{1\leq i,j\leq \dim V}$ defined by

$$\boldsymbol{f}_j = \sum_i T^i_j \boldsymbol{e}_i, \ \ \forall j,$$

has positive determinant. We denote by $\mathrm{Or}_{vect}(V)$ the set of orientations of V. For any orientation $\boldsymbol{or} \in \mathrm{Or}(V)$ we denote by $-\boldsymbol{or}$ the *opposite* orientation, i.e.,

$$\mathrm{Or}(V) = \{\, \boldsymbol{or}, -\boldsymbol{or}\}.$$

For any basis $\underline{\boldsymbol{e}}$ we denote by $[\underline{\boldsymbol{e}}] \in \mathrm{Or}(V)$ the orientation its determines.

A smooth manifold M is called *orientable* if it admits a *volume form*, i.e., a nowhere vanishing top degree form $\omega \in \Omega^{\dim M}(M)$. Two volume forms ω_0 and ω_1 are called *equivalent* if there exists $f \in C^\infty(M)$ such that

$$\omega_1 = e^f \omega_0.$$

An equivalence class of volume forms is called an *orientation* on M, and the set of orientations is denoted by $\mathrm{Or}(M)$. For any volume form ω we denote by $\boldsymbol{or}_\omega$ the orientation defined by ω. For any orientation $\boldsymbol{or}$ on M we define the *opposite orientation* $-\boldsymbol{or}$ so that

$$\boldsymbol{or} = \boldsymbol{or}_\omega \Longleftrightarrow -\boldsymbol{or} = \boldsymbol{or}_{-\omega}.$$

If M is an orientable n-dimensional manifold, then or every point $p \in M$ we have a natural map $\boldsymbol{i}_p : \mathrm{Or}(M) \to Or_{vect}(T_p(M))$ defined as follows. If $\boldsymbol{or} \in \mathrm{Or}(M)$ is defined by a volume form ω, then $\boldsymbol{i}_p\boldsymbol{or}$ is described by an ordered basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of T_pM such that

$$\omega_p(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n) > 0.$$

The map $\boldsymbol{i}_p$ is surjective, and if M is connected, then $\boldsymbol{i}_p$ is in fact a bijection.

If M and and N are smooth manifolds then any diffeomorphism $F : M \to N$ induces a bijection $F^* : \mathrm{Or}(N) \to \mathrm{Or}(M)$, where for every volume form ω on N we have

$$F^*[\boldsymbol{or}_\omega] = [\boldsymbol{or}_{F^*\omega}].$$

If $F : M_0 \to M_1$ is a diffeomorphism, and $\boldsymbol{or}_i$ is an orientation on M_i, $i = 0, 1$, then F is called *orientation preserving* if

$$F^*\boldsymbol{or}_1 = \boldsymbol{or}_0.$$

Example 1.5. (a) A vector space V can also be regarded as a smooth manifold. As such, it is orientable, and the map

$$\boldsymbol{i}_0 : \mathrm{Or}(V) \to \mathrm{Or}_{vect}(V)$$

is a bijection. For this reason, in the sequel we will freely identify $\mathrm{Or}_{vect}(V)$ with $\mathrm{Or}(V)$.

The Euclidean vector space $\mathbb{R}^n$. It has a canonical orientation given by the canonical basis

$$\boldsymbol{e}_j = \begin{bmatrix} \delta^1_j \\ \vdots \\ \delta^i_j \\ \vdots \\ \delta^n_j \end{bmatrix}, \ \ 1 \leq j \leq n, \ \ \delta^i_j := \begin{cases} 1 & i = i \\ 0 & i \neq j \end{cases}.$$

We say that this is the *canonical orientation on* $\mathbb{R}^n$ and we will denote it by $\boldsymbol{or}_{\mathbb{R}^n}$.

We can regard $\mathbb{R}^n$ as a smooth manifold. If we denote by $(x^1, \dots, x^n)$ the Cartesian coordinates determined by the canonical basis, then

$$dV_n := dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$$

is a volume form defining the canonical orientation.
(b) The unit sphere

$$S^n := \left\{ (x^0, \dots, x^n) \in \mathbb{R}^{n+1};\ \sum_{i=0}^n (x^i)^2 = 1 \right\}$$

is an orientable manifold. Consider the point $p_0 = (1, 0, \dots, 0) \in S^n$. To specify an orientation on S^n its suffices to specify an orientation on

$$T_{p_0} S^n = \left\{ (x^0, \dots, x^n) \in \mathbb{R}^{n+1};\ x^0 = 0 \right\}.$$

The *canonical orientation of* S^n is defined by orienting $T_{p_0} S^n$ using the ordered basis

$$\boldsymbol{e}_1(0, 1, 0, \dots, 0), \dots, \boldsymbol{e}_n(0, 0, \dots, 0, 1).$$

We will denote by $\boldsymbol{or}_{S^n}$ the canonical orientation of S^n. □

Theorem 1.6 (Existence of integral)**.** *Let n be a positive integer. To any smooth manifold M of dimension n, and any orientation $\boldsymbol{or}$ on M we can associate a linear map*

$$\int_{[M, \boldsymbol{or}]} : \Omega^n_c(M) \to \mathbb{R},\ \ \Omega^n_c(M) \ni \omega \mapsto \int_{[M, \boldsymbol{or}]} \omega,$$

uniquely defined by the following conditions.
(a)

$$\int_{[M, -\boldsymbol{or}]} = -\int_{[M, \boldsymbol{or}]}.$$

(b) For any diffeomorphism $F : M_0 \to M_1$, and any $\boldsymbol{or}_1 \in \mathrm{Or}(M_1)$ we have

$$\int_{[M_0, F^* \boldsymbol{or}_1]} F^* \omega = \int_{[M_1, \boldsymbol{or}_1]} \omega,\ \ \forall \omega \in \Omega^n_c(M_1).$$

(c) For any orientable smooth manifold M, any $\boldsymbol{or} \in \mathrm{Or}_M$, and any open subset $U \subset M$ we have

$$\int_{[U, \boldsymbol{or}|_U]} \omega|_U = \int_{[M, \boldsymbol{or}]} \omega,\ \ \forall \omega \in \Omega^n_c(M),\ \ \operatorname{supp} \omega \subset U.$$

(d) For any compactly supported smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{[\mathbb{R}^n, \boldsymbol{or}_{\mathbb{R}^n}]} f dx^1 \wedge \cdots \wedge dx^n = \underbrace{\int_{\mathbb{R}^n} f dx^1 \cdots dx^n}_{\text{Riemann integral}}.$$ □

Recall that a n-dimensional manifold with boundary is a topological space M such that there exists a smooth n-dimensional manifold $\tilde{M}$ and a smooth function $f : \tilde{M} \to \mathbb{R}$ with the following properties

$$0 \in \operatorname{int} f(\tilde{M}),\ \ M = \left\{ p \in \tilde{M};\ \ f(p) \leq 0 \right\},$$

$$df(p) = 0 \Longrightarrow f(p) \neq 0,$$

The interior of a manifold with boundary is the set

$$M^0 := \left\{ p \in \tilde{M};\ \ f(p) < 0 \right\}$$

and its boundary is the set

$$\partial M := \{ p \in \tilde{M};\ \ f(p) = 0 \}.$$

Both M^0 and ∂M are smooth manifolds, and they are independent of the choice of $\tilde{M}$ and f.

The manifold with boundary M is called *orientable* if we can choose $\tilde{M}$ to be orientable. Any orientation $\boldsymbol{or}$ on M induces an orientation $\partial\boldsymbol{or}$ on ∂M as uniquely characterized by the *outer-normal-first* condition

$\forall p \in \partial M$ the frame $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1}$ of $T_p\partial M$ is positively oriented with respect to $\partial\boldsymbol{or}$ if and only if there exists $\boldsymbol{e}_0 \in T_p\tilde{M}$ such that $df(\boldsymbol{e}_0) > 0$ and the ordered frame $\{ \boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1}\}$ of $T_p\tilde{M}$ is positively oriented with respect to $\boldsymbol{or}$. (The condition $df(\boldsymbol{e}_0)$ indicates that the vector $\boldsymbol{e}_0$ points towards the exterior of M.)

Example 1.7. The closed unit ball

$$\boldsymbol{B}^{n+1} = \{ (x^0, \dots, x^n) \in \mathbb{R}^{n+1};\ \ \sum_i (x^i)^2 \le 1 \}$$

is an orientable manifold with boundary $\partial\boldsymbol{b}^{n+1} = S^n$. The canonical orientation of $\mathbb{R}^{n+1}$ induces an orientation $\boldsymbol{or}_{n+1}$ on $\boldsymbol{B}^{n+1}$ and $\partial\boldsymbol{or}_{n+1} = S^n$. □

Theorem 1.8 (Stokes formula)**.** *Suppose M is an n-dimensional orientable manifold with boundary defined by a pair $(\tilde{M}, f)$. Then for every orientation $\boldsymbol{or}$ on M and for every $\omega \in \Omega_c^{n-1}(\tilde{M})$ we have*

$$\int_{[M^0, \boldsymbol{or}]} d\omega = \int_{[\partial M, \partial\boldsymbol{or}]} \omega. \qquad \square$$

1.2. **Elementary homological algebra.** We introduce the notion of a *(co)chain complex.* Recall that a *graded* real vector space is a real vector space equipped with a direct sum decomposition

$$V^\bullet := \bigoplus_{n\in\mathbb{Z}} V^n.$$

If $V^\bullet$ is a graded vector space, then a homogeneous element of $V^\bullet$ is an element v which belongs to one of the summands V^n. In this case we say that n is the degree of v and we write $|v| := n$.

A linear map between two graded vector spaces $L : U^\bullet \to V^\bullet$ is called homogeneous of degree k if

$$\forall n \in \mathbb{Z},\ \ u \in U^n \Longrightarrow Lu \in V^{n+k}.$$

If $V^\bullet$ is a graded vector space, then for every $k \in \mathbb{Z}$ we defined its *translate* $V^\bullet[k]$ to be the graded vector space

$$V^\bullet[k] = \bigoplus V[k]^n,\ \ V[k]^n := V^{n+k}.$$

A *chain* (respectively *cochain*) *complex* is a pair $(A^\bullet, d)$ where $A^\bullet$ is a graded vector space and $d : A^\bullet \to A^\bullet$ is a linear homogeneous map of degree -1 (respectively 1) such that $d^2 = 0$. For simplicity, in the sequel we will concentrate exclusively on cochain complexes. Often one thinks of a cochain complex as a sequence of linear maps

$$\dots \longrightarrow A^{n-1} \xrightarrow{d_{n-1}} A^n \xrightarrow{d_n} A^{n+1} \dots,$$

so that $d_n \circ d_{n-1} = 0$, $\forall n \in \mathbb{Z}$

The *cohomology* of a cochain complex $(A^\bullet, d)$ is the graded vector space $H^\bullet(A) = H^\bullet(A^\bullet, d)$ where

$$H^i(A) = \frac{\ker d_i}{\operatorname{im} d_{i-1}}, \ \ \forall i \in \mathbb{Z}$$

It is convenient to rephrase the last definition in the language of *cocycles*. A cocycle of degree n of $(A^\bullet, d)$ is a degree n element a such that $da = 0$, i.e., $a \in \ker d_n$. Two cocycles $a', a'' \in A^n$ are called *cohomologous* if $a' - a''$ is a *coboundary*, i.e., there exists $x \in A^{n-1}$ such that

$$a' - a'' = dx.$$

Thus, we can identify $H^n(A)$ with the set of cohomology classes of cocycles of degree n.

A *morphism* between the cochain complexes $(A^\bullet, d_A)$, $(B^\bullet, d_B)$ is a homogeneous linear map of degree zero

$$f : A^\bullet \to B^\bullet$$

such that, for every $n \in \mathbb{Z}$ the diagram below is commutative

$$\begin{tikzcd} A^n \arrow[r, "d_A"] \arrow[d, "L"'] & B^{n+1} \arrow[d, "L"] \\ B^n \arrow[r, "d_B"'] & B^{n+1} \end{tikzcd} \tag{1.5}$$

Note that a chain morphism maps cocycles to cocycles and coboundaries to coboundaries so that we deduce the following result.

Proposition 1.9. *Any cochain morphism $f : (A^\bullet, d_A) \to (B^\bullet, d_B)$ induces a linear map $H(f) : H^\bullet(A) \to H^\bullet(B)$ which is homogeneous of degree zero. Moreover*

$$H(\mathbb{1}_{A^\bullet}) = \mathbb{1}_{H(A)},$$

and if $g : (B^\bullet, d_B) \to (C^\bullet, d_C)$ is another is a cochain morphism, then

$$H(g \circ f) = H(g) \circ H(f). \qquad \square$$

Two cochain morphisms $f_0, f_1 : (A^\bullet, d_A) \to (B^\bullet, d_B)$ are called *homotopic* if there exists a *cochain homotopy* between them, i.e., a homogeneous linear map of degree -1

$$K : A^\bullet \to B^\bullet$$

such that

$$f_1 - f_0 = K d_A + d_B K.$$

Proposition 1.10. *If $f_0, f_1 : (A^\bullet d_A) \to B^\bullet, d_B)$ are two homotopic cochain morphisms then*

$$H(f_0) = H(f_1).$$

Proof. Suppose $a_0, a_1 \in A^\bullet$ are two cohomologous cocycles, ie.e., $da_0 = da_1 = 0$ and there exists $x \in A^\bullet$ such that $d_A x = a_1 - a_0$.

Then

$$f_1(a_1) - f_0(a_1) = d_B K a_1$$

and

$$f_0(a_1) - f_0(a_0) = f_0(a_1 - a_0) = d_B f_0(x)$$

so that

$$f_1(a_1) - f_0(a_0) = d_B((Ka_1 + f_0(x)).$$

The last equality shows that if a_0 and a_1 are cohomologous, then so are $f_0(a_0)$ and $f_1(a_1)$. □

1.3. **The DeRham complexes.** To any smooth manifold M we can now associate in a canonical fashion a cochain complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0. \qquad (\boldsymbol{DR})$$

We note that this is indeed a cochain complex because $d^2 = 0$. It is called the *De Rham complex*.

Definition 1.11. Let ω be a degree r differential form on M. The form ω on M is called *closed* if $d\omega = 0$. It is called *exact* if there exists $\eta \in \Omega^{r-1}(M)$ such that $d\eta = \omega$. □

We set $d^r := d|_{\Omega^r}$, $\forall r$. We see that the space of closed r-forms coincides with the kernel of d^r, while the space of exact r-forms coincides with the image of d^{r-1}. The cohomology of the De Rham complex is called the *De Rham cohomology* of M and it is denoted by $H^\bullet(M)$.

The *De Rham complex with compact supports* is the complex

$$0 \to \Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_c(M) \to 0. \qquad (\boldsymbol{DR}_c)$$

We denote by $H^\bullet_c(M)$ the cohomology of this complex and we will refer to it as the *De Rham cohomology of M with compact supports*.

Example 1.12 (The De Rham cohomologies of $\mathbb{R}$)**.** We first compute $H^\bullet(\mathbb{R})$. Clearly, $H^k(\mathbb{R}) = 0$ if $k \geq 2$. From the definitions, we have that $H^0(\mathbb{R}) = \ker d^0$. Note that

$$f \in \ker d^0 \iff df = 0 \iff f \text{ is a constant function.}$$

Therefore

$$H^0(\mathbb{R}) = \{\text{constant functions on } \mathbb{R}\} \cong \mathbb{R}.$$

We now compute $H^1(\mathbb{R})$. Clearly, $\ker d_1 = \Omega^1(\mathbb{R})$. Let $\omega = g(x)dx$ be a 1-form. Define a function $f \in \Omega^0(M)$ by:

$$f(x) := \int_0^x g(u)du.$$

Then

$$df = f'(x)dx = g(x)dx = \omega,$$

by the Fundamental Theorem of Calculus. This shows $\operatorname{im} d^0 = \Omega^1(\mathbb{R})$. Therefore, $H^1(\mathbb{R}) = 0$. Thus, we conclude $H^k(\mathbb{R}) = \mathbb{R}$ if $k = 0$ and it is 0 otherwise. In particular,

$$H^k(\mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k > 0 \end{cases}$$

We now compute $H^\bullet_c(\mathbb{R})$. The above argument shows that $H^0_c(\mathbb{R})$ is equal to the constant functions with compact support, and therefore,

$$H^0_c(\mathbb{R}) \cong 0.$$

To compute $H^1_c(\mathbb{R})$ we consider the period map,

$$\int_{\mathbb{R}} : \Omega^1_c(\mathbb{R}) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}} \omega.$$

Clearly, this map is surjective. We then conclude that

$$\frac{\ker d_1}{\ker \int_{\mathbb{R}}} = \frac{\Omega_c^1(\mathbb{R})}{\ker \int_{\mathbb{R}}} \cong \mathbb{R}.$$

We will prove that

$$\ker \int_{\mathbb{R}} = \operatorname{im}\left(d_0 : \Omega_c^0(\mathbb{R}) \to \Omega_c^1(\mathbb{R})\right),$$

and thus conclude that $H_c^1(\mathbb{R}) \cong \mathbb{R}$.

Assume $\omega \in \operatorname{im}\left(d_0 : \Omega_c^0(\mathbb{R}) \to \Omega_c^1(\mathbb{R})\right)$. Then there exists an $f \in \Omega_c^0(\mathbb{R})$ so $\omega = df$. Since ω and f have compact supports, we can assume $\operatorname{supp}(\omega) \subset [a, b]$, and $f(a) = f(b) = 0$. We then have

$$\int_{\mathbb{R}} \omega = \int_{\mathbb{R}} \frac{df}{dx} dx = \int_a^b \frac{df}{dx} dx = f(a) - f(b) = 0.$$

Therefore

$$\operatorname{im}\left(d_0 : \Omega_c^0(\mathbb{R}) \to \Omega_c^1(\mathbb{R})\right) \subset \ker \int_{\mathbb{R}}.$$

Conversely, let $\omega = g(x)dx \in \ker(\int_{\mathbb{R}})$. We will prove show that there exists $f \in \Omega_c^0(\mathbb{R})$ so $df = \omega$. We define f by

$$f(x) := \int_{-\infty}^{x} g(u)du.$$

By the Fundamental Theorem of Calculus, $df = g$. Also, let $\operatorname{supp}(g) \subset [a, b]$. Then we have, for $x < a$,

$$f(x) = \int_{-\infty}^{x} g(u)du = 0$$

since $g(u)$ is zero on the interval $(-\infty, x)$. Also, if $x > a$, we have

$$f(x) = \int_{-\infty}^{x} g(u)du = \int_{\mathbb{R}} g(u)du = 0,$$

since $\omega \in \ker(\int_{\mathbb{R}})$. Therefore, by the above, we get $H_c^1(\mathbb{R}) = \mathbb{R}$ and we conclude

$$H_c^k(\mathbb{R}) = \begin{cases} \mathbb{R}, & k = 1 \\ 0, & k \neq 1 \end{cases}$$ □

Proposition 1.4 implies that for any smooth map $F : M \to N$ the pullback $F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ is a cochain morphism. In particular, it induces a morphism. Therefore, by Proposition 1.9, we have an induced map in cohomology

$$H(F^*) : H^\bullet(N) \to H^\bullet(M),$$

which, for simplicity, we will continue to denote by F^*.

Corollary 1.13. *If M and N are diffeomorphic smooth manifolds then their De Rham groups $H^\bullet(M)$ and $H^\bullet(N)$ are isomorphic.*

Proof. Let $f : M \to N$ and $g : N \to M$ be diffeomorphisms which are inverse to each other. Then we have

$$f^* : H^\bullet(N) \to H^\bullet(M), \qquad g^* : H^\bullet(N) \to H^\bullet(N).$$

By above,

$$\mathbb{1}_{H^\bullet(N)} = (f \circ g)^* = g^* \circ f^* \qquad \mathbb{1}_{H^\bullet(M)} = (g \circ f)^* = f^* \circ g^*.$$

Therefore, f^* and g^* are both isomorphisms, so that

$$H^\bullet(M) \cong H^\bullet(N).$$

□

1.4. **The Poincaré lemma.** To any smooth manifold M we associate the cylinder

$$\widehat{M} := [0,1] \times M.$$

In particular, note that for all $\eta \in \Omega^k\big(\widehat{M}\big)$, we have a decomposition

$$\eta = dt \wedge \eta_0 + \eta_1,$$

where $\eta_0 = \partial_t \lrcorner \eta$ is the contraction of η by dt. We can view η_0 as a t-dependent $(k-1)$-form on M. We define an operator $K : \Omega^k\big(\widehat{M}\big) \to \Omega^{k-1}(M)$ by the following:

$$K(\eta) = \int_0^1 \eta_0(t)dt.$$

More precisely, we can view this as saying that $K(\eta)$ is the form such that

$$K(\eta)_x = \int_0^1 \eta_0(x,t)dt.$$

We now have the following proposition.

Proposition 1.14. *If we denote by i_t the inclusion $M \to [0,1] \times M$ so that $i_t(x) = (t,x)$, then we have, for all $\eta \in \Omega^k(M)$*

$$i_1^*\eta - i_0^*\eta = (dK + K\delta)\eta$$

Proof. Clearly, the above equation is linear in η, therefore, by using partitions of unity, we can reduce to the case when $\operatorname{supp}(\eta) \subset [0,1] \times \mathcal{U}$, where $\mathcal{U}$ is a coordinate neighborhood of M. More specifically, let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be a locally finite cover of M by coordinate charts, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. From this, we can get a partition of unity on $\widehat{M}$ by extending each ρ_α to be constant in the time variable. To prove our claim, it now suffices to show

$$K(\sum_{\alpha\in\mathcal{A}} \rho_\alpha \eta) = \sum_{\alpha\in\mathcal{A}} \rho_\alpha K(\eta) = K(\eta).$$

This can be seen easily using local finiteness. Clearly, we can break $K(\eta)$ as $\sum_\alpha K(\eta|_{U_\alpha})$. We therefore only need to check that if U is a coordinate chart and $\mathcal{B} \subset \mathcal{A} = \{\alpha \in \mathcal{A} : \operatorname{supp}\rho_\alpha \subset U\}$,

$$K(\sum_{\alpha\in\mathcal{B}} \rho_\alpha \eta) = \sum_{\alpha\in\mathcal{B}} \rho_\alpha K(\eta).$$

But this equality is obvious, since K is linear and the sum is finite. Therefore, we now assume that $\operatorname{supp}(\eta) \subset [0,1] \times \mathcal{U}$.

Fix $x^1, \ldots, x^m$ coordinates on $\mathcal{U}$. We know that any η with support on $[0,1] \times \mathcal{U}$ is a linear combination of forms of type I, which look like $f(t,x)dt \wedge dx^I$ and forms of type II, which look like $g(x,t)dx^J$, where I is an index of size $q-1$ and J is of size q. Thus, the proof reduces to checking the homotopy formula in the case where η is either one of these two forms.

We first check the case where η is of type I. Without loss of generality, we will say $\eta = f(t,x)dt \wedge dx^1 \wedge \ldots \wedge dx^{q-1}$. We conclude that in our earlier breakdown, $\eta_1 = 0$ and $\eta_0 = f(t,x)dx^1 \wedge \ldots \wedge dx^{q-1} = f(t,x)dx^I$. We therefore have:

$$dK(\eta) = d((\int_0^1 f(t,x)dt)dx^I)$$

$$= (-1)^q (\sum_{j=q}^m \int_0^1 \frac{\partial f}{\partial x_j} dt) dx^I \wedge dx^j + (f(1,x) - f(0,x))dt \wedge dx^I$$

Also, we can easily compute

$$\hat{d}\eta = (-1)^q \sum_{j=q}^m \frac{\partial f}{\partial x_i} dt \wedge dx^I \wedge dx^j$$

Taking K, we get

$$K(\hat{d}\eta) = (-1)^q (\sum_{j=q}^m \int_0^1 \frac{\partial f}{\partial x_j} dt dx^I \wedge dx^j)$$

We can now clearly see that

$$(dK - K\hat{d})(\eta) = f(1,x)dt \wedge dx^I - f(0,x)dt \wedge dx^I = i_1^*(\eta) - i_0^*(\eta)$$

as required.

We now check the result for forms of type II. We can assume without loss of generality that $\eta = g(t,x)dx^J = g(t,x)dx^1 \wedge \ldots \wedge dx^q$. Using our earlier breakdown, we see that $\eta_0 = 0$ and $\eta_1 = g(t,x)dx^J$. Therefore, we have by definition of K that $K(\eta) = 0$. We have the computation

$$\hat{d}\eta = \frac{\partial g}{\partial t} dt \wedge dx^J + (-1)^q \sum_{j=q+1}^m \frac{\partial g}{\partial x^j} dx^J dx^j$$

We can then easily compute

$$K(\hat{d}\eta) = \int_0^1 \frac{\partial g}{\partial t} dt \wedge dx^J = g(1,x)dx^J - g(0,x)dx^J = i_1^*(\eta) - i_0^*(\eta).$$

Therefore, since $d(K\eta) = 0$, we have $(K\hat{d} - dK)(\eta) = i_1^*(\eta) - i_0^*(\eta)$, as required. $\square$

We note that above implies that if we consider the maps i_0^* and i_1^* as maps of cohomology, then they are equal. Using this, we can prove the following useful theorem.

Theorem 1.15. *Homotopic maps induce identical maps in cohomology.*

Proof. We recall that if $f, g : M \to N$ are homotopic, then there exists $F : \mathbb{R} \times M \to N$ so that

$$f(t,x) = \begin{cases} f(x) & t \geq 1 \\ g(x) & t \leq 0. \end{cases}$$

We can also say this by saying

$$f = F \circ i_1, \qquad g = F \circ i_0$$

where i_t is defined as before in $\mathbb{R} \times M$. Therefore,

$$f^* = i_1^* \circ F^*, \qquad g^* = i_0^* \circ F^*.$$

But by the above argument, we know that $i_0^* = i_1^*$ at the level of cohomology. Therefore, at the level of cohomology, $f^* = g^*$. □

Corollary 1.16. *Two manifolds of the same homotopy type have isomorphic De Rham cohomology*

Proof. If two manifolds have the same homotopy type, we get maps

$$f : M \to N, \quad g : N \to M$$

so that $f \circ g$ and $g \circ f$ are homotopic to the identity. Therefore, by the above theorem, we have

$$g^* \circ f^* = (f \circ g)^* = \mathbb{1}_N^* = \mathbb{1}_{H^\bullet(M)}, \quad f^* \circ g^* = (g \circ f)^* = \mathbb{1}_M^* = \mathbb{1}_{H^\bullet(N)}$$

Therefore, we have that the maps f^* and g^* are inverses of each other, as required. □

Finally, we can prove the Poincaré Lemma

Theorem 1.17. *Let $U \subset \mathbb{R}^n$ be convex. Then*

$$H^k(U) = \begin{cases} 0 & k > 1 \\ \mathbb{R} & k = 0. \end{cases}$$

Proof. Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth map so that

$$\eta(t) = \begin{cases} 1 & t \geq 1 \\ 0 & t \leq 0. \end{cases}$$

We then can define a map $H : \mathbb{R} \times U \to U$ by $H(t, x) = \eta(t)x$. Then H is smooth and

$$H(t, x) = \begin{cases} 0 & t \leq 0 \\ x & t \geq 1. \end{cases}$$

Therefore, we have that $\mathbb{1}_U$ is homotopic to the 0 map.

Now consider $\mathbb{R}^0$, the space of a single point. We can define maps $i : \mathbb{R}^0 \to U$ and $\pi : U \to \mathbb{R}^0$ by $i(pt) = 0$ and $\pi(x) = pt$. Then we have $\pi \circ i = \mathbb{1}_{\mathbb{R}^0}$ and $i \circ \pi = 0$, which is homotopic to the identity. Therefore, we have that U is homotopy equivalent to a point, and therefore, by homotopy invariance, the desired result follows. □

The previous result implies that if we have $U \subset \mathbb{R}^n$ and $0 < k < n$, then there is a map $L : Z^k(U) \to \Omega^{k-1}(U)$ so that $d \circ L\eta = \eta$ for all $\eta \in Z^k(U)$. We now seek to produce this map explicitly. First, we define a map $C : [0, 1] \times U \to U$ as

$$H(s, x_1, \ldots, x_n) = (sx_1, \ldots, sx_n)$$

Now let η be a closed k-form. We know $\eta = \sum \eta_I dx^I$. Let V be the vector field defined by

$$V = \sum_i \partial_{x^i}$$

We compute $C^*(\eta)$. We have

$$C^*(\eta) = \sum_I \eta_I(sx) d(sx^{i_1}) \wedge \ldots \wedge d(sx^{i_k})$$

$$= \sum_I \eta_I s^k dx^I + \sum_I (\eta_I(sx)s^{k-1} ds^(V \lrcorner dx^I)$$

where we have

$$V \lrcorner dx^I = \sum_{j=1}^{k} (-1)^j dx^{j_1} \wedge \ldots \wedge \widehat{dx^{i_j}} \wedge \ldots \wedge dx^{i_k}$$

We can then say that

$$L\eta = \sum_I \left(\int_0^1 \eta_I(sx) s^{k-1} ds \right) V \lrcorner dx^I.$$

We leave it to the reader to check that $d \circ L(\eta) = \eta$.

2. Simplicial Complexes

2.1. **Basic Concepts.** First, we establish a notation. If S is any set, we define $\mathfrak{2}^S$ as the collection of subsets of S, and $\mathfrak{2}^S_*$ as the collection of non-empty subsets of S.

A *simplicial complex* is a finite collection of non-empty finite sets K such that if $T \in K$, $S \subset T$, and $S \neq \emptyset$, then $S \in K$. We define the vertex set of K, $V(K)$ as

$$V(K) = \bigcup_{S \in K} S$$

We will say that if $v \in V(K)$, v is a *vertex*, and if $S \in K$, S is a *face* of K

Example 2.1. For every set V, the collection $\mathfrak{2}^V_*$ is a simplicial complex called the standard simplex with vertex set V. If $V = \{v_0, v_1, v_2\}$, we have

$$\mathfrak{2}^V_* = \{\{v_0\}, \{v_1\}, \{v_2\}, , \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1\}, \{v_0, v_1, v_2\}\}$$

and clearly

$$V(\mathfrak{2}^V_*) = V$$ □

Example 2.2 (The Nerve of a Cover)**.** Any time we have a manifold M and an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, we can associate a simplicial complex, called the nerve of $\mathcal{U}$ and denoted $N(\mathcal{U})$ as follows. The vertex set of $N(\mathcal{U})$ is the set A, and we have the relation

$$\{\alpha_1, \ldots, \alpha_p\} \subset A \in N(\mathcal{U}) \Leftrightarrow \bigcap_{i=1}^{p} U_{\alpha_i} \neq \emptyset$$

Specifically, if $M = I$ where I is a discrete set, we can consider the cover $\mathcal{U} = \{U_i\}_{i \in I}$ given by

$$U_i = I \setminus \{i\}.$$

In this case, we can see at once that

$$N(\mathcal{U}) = \mathfrak{2}^I_*$$ □

A *morphism of simplicial complexes*, K_0 and K_1, is a map

$$f : V(K_0) \longrightarrow V(K_1)$$

such that if S is a face of K_0, $f(S)$ is a face of K_1. That is to say

$$S \in K_0 \Rightarrow f(S) \in K_1$$

Example 2.3. Let K be a simplicial complex, and let $V = V(K)$. Then the natural inclusion

$$K \hookrightarrow \mathfrak{2}^V_*$$

is a morphism of simplicial complexes. □

We now define $\mathbb{R}^V$ to be the vector space of all functions from V to $\mathbb{R}$. That is,

$$\mathbb{R}^V = \{f : V \to \mathbb{R}\}$$

We can find a basis of $\mathbb{R}^V$ by defining the so-called Dirac functions δ_u, for all $u \in V$:

$$\delta_u(v) := \begin{cases} 1 & u = v \\ 0 & u \neq v \end{cases}$$

We see that the collection $\{\delta_u\}$ does indeed form a basis of $\mathbb{R}^V$. Indeed, if $f \in \mathbb{R}^V$, we have

$$f = \sum_{u \in V} f(u)\delta_u$$

We note that if $V = \{1, \dots, n\}$, then $\mathbb{R}^V$ is just $\mathbb{R}^n$ and $\delta_1, \dots, \delta_n$ is the standard basis.

Let K be a simplicial complex with vertex set $V(K) = V$. Now, let $S \in K$. We recall $S \subset V$. Then we have

$$\Delta_S = \{f \in \mathbb{R}^V : f = \sum_{s \in S} t_s\delta_s,\ t_s \geq 0,\ \sum_{s \in S} t_s = 1\}$$

This is called the *closed simplex* spanned by $\{\delta_s\}_{s \in S}$. We can similarly define the open simplex

$$\dot{\Delta}_S = \{f \in \mathbb{R}^V : \quad f = \sum_{s \in S} t_s\delta_s,\ t_s \geq 0,\ \sum_{s \in S} t_s = 1\}$$

We can also define Aff_S to be the smallest affine plane containing S:

$$\mathrm{Aff}_S = \{f \in \mathbb{R}^V : \quad f = \sum_{s \in S} t_s\delta_s,\ \sum_{s \in S} t_s = 1\}$$

We note $\dim \mathrm{Aff}_S = |S| - 1$. We now define the *geometric realization of* K, denoted by $[K]$,

$$[K] := \bigcup_{S \in K} \Delta_S = \bigcup_{S \in K} \dot{\Delta}_S.$$

Similarly, we define the m-skeleton of K, denoted by $[K^m]$ as

$$[K^m] := \bigcup_{S \in K, |S| \leq m+1} \Delta_S$$

We now describe the notion of a chain of subsets of K. A *chain of subsets of* K is a finite collection $\{S_i\}_{i=1}^n$ such that $S_i \in K$ for all i, and $S_i \subset S_{i+1}$. Using this, we can define the *barycentric subdivision of* K as the simplicial complex $K' \subset 2_*^K$, where K' consists of all chains of subsets of K.

We can define the so-called *barycentric coordinate functions*, b_v, which are continuous functions from $[K]$ to $\mathbb{R}$, as follows. Let $v \in V(K)$, and let $x \in [K]$. We have

$$x = \sum_{v' \in V} t_{v'}\delta_{v'},\ t_{v'} \geq 0,\ \sum_{v' \in V} t_{v'} = 1$$

We note that in the above expression, if $v' \notin S$, then $t_{v'} = 0$. Also, in the above expression there is a t_v corresponding to the vertex v, and we define $b_v(x) = t_v$. We have the following proposition.

Proposition 2.4. *The barycentric coordinate functions b_v have the following properties.*

(1) $b_v(x) \geq 0$ *for all v, and for all $x \in [K]$*
(2) $\sum_{v \in V} b_v(x) = 1$

(3) $x = \sum_{v \in V} b_v(x)\delta_v$
(4) $\exists x \in [K]$ *so that* $b_{v_j}(x) \neq 0$ *for all* $j = 1 \ldots k$ *if and only if* $\{v_1, \ldots, v_k\} \in K$.

If $v \in V(K)$ is a vertex of K. We set

$$\mathrm{St}(v) := \bigcup_{S \ni v} \dot{\Delta}_S$$

This is called the *star of* v. We can similarly define the star of any face of K as follows. If $S \in K$, then

$$\mathrm{St}(S) := \bigcup_{K \ni T \supset S} \dot{\Delta}_T$$

We have the following relation

$$\mathrm{St}(S) = \bigcap_{v \in S} \mathrm{St}(v)$$

We now develop the notion of an oriented simplex. As before, let K be a simplicial complex with vertex set $V(K) = V$. We say an *ordered simplex* is a pair $(S, <)$, where $S \in K$ and $<$ is a total order of S. If $S = \{v_0, v_1, \ldots, v_l\}$ and $v_0 < v_1 < \ldots < v_l$, then we will say

$$(S, <) = [v_0, v_1, \ldots, v_l]$$

We can define an equivalence relation, denoted $\sim$, of the set of ordered simplices. We say that $(S_1, <_1) \sim (S_2, <_2)$ if $S_1 = S_2$ and if the identity map consists of an even number of inversions as a map of ordered sets. Clearly, there are two equivalence classes, and we define an oriented simplex $\overrightarrow{S}$ as a choice of equivalence class. We denote the other choice by $\overrightarrow{S}^{op}$. We also note that using this, we can get an orientation on Δ_S by choosing the orientation of Aff_S given by the ordered basis $\delta_{v_1} - \delta_{v_0}, \ldots, \delta_{v_l} - \delta_{v_0}$. This will be denoted $\overrightarrow{\Delta}_S$

2.2. **Simplicial Homology.** We eventually wish to develop the theory of simplicial cohomology, which we will then compare to the De Rham cohomology. To do this, we first discuss simplicial homology. As discussed in the beginning of the paper, in order to get homology, we first must produce a chain complex. To this end, we introduce the groups $C_l(K)$ for any oriented simplicial complex K. Specifically, we will say that the group $C_l(K)$ is the free abelian group with generators $\overrightarrow{S}$ and $\overrightarrow{S}^{op}$, where $S \in K$ has $|S| = l + 1$, with the relations

$$\overrightarrow{S} + \overrightarrow{S}^{op} = 0$$

In the sequel, we will use the notation $-\overrightarrow{S}$ for $\overrightarrow{S}^{op}$

We turn this into a chain complex by introducing the boundary operator:

$$\delta : C_l(K) \longrightarrow C_{l-1}(K)$$

We define δ by its action on the generators.

$$\delta([v_0, v_1, \ldots, v_l]) = \sum_{j=0}^{l} (-1)^j [v_0, v_1, \ldots, \hat{v}_j, \ldots, v_l]$$

We note that it is necessary to check that this definition is well defined. To do this, we must show that if $\sigma \in S_{l+1}$,

$$\delta([v_{\sigma(0)}, \ldots, v_{\sigma(l)}]) = \epsilon(\sigma)\delta([v_0, \ldots, v_l]),$$

where $\epsilon(\sigma)$ is the sign of σ. We further notice that it is sufficient to check this for all transpositions. We first check that

$$\delta([v_0,\ldots,v_{i-1},v_{i+1},v_i,v_{i+2},\ldots,v_l]) = -\delta([v_0,\ldots,v_l])$$

Indeed, we have

$$\delta([v_0,\ldots,v_{i-1},v_{i+1},v_i,v_{i+2},\ldots,v_l])$$

$$= \sum_{j=0}^{i-1}(-1)^j[v_0,v_1,\ldots,\hat{v}_j,\ldots,v_{i+1},v_i,\ldots,v_l]+(-1)^{i+1}[v_0,\ldots,\hat{v}_i,\ldots,v_l]+(-1)^{i+2}[v_0,\ldots,\hat{v}_{i+1}\ldots,v_l]$$

$$+\sum_{j=i+2}^{l}(-1)^j[v_0,v_1,\ldots,v_{i+1},v_i,\ldots,\hat{v}_j,\ldots,v_l] = \sum_{j=0}^{l}(-1)^{j+1}[v_0,v_1,\ldots,\hat{v}_j,\ldots,v_l] = -\delta([v_0,\ldots,v_l])$$

Using this, we have clearly that

$$\delta([v_0,\ldots,v_{i-1},v_j,v_{i+1},\ldots,v_{j-1},v_i,v_{j+1},\ldots,v_l]) = (-1)^{2(j-i)-1}\delta([v_0,\ldots,v_l]) = -\delta([v_0,\ldots,v_l]).$$

It is now clear, that if $\sigma \in S_{l+1}$, then since σ is a product of transpositions,

$$\delta([v_{\sigma(0)},\ldots,v_{\sigma(l)}]) = \epsilon(\sigma)\delta([v_0,\ldots,v_l]),$$

as required, so that δ is well defined. We leave with the reader the basic exercise that $\delta \circ \delta = 0$. The proof is not hard, and involves rearranging sums. This shows that the pair $(C_\bullet, \delta)$ is a chain complex, and thus we can form the simplicial homology of an oriented simplicial complex K, which will be denoted $H^\bullet(K)$.

Example 2.5. We compute the simplicial homology of $\mathfrak{2}_*^V$, where V is a finite set, $|V| = n$. Define a map $\epsilon : C_0(\mathfrak{2}_*^V) \to \mathbb{Z}$ as follows. We recall that

$$C_0(\mathfrak{2}_*^V) = \bigoplus_{v\in V}\mathbb{Z} < v > .$$

We therefore define ϵ by saying that $\epsilon(v) = 1$ for all $v \in V$. We claim that the sequence

$$0 \longrightarrow \mathbb{Z} \xleftarrow{\epsilon} C_0(\mathfrak{2}_*^V) \xleftarrow{\delta} C_1(\mathfrak{2}_*^V) \xleftarrow{\delta} \cdots \tag{2.1}$$

is exact. First, we note that the surjectivity of ϵ is apparent. To see the exactness elsewhere, we will construct a linear map

$$L_q : C_q(\mathfrak{2}_*^V) \to C_{q+1}(\mathfrak{2}_*^V)$$

such that

$$\delta_{q+1}L_q + L_{q-1}\delta_q = \mathbb{1},$$

which would imply that $\mathbb{1}$ is homotopic to the zero map, which gives that sequence is exact at all other required points. Now define

$$L_q([v_0,v_1,\ldots,v_q]) = \sum_{v\in V}[v,v_0,\ldots,v_q].$$

One can readily check that this definition of L_q has all of the required properties, so that the sequence above is exact, as required. But the sequence being exact implies

$$H^n(K) = \begin{cases}\mathbb{Z} & n = 0\\ 0 & n \geq 1\end{cases}$$

We now define the notion of simplicial cohomology. We can define a cochain complex by taking duals and adjoints of the above chain complex. That is, we have the cochain complex $(C^\bullet(K), \delta^*)$, where $C^\bullet = C_\bullet^* = \mathrm{Hom}(C_\bullet(K), \mathbb{R})$. We have the map

$$\delta^* : C^l(K) \to C^{l+1}(K)$$

defined as follows. If $f \in C^l(K)$, then we have

$$\delta^*(f) = f \circ \delta$$

We see that $\delta^* \circ \delta^* = 0$, since $\delta \circ \delta = 0$. We will call the cohomology of this complex the simplicial cohomology of K.

We now seek to describe the effect of δ^*. First, we define a basis of $C^l(K)$. If $\overrightarrow{S} \in C_l(K)$, then we define a function $\phi_S \in C^l(K)$ by

$$\phi_S(\overrightarrow{T}) = \begin{cases} 1 & \overrightarrow{T} = \overrightarrow{S} \\ -1 & \overrightarrow{T} = -\overrightarrow{S} \\ 0 & S \neq T \end{cases}$$

We see that if $\{\overrightarrow{S_i}\}$ is a basis of $C_l(K)$, then $\{\phi_{S_i}\}$ forms the dual basis of $C^l(K)$ corresponding to $\{\overrightarrow{S_i}\}$. Since δ^* is linear, it will be enough to compute δ^* of these basis vectors. We have

$$\delta^*(\phi_S) = \sum_{v, v \notin S} \phi_{v \cup S}$$

Finally, as a piece of notation, we will denote by c^0 the 0-cochain so that $c^0(v) = 1$ for all $v \in V(K)$.

3. A Simplicial Approach to the De Rham Theorem

3.1. **A Simplicial Approach to the De Rham Theorem.** We must first establish more language before we can discuss the De Rham theorem. In particular, we must discuss the notion of a smoothly triangulated manifold.

We say that a *smoothly triangulated manifold* is a triple (M, K, h), where M is a smooth manifold, K is a simplicial complex, and

$$h : [K] \longrightarrow M$$

is a homeomorphism with the following property. If $S \in K$, the map $h|_{\Delta_S} : \Delta_S \to M$ has a smooth extension $\tilde{h}_S$ to a neighborhood of Δ_S, $U \subset \mathrm{Aff}_S$, $\tilde{h}_S : U \to M$. Additionally, we will say that $V(K) = V$ is the vertex set of K

Here I will provide an example with a picture.

Our general goal for the section is, for any smoothly triangulated manifold (M, K, h), to find an isomorphism from $H^l(M)$ to $H^l(K)$. We will do this by first producing a morphism of chain complexes from the De Rham complex to simplicial cohomology complex, and then checking that this morphism induces isomorphisms in cohomology using two lemmas. We will assume that $\dim M = n$

We need to produce a sequence of maps $\mathcal{P}_l : \Omega^l(M) \to C_l^*(K)$ such that

$$\partial^* \circ \mathcal{P}_l = \mathcal{P}_{l+1} \circ d.$$

We define these maps now.

We seek to define a map

$$\mathcal{P}_l : \Omega^l(M) \to C^l(K).$$

To each l-form $\omega \in \Omega^l(M)$, we must associate a linear function from $C_l(K) \to \mathbb{R}$. We do this as follows. Let $\overrightarrow{S} \in C_l(K)$ be an oriented l-simplex. Consider the smooth extension $\tilde{h}_S : U \to M$. Taking the pullback gives us a smooth map $\tilde{h}_S^* : \Omega^l(M) \to \Omega^l(U)$. Thus, for every $\omega \in \Omega^l(M)$, we get a form $\tilde{h}_S^*(\omega)$ which is an l-form on U, which was a neighborhood of Δ_S in Aff_S. We can then define

$$< \mathcal{P}_k(\omega), \overrightarrow{S} >:= \int_{\overrightarrow{\Delta}_S} h_s^*(\omega) =: \int_{\overrightarrow{\Delta}_S} \omega,$$

Where $<,>$ denotes the canonical pairing between a vector space and its dual. These maps are called the *period maps*. We note that we can define all of the above terms on all of $C_l(K)$ by extending by linearity.

Now that we have this definition and can talk about integrating over a simplex, we will state and prove a replacement for Stokes' Theorem, which will be useful to us later.

Lemma 3.1. *Let Δ_n be the standard n-simplex in $\mathbb{R}^n$, i.e. the oriented simplex $[x_0, \ldots, x_n]$ where x_i is the i-th standard basis vector of $\mathbb{R}^n$ and x_0 is the origin in $\mathbb{R}^n$, and let $\omega \in \Omega^{n-1}(M)$. Then we have*

$$\int_{\Delta_n} d\omega = \int_{\partial\Delta_n} \omega.$$

Proof. Let $\omega = \sum_{i=1}^n \omega_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \ldots \wedge dx_n$. Then we have

$$d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{dx_i} dx_1 \wedge \ldots \wedge dx_n$$

We know by definition that

$$\partial([x_0, \ldots, x_n]) = \sum_{i=0}^n [x_0, \ldots, \hat{x}_i, \ldots, x_n]$$

Also, we have for each i, an orientation preserving diffeomorphism

$$\phi_i : \{(t_1, \ldots, t_{n-1}) \in (0,1)^{n-1} : \sum_i t_i \leq 1\} = A \to [x_0, \ldots, \hat{x}_i, \ldots, x_n]$$

Where

$$\phi_i(t_1, \ldots, t_{n-1}) = \begin{cases} (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) & i \neq 0 \\ (1 - t_1 - \ldots - t_{n-1}, t_1, \ldots, t_{n-1}) & i = 0 \end{cases}$$

Using this, we can compute, if $i = 0$

$$\int_{[x_0, \ldots, \hat{x}_i, \ldots, x_n]} \phi_i^*(\omega)$$

$$= \sum_{i=1}^n (-1)^{i-1} \int_A \omega_i(x_1, \ldots, v_{x-1}, 1 - x_1 - \ldots - \hat{x}_i - \ldots - x_n, x_{i+1}, \ldots, x_n) dx_1 \ldots \widehat{dx_i} \ldots dx_n$$

And if $i \neq 0$,

$$\int_A \omega_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) dx_1 \ldots \widehat{dx_i} \ldots dx_n$$

Using all this, we finally conclude

$$\int_{[x_0, \ldots, x_n]} d\omega = \sum_{i=1}^n (-1)^{i-1} \int_A \int_0^{1 - v_1 - \ldots - \hat{v}_i - \ldots - v_n} \frac{\partial \omega_i}{\partial x_i}(x_1, \ldots, x_n) dx_i dx_1 \ldots \widehat{dx_i} \ldots dx_n$$

$$= \sum_{i=1}^{n}((-1)^{i-1}\int_A \omega_i(x_1,\dots,x_{i-1},1-x_1-\dots-\hat{x}_i-\dots-x_n,x_{i+1},\dots,x_n)dx_1\dots\widehat{dx_i}\dots dx_n$$

$$-\sum_{i=1}^{n}((-1)^{i-1}\int_A \omega_i(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)dx_1\dots\widehat{dx_i}\dots dx_n$$

$$= \sum_{i=1}^{n}(-1)^{i-1}\int_A \omega_i(x_1,\dots,x_{i-1},1-x_1-\dots-\hat{x}_i-\dots-x_n,x_{i+1},\dots,x_n)dx_1\dots\widehat{dx_i}\dots dx_n$$

$$+\sum_{i=1}^{n}(-1)^{i}\int_A \omega_i(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)dx_1\dots\widehat{dx_i}\dots dx_n$$

$$= \sum_{i=0}^{n}(-1)^{i}\int_{[x_0,\dots,\hat{x}_i,\dots,x_n]}\phi_i^*(\omega) = \int_{\partial[x_0,\dots,x_n]}\omega$$

□

Proposition 3.2. *The maps $\mathcal{P}_l$ satisfy the relation $\partial^* \circ \mathcal{P}_l = \mathcal{P}_{l+1} \circ d$ for all l.*

Proof. Let ω be a smooth l-form, and let $\overrightarrow{\Delta}_S$ be an oriented $(l+1)$-simplex. Then we have by Stokes' Theorem

$$< \mathcal{P}_{l+1} \circ d(\omega), \overrightarrow{S} >= \int_{\overrightarrow{\Delta}_S}(h_S)^*(d\omega)$$

$$= \int_{\overrightarrow{\Delta}_S} d(h_S^*(\omega)) = \int_{\partial\overrightarrow{\Delta}_S} h_S^*(\omega)$$

$$=< \mathcal{P}_l(\omega), \partial\overrightarrow{S} >=< \partial^* \circ \mathcal{P}_l(\omega), \overrightarrow{S} >$$

Thus, $\partial^* \circ \mathcal{P}_l = \mathcal{P}_{l+1} \circ d$ for all l, as required. □

Therefore, we get maps

$$\boldsymbol{P}_l : H^l(M) \longrightarrow H^l(K)$$

Theorem 3.3 (De Rham Theorem)**.** *$\boldsymbol{P}_l$ is an isomorphism for all l.*

We will prove the De Rham theorem using two lemmas. We first state the lemmas, then use them to prove the De Rham Theorem, then we go back and prove the lemmas.

Lemma 3.4. *There exists a sequence of linear maps*

$$\alpha_l : C^l(K) \to \Omega^l(M)$$

for all $0 \le l \le n$ with the following properties.

(1) $d \circ \alpha_l = \alpha_{l+1} \circ \partial^*$
(2) $\mathcal{P}_l \circ \alpha_l = \mathbb{1}_{C^l(K)}$
(3) $\alpha_0(c^0) = 1$
(4) $\operatorname{supp} \alpha_l(\phi_S) \subset \operatorname{St}(S)$

Lemma 3.5. *Let $\omega \in \Omega^l(M)$ be closed. Suppose $\mathcal{P}_l(\omega) = \partial^*(c)$, for some $c \in C^{l-1}(K)$. Then there exists a $\tau \in \Omega^{l-1}(M)$ so that $d\tau = \omega$ and $\mathcal{P}_{l-1}(\tau) = c$.*

Proof. (De Rham Theorem) We first show that $\boldsymbol{P}_l$ is surjective. Let $z \in C^l(K)$ be a cocycle, i.e. $\partial^*(z) = 0$. Choose a sequence α_l for all $l = 0, \ldots, n$, as in Lemma 3.4, and let $\omega = \alpha_l(z)$. We have

$$d\omega = d(\alpha_l(z)) = \alpha_{l+1}(\partial^*(z)) = \alpha_{l+1}(0) = 0$$

Also, we have

$$\mathcal{P}_l(\omega) = \mathcal{P}_l(\alpha_l(z)) = z$$

Therefore, by passing to cohomology, we clearly see that $\boldsymbol{P}_l(\omega) = z$, so that $\boldsymbol{P}_l$ is surjective for all l.

We now show $\boldsymbol{P}_l$ is injective. It is enough to show that if $\omega \in \Omega^l(M)$ is closed and $\mathcal{P}_l(\omega) = \partial^*(c)$ for some $c \in C^{l-1}(K)$, then there exists a $\tau \in \Omega^{l-1}(M)$ so that $d\tau = \omega$. But this is exactly what we get by Lemma 3.5. Therefore $\boldsymbol{P}_l$ is injective for all l, and we conclude that $\boldsymbol{P}_l$ is an isomorphism for all l.

□

Proof. (Lemma 3.4) Without loss of generality, and for notational convenience, we assume that $[K] = M$ and that $h = \mathbb{1}_M$.

We first construct a partition of unity subordinate to the open cover of M given by the collection

$$\{\mathrm{St}(v) : v \in V\}$$

Consider the functions b_v defined earlier, the barycentric coordinate functions. Consider the sets F_v and G_v defined as follows for all $v \in V$:

$$F_v := \{x \in M : \; b_v(x) \geq \frac{1}{n+1}\} \quad (n = \dim M)$$

$$G_v := \{x \in M : \; b_v(x) \leq \frac{1}{n+2}\}$$

Clearly, F_v and G_v are disjoint closed sets of M with

$$F_v \subset \mathrm{St}(v), \quad M - \mathrm{St}(v) \subset G_v.$$

Hence, for all $v \in V$, we can produce a function $f_v : M \to \mathbb{R}$ such that $f_v(x) > 0$ on F_v, and $f_v(x) = 0$ on G_v. We notice that the collection $\{F_v : v \in V\}$ forms a cover of M. Indeed, if $x \in M$, then there exists an $S \in K$ so that $x \in \dot{\Delta}_S$. We conclude that $b_v(x) = 0$ if $v \in V \setminus S$, and we recall that

$$\sum_{v \in V} b_v(x) = 1$$

But since $S \subset V$, $|S \leq n|$ and we conclude that there exists some $v \in S$ so that $b_v(x) \geq \frac{1}{n+1}$, so that $\{F_v\}$ is an open cover of M. This implies that $M - G_v$ is an open cover of M. The above also shows that for all $x \in M$, there exists a v so that $f_v(x) \neq 0$. We conclude that for all $x \in M$,

$$\sum_{v \in V} f_v(x) > 0.$$

Therefore, we can define functions $g_v : M \to \mathbb{R}$ as

$$g_v(x) = \frac{f_v(x)}{\sum_{v \in V} f_v(x)}$$

But then, by definition of the g_v, we clearly have that $\{g_v\}$ is a partition of unity subordinate to the open cover $\{M \setminus G_v\}$, and therefore it is also subordinate to the open cover

$$\{\mathrm{St}(v) : v \in V\}.$$

Using this partition of unity, we can define our desired sequence of maps,

$$\alpha_l : C^l(K) \to \Omega^l(M).$$

It suffices to define these maps on the generators of $C^l(K)$. If $S \in K$, $|S| = l$, and $\overrightarrow{S} = [v_0, v_1, \dots, v_l]$, then we say

$$\alpha_l(\phi_S) = l! \sum_{i=0}^{l} (-1)^i g_{v_i} dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

It remains to check that these functions satisfy properties $(1) - (4)$

We first prove property (1). We clearly see that

$$d \circ \alpha_l(\phi_S) = (l+1)! dg_{v_0} \wedge \ldots \wedge dg_{v_l}$$

We also have

$$\alpha_{l+1} \circ \partial^*(\phi_S) = \alpha_{l+1}\big(\sum_{v, v \cup S \in K} \phi_{v \cup S}\big)$$

$$= (l+1)! \sum_{v, v\cup S \in K} \left[g_v dg_{v_0} \wedge \ldots \wedge dg_{v_l} - \sum_{i=0}^{l} (-1)^i g_{v_i} dg_v dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l} \right]$$

$$= (l+1)!(I - II)$$

Where I and II are the sums of the previous line. We have the following:

Claim. If $\{v, v_0, \dots, v_l\}$ is not in K, then

$$g_v dg_{v_0} \wedge \ldots \wedge dg_{v_l} = 0 \text{ on } M$$

To see this, assume first that x is not in $\mathrm{St}(v)$, so that $g_v(x) = 0$ and the claim is obvious.

Now, assume $x \in \mathrm{St}(v)$. Then we have $b_v(x) \neq 0$. Therefore, there exists a j so that $b_{v_j}(x) = 0$, since otherwise $\{v, v_0, \dots, v_l\} \in K$. Now define a set

$$U = \{y \in M : b_{v_j}(y) < \frac{1}{n+2}\}$$

Then U is open and $x \in U$. Also, by definition, $g_{v_j} = 0$ on U since $U \subset G_{v_j}$. Therefore, we conclude that dg_{v_j} is zero on U. Hence, $dg_{v_j}(x) = 0$, so that $g_v dg_{v_0} \wedge \ldots \wedge dg_{v_l} = 0$, and the claim is proved.

We now use this claim to rewrite the sums I and II.

First, we consider I. Using the claim, we trivially have

$$I = \sum_{v, v\cup S \in K} g_v dg_{v_0} \wedge \ldots \wedge dg_{v_l} = \sum_{v \notin S} g_v dg_{v_0} \wedge \ldots \wedge dg_{v_l}$$

We now consider II. We have

$$II = \sum_{v, v\cup S \in K} \sum_{i=0}^{l} (-1)^i g_{v_i} dg_v dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

$$= \sum_{i=0}^{l} (-1)^i \sum_{v, v\cup S \in K} g_{v_i} dg_v dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

$$= \sum_{i=0}^{l}(-1)^i \sum_{v \notin S} g_{v_i} dg_v dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

$$= \sum_{i=0}^{l}(-1)^i \sum_{v \neq v_i} g_{v_i} dg_v dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

$$= \sum_{i=0}^{l}(-1)^i g_{v_i} \left(\sum_{v \neq v_i} g_{v_i} \right) \wedge dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

$$= \sum_{i=0}^{l}(-1)^i g_{v_i}(-dg_{v_i}) \wedge dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

$$= -\sum_{i=0}^{l} g_{v_i} dg_{v_0} \wedge \ldots \wedge dg_{v_l}.$$

Note, we have used that

$$\sum_{v \in V} g_v = 1 \Rightarrow \sum_{v \in V} dg_v = 0.$$

Combining, we get

$$\alpha_{l+1} \circ \partial^*(\phi_S) = (l+1)![I - II] = (l+1)!(\sum_{v \in V} g_v) dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

$$= (l+1)! dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l} = d \circ \alpha_l(\phi_S).$$

From this, property (1) follows.

We now check property (3). We note that $\alpha_0(\phi_v) = g_v$. We therefore have

$$\alpha_0(c^0) = \alpha_0 \left(\sum_{v \in V} \phi_v \right) = \sum_{v \in V} g_v = 1$$

We now check property (4). Suppose, as before, that we have

$$S \in K, \overrightarrow{S} = [v_0, \ldots, v_l].$$

Then we have

$$\alpha_l(\phi_S) = l! \sum_{i=0}^{l}(-1)^i(-1)^i g_{v_i} dg_{v_0} \wedge \ldots \wedge \widehat{dg_{v_i}} \wedge \ldots \wedge dg_{v_l}$$

We note that if $x \in M$ has

$$b_{v_k}(x) < \frac{1}{n+2},$$

then $x \in G_{v_k}$, so that g_{v_k} and dg_{v_k} are zero at x, which means $\alpha_l(\phi_S)$ is zero at x. Therefore, $\alpha_l(\phi_S)$ is identically zero on the set

$$\{x \in M : b_v(x) < \frac{1}{n+2}, v \in S\}.$$

But we have that this is an open set containing $M \setminus \mathrm{St}(S)$, so that

$$\operatorname{supp} \alpha_l(\phi_S) \subset \mathrm{St}(S)$$

We now verify property (2). The proof will be by induction on l. If $l = 0$, then we have $\mathcal{P}_0 \circ \alpha_0(\phi_v), v \in V$ is the 0-cochain given by, if $v' \in V$

$$< \mathcal{P}_0 \circ \alpha_0(\phi_v), v' >=< \mathcal{P}_0(g_v), v' >= g_v(v')$$

We see that $g_v(v') = 0$ if $v \neq v'$, since $v' \notin \mathrm{St}(v)$ if $v \neq v'$ and $g_v = 0$ outside of $\mathrm{St}(v)$. Also,

$$1 = \sum_{v \in V} g_v(v') = g_{v'}(v')$$

for all $v' \in V$. We conclude that

$$< \mathcal{P}_0 \circ \alpha_0(\phi_v), v' >= \begin{cases} 1 & v = v' \\ 0 & v \neq v' \end{cases}$$

$$= \phi_v(v')$$

But the above holds for all $v, v' \in K$, so that $\mathcal{P}_0 \circ \alpha_0 = \mathbb{1}$, as required.

Now assume that property (2) holds for $l - 1$. We have that if $S, T \in K$, then

$$< \mathcal{P}_l \circ \alpha_l(\phi_S), \overrightarrow{T} >= \int_{\overrightarrow{\Delta}_T} \alpha_l(\phi_S).$$

Therefore, it suffices to show that the above equals 1 if $\overrightarrow{S} = \overrightarrow{T}$ and equals 0 if $S \neq T$.

We first note that if $S \neq T$, then since

$$\Delta_T \subset M \setminus \mathrm{St}(S)$$

and by property (4), we have that

$$< \mathcal{P}_l \circ \alpha_l(\phi_S), \overrightarrow{T} >= \int_{\overrightarrow{\Delta}_T} \alpha_l(\phi_S) = 0.$$

It remains only to check

$$\int_{\overrightarrow{\Delta}_S} \alpha_l(\phi_S) = 1$$

Now let $\overrightarrow{S} = [v_0, v_1, \ldots, v_l]$, and let $\overrightarrow{R} = [v_1, \ldots, v_l]$, with the $v_i \in V$. We have

$$\int_{\overrightarrow{\Delta}_S} \alpha_l(\partial^* \phi_R) = \int_{\overrightarrow{\Delta}_S} d\alpha_{l-1}(\phi_R) = \int_{\partial \overrightarrow{\Delta}_S} \alpha_{l-1}(\phi_R).$$

But $\partial \overrightarrow{S} = \overrightarrow{R}$ plus some sum of other oriented $(l-1)$-simplices, so that by induction, we have

$$\int_{\partial \overrightarrow{\Delta}_S} \alpha_{l-1}(\phi_R) = \int_{\overrightarrow{\Delta}_R} \alpha_{l-1}(\phi_R) = 1$$

We finally conclude that

$$1 = \int_{\overrightarrow{\Delta}_S} \alpha_l(\partial^* \phi_R) = \int_{\overrightarrow{\Delta}_S} \alpha_l(\phi_S + \text{terms of form } \phi_T, \text{ with } S \neq T)$$

$$= \int_{\overrightarrow{\Delta}_S} \alpha_l(\phi_S),$$

as required. Thus, the proof of Lemma 3.4 is complete □

With this, we now see that the proof of the De Rham Theorem will be finished by a proof of Lemma 3.5. Before we can do this, we must introduce and prove yet another lemma.

Lemma 3.6. *Let S be a k-simplex in $\mathbb{R}^n$*

(a_r) Suppose $r \geq 0$ and $k \geq 1$. Let ω be a smooth, closed r-form defined "near" $\mathrm{Fr}(\Delta_S)$*, i.e. in a neighborhood of* $\mathrm{Fr}(\Delta_S)$*, where* $\mathrm{Fr}(\Delta_S) := \Delta_S - \dot{\Delta}_S$*. If $k = r+1$, further assume that $\int_{\partial\overrightarrow{\Delta}_S} = 0$. Then there exists a smooth closed r-form τ defined near Δ_S so that $\tau = \omega$ near* $\mathrm{Fr}(\Delta_S)$

(b_r) Suppose $r \geq 1$ and $k \geq 1$. Let ω be a smooth closed r-form defined near Δ_S. Suppose τ is a smooth $(r-1)$-form defined near $\mathrm{Fr}(\Delta_S)$ *so that $d\tau = \omega$ near* $\mathrm{Fr}(\Delta_S)$*. If $k = r$, further assume that $\int_{\partial\overrightarrow{\Delta}_S} \tau = \int_{\overrightarrow{\Delta}_S} \omega$. Then there exists a smooth $(r-1)$-form τ' defined near Δ_S so that $\tau' = \tau$ near* $\mathrm{Fr}(\Delta_S)$*, and $d\tau' = \omega$ near Δ_S.*

Proof. The proof will proceed by induction, first showing (a_0), then showing that $(a_{r-1}) \Rightarrow (b_r)$, and finally showing that $(b_r) \Rightarrow (a_r)$.

(a_0): $r = 0$, therefore, ω is a smooth function defined near $\mathrm{Fr}(\Delta_S)$ so that $d\omega = 0$. Therefore, ω is constant on each connected component of its domain. If $k > 1$, then $\mathrm{Fr}(\Delta_S)$ is connected, and hence we have that ω is a constant function, and therefore has an obvious extension to a function near Δ_S. If $k = 1$, then let $\overrightarrow{\Delta}_S =< v_0, v_1 >$, with $v_0, v_1 \in V(S)$. We also have, by assumption

$$0 = \int_{\partial\overrightarrow{\Delta}_S} = \omega(v_1) - \omega(v_0).$$

Therefore, we have $\omega(v_0) = \omega(v_1)$, and once again, ω is constant near $\mathrm{Fr}(\Delta_S)$, so that it has an obvious extension to a function near Δ_S. This proves (a_0).

$(a_{r-1}) \Rightarrow (b_r)$: ω is a closed r-form $(r \geq 1)$ defined on an open set containing Δ_S. By the Poincaré Lemma, we have that ω is exact near Δ_S. That is, there exists a smooth $(r-1)$-form τ_1 defined near Δ_S so that $d\tau_1 = \omega$ near Δ_S. In general, we will not have $\tau_1 = \tau$ near $\mathrm{Fr}(\Delta_S)$. However, we do have that near $\mathrm{Fr}(\Delta_S)$, $\tau_1 - \tau$ is closed. Indeed, we have

$$d(\tau_1 - \tau) = d\tau_1 - d\tau = \omega - \omega = 0.$$

Also, if $k = (r-1)+1 = r$, then we have

$$\int_{\partial\overrightarrow{\Delta}_S} (\tau_1 - \tau) = \int_{\partial\overrightarrow{\Delta}_S} \tau_1 - \int_{\partial\overrightarrow{\Delta}_S} \tau$$
$$= \int_{\overrightarrow{\Delta}_S} d\tau_1 - \int_{\partial\overrightarrow{\Delta}_S} \tau = \int_{\overrightarrow{\Delta}_S} \omega - \int_{\partial\overrightarrow{\Delta}_S} \tau = 0,$$

where the last equality is by hypothesis. Therefore, we can apply (a_{r-1}) to $\tau_1 - \tau$ to get a smooth closed $(r-1)$form μ defined near Δ_S so that $\mu = \tau_1 - \tau$ near $\mathrm{Fr}(\Delta_S)$. Let $\tau' = \tau_1 - \mu$. Then we clearly have that τ' is a smooth closed $(r-1)$-form defined near Δ_S, $\tau' = \tau_1 - \tau_1 + \tau = \tau$ near $\mathrm{Fr}(\Delta_S)$, and, near $\mathrm{Fr}(\Delta_S)$ we have

$$d\tau' = d\tau_1 - d\mu = \omega - 0 = \omega.$$

This completes the proof of (b_r).

$(b_r) \Rightarrow (a_r)$: Let $\overrightarrow{S} = [v_0, \ldots, v_k]$ for some vertices in K, and define a simplex $T \subset S$ as $\overrightarrow{T} = [v_1, \ldots, v_k]$. Furthermore, define F as $\mathrm{Fr}(\Delta_S) \setminus \Delta_T$. Since ω is closed, we can apply the Poincaré Lemma to ω to obtain a smooth $(r-1)$-form μ defined near F so that $d\mu = \omega$. In particular, $d\mu = \omega$ near $\mathrm{Fr}(\Delta_T)$. If $k > 1$, we seek to use (a_r) on μ and ω. We must therefore check that if $k - 1 = r$, then

$$\int_{\overrightarrow{\Delta}_T} \omega - \int_{\partial\overrightarrow{\Delta}_T} \mu = 0$$

Now let $\overrightarrow{C} = \partial\overrightarrow{S} - \overrightarrow{T}$, so that $\partial\overrightarrow{C} = -\partial\overrightarrow{T}$. Then, noting that each simplex of $\overrightarrow{C}$ is contained in F, and that $d\mu = \omega$ near F, we have

$$\begin{aligned}\int_{\overrightarrow{\Delta}_T}\omega - \int_{\partial\overrightarrow{\Delta}_T}\mu &= \int_{\overrightarrow{\Delta}_T}\omega + \int_{\partial\overrightarrow{\Delta}_C}\mu \\ &= \int_{\overrightarrow{\Delta}_T}\omega - \int_{\overrightarrow{\Delta}_C} d\mu \\ &= \int_{\overrightarrow{\Delta}_T}\omega - \int_{\partial\overrightarrow{\Delta}_T}\omega \\ &= \int_{\partial\overrightarrow{\Delta}_S}\omega = 0,\end{aligned}$$

Where the last equality is by hypothesis. Thus, we can apply (b_r) to get a form μ' defined near Δ_T so that $\mu' = \mu$ near $\mathrm{Fr}(\Delta_T)$ and $d\mu' = \omega$ near Δ_T. Thus, we can define a form μ_2 near $\mathrm{Fr}(\Delta_S)$ by gluing together μ' and μ, which we can do because they are equal on their common domain. Clearly, since μ' and μ have the property, we also have $d\mu_2 = \omega$ near $\mathrm{Fr}(\Delta_S)$.

Now let $k = 1$. Then $\mathrm{Fr}(\Delta_S)$ consists of 2 vertices, v_0 and v_1. Since ω is closed, the Poincaré Lemma again guarantees the existence of smooth $(r-1)$-forms μ_i near v_i for $i = 0, 1$, where $d\mu_i = \omega$. Shrinking domains if necessary, we can assume that the domain of μ_0 and the domain of μ_1 are disjoint. This again defines a μ_2 near $\mathrm{Fr}(\Delta_S)$ with $d\mu_2 = \omega$ near $\mathrm{Fr}(\Delta_S)$.

Now, let f be a smooth function which is identically 1 near $\mathrm{Fr}(\Delta_S)$ and is zero outside of the domain of μ_2. Then we have that $f\mu_2$ is a smooth $(r-1)$-form defined near Δ_S. Look at $\tau = d(f\mu_2)$. Clearly, τ is a closed r-form defined near Δ_S, and near $\mathrm{Fr}(\Delta_S)$, we have

$$\tau = d(f\mu_2) = df \wedge \mu_2 + f d\mu_2 = d\mu_2 = \omega$$

since $f = 1$ near $\mathrm{Fr}(\Delta_S)$, and therefore $df = 0$ near $\mathrm{Fr}(\Delta_S)$. This completes the proof of (a_r), and therefore of Lemma 3.6. □

We are now ready to prove Lemma 3.5, and thus finish the proof of the De Rham Theorem.

Proof. (Lemma 3.5) We shall inductively construct a sequence $\tau_0, \ldots, \tau_n = \tau$ of $(l-1)$-forms so so that

(1) τ_k is defined in a neighborhood of $[K^k]$
(2) $d\tau_k = \omega$ near $[K^k]$
(3) $\tau_k = \tau_{k-1}$ near $[K^{k-1}]$
(4) $\mathcal{P}_{l-1}(\tau_{l-1}) = c$

We see that this will complete the proof, since for each oriented $(l-1)$-simplex $\overrightarrow{S}$ of $[K]$ and for each $k \geq (l-1)$, we have

$$< \mathcal{P}_{l-1}(\tau_k), \overrightarrow{S} >= \int_{\overrightarrow{\Delta}_S}\tau_k = \int_{\overrightarrow{\Delta}_S}\tau_{l-1} =< \mathcal{P}_{l-1}(\tau_{l-1}), \overrightarrow{S} >= c(\overrightarrow{S}).$$

so that $\tau = \tau_n$ has all the required properties.

We first construct τ_0. Cover $[K^0]$ by a collection of mutually disjoint balls. Since ω is closed, we get by the Poincaré Lemma that ω is exact on each ball. Therefore, by gluing, we get a form τ_0' defined on the union of these balls so that $d\tau_0' = \omega$. If $l - 1 \neq 0$, we can set $\tau_0 = \tau_0'$ and we are done. Now assume $l-1=0$. We need $\mathcal{P}_0(\tau_0) = c$. If $v \in V = V(K)$, we have

$$< \mathcal{P}_0(\tau_0'), \overrightarrow{v} >= \int_{\overrightarrow{\Delta}_v} \tau_0' = \tau_0'(v)$$

Let $a_v = c(v) - \tau_0'(v)$, and define, near v,

$$\tau_0 = \tau_0' + a_v.$$

Clearly, $d\tau_0 = d\tau_0' = \omega$ near $[K^0]$ and $\mathcal{P}_0(\tau_0) = c$, as required.

Now, assume τ_{k-1} has been constructed with properties $(1)-(4)$. We seek to construct τ_k with properties $(1)-(4)$. Assume that, for all oriented k-simplices $\overrightarrow{S}$, we can find a smooth $(l-1)$-form $\tau_k(S)$ defined near Δ_S so $d(\tau_k(S)) = \omega$ near Δ_S and $\tau_k(S) = \tau_{k-1}$ near $\mathrm{Fr}(\Delta_S)$. By gluing them together, we then get a smooth $(l-1)$ form τ_k' satisfying $(1)-(3)$.

To construct $\tau_k(S)$, we seek to apply (b_l) of Lemma 3.6. Note that ω is a smooth closed l-form defined near Δ_S and that τ_{k-1} is a smooth $(l-1)$-form defined near $\mathrm{Fr}(\Delta_S)$ so that $d\tau_{k-1} = \omega$ near $\mathrm{Fr}(\Delta_S)$. Also, if $k=l$, we have by (4) and by hypothesis

$$\int_{\overrightarrow{\Delta}_S} \omega = \mathcal{P}_l(\omega)(\overrightarrow{\Delta}_S)$$

$$= \partial^* c(\overrightarrow{\Delta}_S) = c(\partial\overrightarrow{\Delta}_S) = \mathcal{P}_{k-1}(\tau_{k-1})(\partial\overrightarrow{\Delta}_S) = \int_{\partial\overrightarrow{\Delta}_S} \tau_{k-1}.$$

Thus, we can apply (b_l) to get a smooth $(l-1)$-form $\tau_k(S)$ defined near Δ_S so that $\tau_k(S) = \tau_{k-1}$ near $\mathrm{Fr}(\Delta_S)$ and $d(\tau_k(S)) = \omega$ near Δ_S.

Thus, we have τ_k' satisfying $(1)-(3)$. If $k \neq l-1$, set $\tau_k = \tau_k'$. Now assume $k = l-1$. We know that τ_{l-1}' satisfies $(1)-(3)$, and we want τ_{l-1} to have $\mathcal{P}_{l-1}(\tau_{l-1}) = c$. Choose a sequence α_l for all $l = 0, \ldots, n$, as in Lemma 3.4, and let $c_1 = c - \mathcal{P}_{l-1}(\tau_{l-1}')$. Then define τ_{l-1} in a neighborhood of $[K^{l-1}]$ by

$$\tau_{l-1} = \tau_{l-1}' + \alpha_{l-1}(c_1).$$

For each r and each oriented r-simplex $\overrightarrow{S}$, we note that $\alpha_r(\phi_S)$ is identically zero in a neighborhood of $M - \mathrm{St}(S)$. In particular, $\alpha_r(\phi_S)$ is zero near $[K^{r-1}]$. Since each $c \in C^r(K)$ is a linear combination of such ϕ_S, we have that $\alpha_r(c)$ is zero near $[K^{r-1}]$ for all r-cochains c.

Applying this with $r = l$ and $r = l-1$, we get that near $[K^{l-1}]$,

$$d\tau_{l-1} = d\tau_{l-1}' + d \circ \alpha_{l-1}(c_1) = d\tau_{l-1}' + \alpha_l \circ \partial^*(c_1) = d\tau_{l-1}' = \omega,$$

and that near $[K^{l-2}]$,

$$\tau_{l-1} = \tau_{l-1}' + \alpha_{l-1}(c_1) = \tau_{l-1}' = \tau_{l-2}.$$

Therefore, τ_{l-1} satisfies $(1)-(3)$. But property (4) is also satisfied. Indeed we have by definition of c_1 and Lemma 3.4

$$\mathcal{P}_{l-1}(\tau_{l-1}) = \mathcal{P}_{l-1}(\tau_{l-1}') - \mathcal{P}_{l-1} \circ \alpha_{l-1}(c_1)$$

$$= (c - c_1) + c = c.$$

Therefore, we have constructed τ_k satisfying properties $(1)-(4)$, as required. $\square$

4. Singular Cohomology Theory

4.1. **Singular Homology and Cohomology.** Our goal this section is to establish the theory of singular cohomology. In order to talk about singular cohomology, we first introduce singular homology.

Let $\mathbb{R}^\infty$ be the space

$$\mathbb{R}^\infty = \bigoplus_{i=0}^{\infty} \mathbb{R}.$$

We topologize $\mathbb{R}^\infty$ by declaring a subset $C \subset \mathbb{R}^\infty$ closed if and only if $C \cap \mathbb{R}^n$ is closed in $\mathbb{R}^n$, for any $n \geq 0$.

Let $P_i \in \mathbb{R}^\infty$ denote the i-th standard basis element, that is the element which is 1 in the i position and 0 everywhere else. We define the *standard q-simplex* Δ_q as

$$\Delta_q := \Big\{ \sum_{j=0}^{q} t_j P_j : \sum_{j=0}^{q} t_j = 1, t_j \geq 0 \Big\}.$$

We see that Δ_q clearly lies in the affine plane $\mathrm{Aff}(\Delta_q)$, defined as follows:

$$\mathrm{Aff}(\Delta_q) = \{\sum_{j=0}^{q} t_j P_j : \quad \sum_{j} t_j = 1\}.$$

Now let M be a smooth manifold. We clearly see that Δ_q is not a smooth manifold, so we cannot a priori talk about a smooth map from Δ_q to M. Therefore, we will say that a map

$$f : \Delta_q \longrightarrow M$$

is smooth if there exists a neighborhood U of Δ_q in $\mathrm{Aff}(\Delta_q)$ and a smooth function $\tilde{f} : U \longrightarrow M$ such that $\tilde{f}|_{\Delta_q} = f$. We now define

$$S_q(M) = C^\infty(\Delta_q, M), \;\; C_q(M) = \bigoplus_{s \in S_q(M)} \mathbb{R}\langle s \rangle.$$

The elements of $S_q(M)$ are called *(smooth) singular q-simplices.* The elements of $C_q(M)$ are called *(smooth) singular q-chains* in M. They are finite linear combinations with integral coefficients of singular q-simplices.

We define the i-th *face map* of the standard q-simplex to be the unique affine function

$$\partial_q^i : \Delta_{q-1} \to \Delta_q, \;\; \partial_q^i(\sum_{j=0}^{q-1} t_j P_j) = \sum_{j=0}^{i-1} t_j P_j + \sum_{j=i+1}^{q} t_{j-1} P_j$$

We can then further define maps ∂_i using these face maps. In particular, if $\phi : \Delta_q \to M$, we can define

$$\partial_i : S_q(M) \to S_{q-1}(M), \;\; \partial_i(\phi) = \phi \circ \partial_q^i.$$

We can use these maps to turn $C_\bullet(M)$ into a chain complex by defining a boundary operator

$$\partial : C_q(M) \to C_{q-1}(M), \;\; \partial(\phi) = \sum_{i=0}^{q} (-1)^i \partial_i(\phi)$$

It can be checked that $\partial^2 = 0$, but we omit the computation here. We will call the homology of this complex the *singular homology* of M with integral coefficients and we denote it by $H_\bullet(M)$.

As an example, we now compute the singular homology of $\mathbb{R}^n$. First, let $s \in S_q(\mathbb{R}^n)$. We can define the *cone* over s to be the $(q+1)$-simplex Ks in $S_{q+1}(\mathbb{R}^n)$ as

$$Ks(\sum_{j=0}^{q+1} t_j P_j) = (1 - t_{q+1}) s(\sum_{j=0}^{q} \frac{t_j}{1 - t_{q+1}} P_j).$$

This is the cone in $\mathbb{R}^n$ with vertex origin and base the simplex s. Intuitively, we can view t_{q+1} as a time variable, and as it varies from 0 to 1, our cone varies from s to the origin.

Proposition 4.1. *Let $K : S_q(\mathbb{R}^n) \to S_{q+1}(\mathbb{R}^n)$ be the cone construction, for any q. This induces a linear map $K : C_q(\mathbb{R}^n) \to C_{q+1}(\mathbb{R}^n)$ satisfying*

$$\partial K - K\partial = (-1)^{q+1}\mathbb{1}$$

Proof. The proof is a simple matter writing out ∂Ks and $K\partial s$ and comparing the two sides. It is left to the reader. □

We see that an immediate consequence of this proposition is that the cone construction K is a homotopy operator between the identity map and the zero map on $S_q(\mathbb{R}^n)$. We see that an immediate consequence of this is

$$H_q(\mathbb{R}^n) = \begin{cases} 0 & q \geq 1 \\ \mathbb{Z} & q = 0 \end{cases}$$

We now discuss singular cohomology. We can define a group

$$C^q(M) = \mathrm{Hom}(C_q(M), \mathbb{R}).$$

The elements of $C^q(M)$ are called *singular q-cochains*. We can also define a coboundary operator ∂^* as follows. If $\omega \in C^q(M)$, we define $\partial^*\omega \in \mathrm{Hom}(C_q(M), \mathbb{R})$ by

$$\partial^*\omega(c) := \omega(\partial c)$$

Then it is clear that $\partial^*\omega \in C^{q+1}(M)$ and that $(\partial^*)^2 = 0$. Thus, we have that $C^\bullet$ is a cochain complex, and thus we can form its cohomology. We will call this the *singular cohomology* of M, and we will denote it by $H^\bullet_{sing}(M)$.

Any 0-cochain can be identified with a (possibly discontinuous function $\omega : M \to \mathbb{R}$. We note that a function ω on M is a 0-cocycle, i.e. $\partial^*\omega = 0$, if and only if $\omega(\partial c) = 0$ for all paths c in M. Thus, ω is constant on each path component of M, and we have

$$H^0_{sing}(M) = \mathbb{R}^\nu,$$

where ν is the number of path components of M.

We now compute the singular cohomology of $\mathbb{R}^n$. Define the operator $L : C^q(\mathbb{R}^n) \to C^{q+1}(\mathbb{R}^n)$ to be the adjoint of the cone operator K. That is, if $\omega \in C^q(\mathbb{R}^n)$, and $c \in C_{q-1}(\mathbb{R}^n)$, then

$$L\omega(c) = \omega(Kc).$$

Then, for $\omega \in C^q(\mathbb{R}^n)$ and $c \in C_q(\mathbb{R}^n)$, we have

$$((\delta L - L\delta)\omega)(c) = (\delta(L\sigma))(c) - (L(d\omega))(c) = L\omega(\partial c) - \delta\omega(Kc)$$
$$= \omega(K\partial c) - \omega(\partial Kc) = \omega((K\partial - \partial K)c) = (-1)^{q+1}\omega(c),$$

where the last equality follows from Proposition 4.1. Therefore

$$\mathbb{1} = (-1)^{q+1}(\delta L - L\delta),$$

so that again the identity map is homotopic to the zero map on $C^q(\mathbb{R}^n)$ for $q \geq 1$. Therefore, we again have

$$H^q_{sing}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q \geq 1 \end{cases}$$

4.2. **The General Mayer-Vietoris Principle.** Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of M, where A is a totally ordered set. We will define the groups $C^{\mathcal{U}}_\bullet(M)$, the group of so-called $\mathcal{U}$-*small chains* in M. This is the free Abelian group generated by the singular chains in M that lie entirely in some open set of the cover $\mathcal{U}$. More precisely, if we define

$$S^{\mathcal{U}}_q(M) = \{s : \Delta_q \to M;\ \ \exists \alpha \in A \ \text{ such that } \ s(\Delta_q) \subset U_\alpha\},$$

then

$$C^{\mathcal{U}}_\bullet(M) := \bigoplus_{s \in S^{\mathcal{U}}_q(M)} \mathbb{Z}\langle s \rangle.$$

The Excision Theorem [3, Prpp. 2.21] implies that the inclusion

$$i : C^{\mathcal{U}}_\bullet(M) \to C_\bullet(M)$$

is a chain homotopy equivalence.

Denote by $N(\mathcal{U})$ the nerve of $\mathcal{U}$ and by $N_p(\mathcal{U})$ the set of p-faces of the nerve. Recall that these are linearly ordered sets

$$\sigma := \{\alpha_0 < \cdots < \alpha_p\}$$

such that

$$U_\sigma := \bigcap_{i=0}^{p} U_{\alpha_i} \neq \emptyset.$$

Following A. Weil [5] we define

$$S_{p,q}(\mathcal{U}) := \big\{(\sigma, s) \in N_p(\mathcal{U}) \times S_q(M);\ \ s(\Delta_q) \subset U_\sigma \big\}$$

and we set

$$C_{p,q}(\mathcal{U}) := \bigoplus_{(\sigma,s) \in S_{p,q}(\mathcal{U})} \mathbb{Z}\langle \sigma, s \rangle.$$

Now define

$$\delta : C_{p,q}(\mathcal{U}) \to C_{p-1,q}(\mathcal{U}),\ \ \delta\langle \sigma, s \rangle = \sum_{i=0}^{p} \langle \delta_i \sigma, s \rangle,$$

where

$$\delta_i(\alpha_0, \ldots, \alpha_p) := (\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_p).$$

We define

$$\partial : C_{p,q}(\mathcal{U}) \to C_{p,q-1}(\mathcal{U}),\ \ \partial\langle \sigma, s \rangle = \sum_{i=0}^{q} (-1)^i \langle \sigma, \partial_i \sigma \rangle.$$

For every $\mathcal{U}$-small singular q-simplex $s : \Delta_q \to M$ we set

$$V(s) := \big\{\alpha \in A;\ \ s(\Delta_q) \subset U_\alpha \big\}.$$

Observe that for any $B \subset V(s)$ we have

$$\bigcap_{\beta \in B} U_\beta \neq \emptyset \Longrightarrow B \in N(\mathcal{U}).$$

This shows that $\mathfrak{Q}_*^{V(s)}$ is a subcomplex of $N(\mathcal{U})$, and moreover, for every $q \geq 0$ we have an *isomorphism of chain complexes*

$$(C_{\bullet,q}(\mathcal{U}), \delta) \cong \bigoplus_{s \in S_q(\mathcal{U})} \big(C_\bullet(\mathfrak{Q}_*^{V(s)}), \delta \big).$$

Observe that we have a natural augmentation

$$\epsilon : C_{0,q}(\mathcal{U}) \to C_q^{\mathcal{U}}(M), \ \ \langle \sigma, s \rangle \mapsto \langle s \rangle, \ \ \forall (\sigma, s) \in S_{0,q}(\mathcal{U}).$$

Using (2.1) we deduce that for every $s \in S_q^{\mathcal{U}}(M)$ we have a long exact sequence

$$0 \longrightarrow \mathbb{Z}\langle s \rangle \xleftarrow{\epsilon} C_0(\mathfrak{Q}_*^{V(s)}) \xleftarrow{\delta} C_1(\mathfrak{Q}_*^{V(s)}) \xleftarrow{\delta} \cdots$$

and thus a long exact sequence

$$0 \longrightarrow \bigoplus_{s \in S_q^{\mathcal{U}}(M)} \mathbb{Z}\langle s \rangle \xleftarrow{\epsilon} \bigoplus_{s \in S_q^{\mathcal{U}}(M)} C_0(\mathfrak{Q}_*^{V(s)}) \xleftarrow{\delta} \bigoplus_{s \in S_q^{\mathcal{U}}(M)} C_1(\mathfrak{Q}_*^{V(s)}) \xleftarrow{\delta} \cdots$$

We have thus proved the following proposition.

Proposition 4.2 (Generalized Mayer-Vietoris Exact Sequence)**.**

$$0 \longleftarrow C_q^{\mathcal{U}}(M) \xleftarrow{\epsilon} C_{0,q}(\mathcal{U}) \xleftarrow{\delta} C_{1,q}(\mathcal{U}) \xleftarrow{\delta} \cdots$$

is an exact sequence for all $q \geq 0$. □

5. André Weil's Approach to the De Rham Theorem

5.1. **Double Complexes.** We now introduce the notion of a *double complex*. First, let $A^{\bullet,\bullet}$ be a doubly graded vector space. That is, we have

$$A^{\bullet,\bullet} = \bigoplus_{m,n} A^{m,n}.$$

Further assume we have two maps, D'_A and D''_A, with the following properties. First, assume that for all p, q, we have

$$D'_A : A^{p,q} \longrightarrow A^{p+1,q}, D''_A : A^{p,q} \longrightarrow A^{p,q+1}.$$

Furthermore, assume that we have the relations

$$D'_A \circ D'_A = D''_A \circ D''_A = 0, \ D'_A \circ D''_A = -D''_A \circ D'_A.$$

Under these circumstances, we say that the triple $(A^{\bullet,\bullet}, D'_A, D''_A)$ form a *double complex*.

We can define the notion of a *morphism of double complexes* in the same was as we defined a morphism of chain complexes. We say that ϕ is a morphism of double complexes if, for double complexes $(A^{\bullet,\bullet}, D'_A, D''_A)$ and $(B^{\bullet,\bullet}, D'_B, D''_B)$

$$\phi : A^{p,q} \longrightarrow B^{p,q}$$

for all p and q, and also

$$\phi \circ D'_A = D'_B \circ \phi, \ \phi \circ D''_A = D''_B \circ \phi.$$

Given any double complex $(A^{\bullet,\bullet}, D'_A, D''_A)$, we can associate a cochain complex called the *total complex* as follows. First, we define the graded vector space $\mathrm{Tot}^\bullet(A)$ as follows:

$$\mathrm{Tot}^n(A) = \bigoplus_{p+q=n} A^{p,q}.$$

We can also define a map $D_A : \mathrm{Tot}^n(A) \longrightarrow \mathrm{Tot}^{n+1}(A)$ by the formula

$$D_A = D'_A + D''_A.$$

It is immediately clear that the relations $D'_A \circ D'_A = D''_A \circ D''_A = 0,\ D'_A \circ D''_A = -D''_A \circ D'_A$ give that $D_A^2 = 0$, so that $(\mathrm{Tot}^\bullet, D_A)$ is a cochain complex. We see that if ϕ is a morphism of double complexes $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$, then we get an induced cochain map

$$\phi : \mathrm{Tot}^\bullet(A) \longrightarrow \mathrm{Tot}^\bullet(B).$$

We note that since D'_A and D''_A are both cochain operators, we can consider the cohomology of columns and rows in our double complex. We set $E_1^{p,\bullet} := H^\bullet(A^{p,\bullet}, D''_A)$. By definition, we see that if we have

$$\phi : A^{\bullet,\bullet} \longrightarrow B^{\bullet,\bullet}$$

a morphism of double complexes then we get an induced morphism

$$\phi_1 : E_1^{p,\bullet}(A) \longrightarrow E_1^{p,\bullet}(B).$$

Theorem 5.1. *Let*

$$\phi : A^{\bullet,\bullet} \longrightarrow B^{\bullet,\bullet}$$

be a morphism of double complexes and

$$\phi_1 : E_1^{p,\bullet}(A) \longrightarrow E_1^{p,\bullet}(B).$$

be the induced morphism. If ϕ_1 is an isomorphism for any p, then the map

$$\phi : \mathrm{Tot}^\bullet(A) \longrightarrow \mathrm{Tot}^\bullet(B)$$

induces isomorphisms in cohomology.

Proof. We follow the approach in [2, Lemma 1.19]. Let $(C^{\bullet,\bullet}, D'_C, D''_C)$ be a double complex with total complex $(\mathrm{Tot}^\bullet(C), D_C)$. Define a subcomplex $F_q^\bullet(C) \subset \mathrm{Tot}^\bullet(C)$ by

$$F_q^l(C) = \bigoplus_{k\geq q} C^{k,l-k}$$

Then clearly, we have

$$\ldots \supset F_{q-1}^\bullet(C) \supset F_q^\bullet(C) \supset F_{q+1}^\bullet(C) \supset \ldots$$

for all l. Also, it is clear that D_C maps $F_q^\bullet(C)$ to itself. From this definition, it is clear that the quotient complex $(F_q^\bullet(C)/F_{q+1}^\bullet(C), D_C)$ is isomorphic to $(C^{\bullet,q}, D'_C)$. Thus, for our map $\phi : A^{\bullet,\bullet} \to B^{\bullet,\bullet}$, we have that the condition that $\phi_1 : E_1^{p,\bullet}(A) \to E_a^{p,\bullet}(B)$ be an isomorphism is equivalent to saying that the map

$$\phi : F_q^\bullet(A)/F_{q+1}^\bullet(A) \to F_q^\bullet(B)/F_{q+1}^\bullet(B)$$

induces isomorphisms in cohomology for all q. We note that we have the following commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 \longrightarrow & F_{q+r}^\bullet(A)/F_{q+r+1}^\bullet(A) & \longrightarrow & F_q^\bullet(A)/F_{q+r+1}^\bullet(A) & \longrightarrow & F_q^\bullet(A)/F_{q+r}^\bullet(A) & \longrightarrow 0 \\
& \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & \\
0 \longrightarrow & F_{q+r}^\bullet(B)/F_{q+r+1}^\bullet(B) & \longrightarrow & F_q^\bullet(B)/F_{q+r+1}^\bullet(B) & \longrightarrow & F_q^\bullet(B)/F_{q+r}^\bullet(B) & \longrightarrow 0
\end{array}$$

Using this diagram, induction on r, and the 5-lemma, we can easily conclude that the map

$$\phi : F_q^\bullet(A)/F_{q+r}^\bullet(A) \to F_q^\bullet(B)/F_{q+r}^\bullet(B)$$

induces isomorphisms in cohomology for all q, for all $r \geq 0$. But, if $C^{\bullet,\bullet}$ is a double complex as before, then we see clearly that

$$F_0^n(C) = \mathrm{Tot}^n(C), \quad F_r^n(C) = 0,$$

where $r > n$. But then we have that

$$\phi : \mathrm{Tot}^\bullet(A) \longrightarrow \mathrm{Tot}^\bullet(B)$$

induces isomorphisms in cohomology, as required. □

We note that by merely switching columns and rows, one can state and prove an equivalent statement using the cohomology of the rows instead of the cohomology of the columns.

5.2. **De Rham Theorem.** Let M be a smooth manifold and let $\mathcal{U}$ be a good cover of the manifold M, and let $N(\mathcal{U})$ denote the nerve of $\mathcal{U}$. Let

$$N_p(\mathcal{U}) = \{\sigma \in N(\mathcal{U}) : |\sigma| = p+1\}$$

be the set of p-simplices of the nerve. As in section 4.1, we have the space $S^q(M)$, and we can form the DeRham cohomology $H_{sing}^\bullet(M)$. Similarly, we have the complex $\Omega^\bullet(M)$ and we can form $H^\bullet(M)$. We define a map

$$\mathcal{P}_M : \Omega^q(M) \longrightarrow S^q(M) = \mathrm{Hom}(S_q(M), \mathbb{R})$$

for all q, called the *period map*, as follows. If $\omega \in \Omega^q(M)$, $S \in S_q(M)$, and $<,>$ denotes the canonical pairing between a space and its dual, then

$$< \mathcal{P}_M(\omega), S >:= \int_{\Delta_q} S^*(\omega).$$

It is easily shown that $\mathcal{P}_M$ is a cochain map. We do not prove this here, but note that the proof is very similar to the proof of Proposition 3.2.

Consider now the double complex

$$C^{p,q}(\mathcal{U}) = \mathrm{Hom}(C_{p,q}(\mathcal{U}), \mathbb{R}) = \mathrm{Hom}\Big(\bigoplus_{(\sigma,s)\in S_{p,q}(\mathcal{U})} \mathbb{Z}\langle(\sigma,s)\rangle, \mathbb{R} \Big),$$

as defined in Section 4.2, and also the double complex $C^{\bullet,\bullet}(\mathcal{U},\Omega)$ defined as follows.

$$C^{p,q}(\mathcal{U},\Omega) := \prod_{\sigma\in N_p(\mathcal{U})} \Omega^q(U_\sigma).$$

The elements of $C^{p,q}(\mathcal{U},\Omega)$ are families $(\omega_\sigma)_{\sigma\in N_p(\mathcal{U})}$, $\omega_\sigma \in \Omega^q(U_\sigma)$.

The differentials

$$D' : C^{p,q}(\mathcal{U},\Omega) \to C^{p+1,q}(\mathcal{U}), \;\; D'' : C^{p,q}(\mathcal{U},\Omega) \to C^{p,q+1}(\mathcal{U})$$

are defined by the equalities .

$$D'(\omega_\sigma)_{\sigma\in N_p(\mathcal{U})} = (\eta_\tau)_{\tau\in N_{p+1}(\mathcal{U})}, \;\; \eta_\tau = \sum_{i=0}^{p+1} (-1)^i \omega_{\delta_i\tau}|_{U_\tau}.$$

$$D''(\omega_\sigma)_{\sigma\in N_p(\mathcal{U})} = ((-1)^p d\omega_\sigma)_{\sigma\in N_p(\mathcal{U})}.$$

It is easy to check that

$$(D' + D'')^2 = (D')^2 = (D'')^2 = 0.$$

Now define another period maps $\mathcal{P} : C^{p,q}(\mathcal{U}, \Omega) \longrightarrow C^{p,q}(\mathcal{U})$ for all p, q as follows. If

$$(\sigma, s) \in S_{p,q}(\mathcal{U}) \quad \text{and} \quad \underline{\omega} := (\omega_\sigma) \in \prod_{\sigma \in N_p(\mathcal{U})} \Omega^q(U_\sigma),$$

then we define

$$\mathcal{P}_{\underline{\omega}}(\langle \sigma, s \rangle) = \int_{\Delta_q} s^*(\omega_\sigma).$$

The map $\mathcal{P}$ is then the natural extension of this to the direct sums. One can easily check that $\mathcal{P}$ is a map of double complexes, again following the methods of Proposition 3.2, so we leave this computation out as well.

We now recall from Section 4.2 that there is an augmentation morphism ϵ

$$\epsilon : C_{0,q}(\mathcal{U}) \longrightarrow C_q^{\mathcal{U}}(M).$$

Taking duals ($\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{R})$), and adjoints, we then get that there exists a map

$$\epsilon^* : C^q_{\mathcal{U}}(M) \longrightarrow C^{0,q}(\mathcal{U}).$$

We recall that the inclusion

$$i : C_\bullet^{\mathcal{U}}(M) \to C_\bullet(M)$$

is a chain map and induces isomorphisms in homology, from which we conclude that

$$i^* : C^\bullet(M) \to C^\bullet_{\mathcal{U}}(M)$$

is a cochain map and induces isomorphisms in cohomology.

Finally, we can define the restriction map

$$r : \Omega^q(M) \to C^{0,q}(\mathcal{U}, \Omega)$$

by sending ω to the collection ω_U, where for each $U \in \mathcal{U}$, $\omega_U := \omega|_U$.

Putting all of this together, we get the following commutative diagram of complexes:

$$\begin{tikzcd}
C^\bullet(M) \arrow[r, "i^*"] & C^\bullet_{\mathcal{U}}(M) \arrow[r, "\epsilon^*"] & C^{\bullet,\bullet}(\mathcal{U}) \\
\Omega^\bullet(M) \arrow[u, "\mathcal{P}_M"] \arrow[r, "r"] & C^{\bullet,\bullet}(\mathcal{U}, \Omega) \arrow[ur, "\mathcal{P}"] &
\end{tikzcd}$$

Passing to total complexes, we get the similar commutative diagram

$$\begin{tikzcd}
C^\bullet(M) \arrow[r, "i^*"] & S^\bullet_{\mathcal{U}}(M) \arrow[r, "\epsilon^*"] & \mathrm{Tot}^\bullet(C(\mathcal{U})) \\
\Omega^\bullet(M) \arrow[u, "\mathcal{P}_M"] \arrow[r, "r"] & \mathrm{Tot}^\bullet(C(\mathcal{U}, \Omega)) \arrow[ur, "\mathcal{P}"] &
\end{tikzcd}$$

Using this, we can now prove the De Rham Theorem.

Theorem 5.2 (De Rham Theorem). *Let M be a smooth manifold. Then the period map $\mathcal{P}_M : \Omega^\bullet(M) \to C^\bullet(M)$ induces isomorphisms in cohomology. In particular, the DeRham cohomology of M is isomorphic with the singular cohomology.*

Proof. Let $\mathcal{U}$ be a good cover of M, $N(\mathcal{U})$ the nerve of $\mathcal{U}$, and $N_p(\mathcal{U})$ the set of p-simplices of the nerve, as before. Then as shown, we get a commutative diagram of cochain complexes

$$\begin{tikzcd} C^\bullet(M) \arrow[r, "i^*"] & C^\bullet_{\mathcal{U}}(M) \arrow[r, "\epsilon^*"] & \mathrm{Tot}^\bullet(C(\mathcal{U})) \\ \Omega^\bullet(M) \arrow[u, "\mathcal{P}_M"] \arrow[r, "r"] & \mathrm{Tot}^\bullet(C(\mathcal{U},\Omega)) \arrow[ur, "\mathcal{P}"] & \end{tikzcd} \tag{5.1}$$

We see that if we can show that in this diagram, i^*, ϵ^*, r, and $\mathcal{P}$ all induce isomorphisms in cohomology, then we can conclude that $\mathcal{P}_M$ induces isomorphisms in cohomology, as required. We already noted that i^* induces isomorphisms in cohomology. We turn to ϵ^*.

View $C^\bullet_{\mathcal{U}}(M)$ as a trivial double complex $A^{\bullet,\bullet}$ where

$$A^{p,q} := \begin{cases} C^q_{\mathcal{U}}(M) := \mathrm{Hom}_{\mathbb{Z}}(C^{\mathcal{U}}_q(M), \mathbb{R}) & p = 0 \\ 0 & p \geq 1 \end{cases}.$$

We recall that by Theorem 4.2 that

$$0 \longleftarrow C^{\mathcal{U}}_q(M) \xleftarrow{\epsilon} C_{0,q}(\mathcal{U}) \xleftarrow{\delta} C_{1,q}(\mathcal{U}) \xleftarrow{\delta} \cdots$$

is an exact sequence, where ϵ is defined in Section 4.2. But, since each term of this is a free abelian group, we conclude by the properties of the Hom functor that

$$0 \longrightarrow C^q_{\mathcal{U}}(M) \xrightarrow{\epsilon^*} C^{0,q} \xrightarrow{\delta^*} C_1, q \xrightarrow{\delta^*} \cdots$$

is an exact sequence, from which it follows that the map of double complexes

$$\epsilon^* : A^{\bullet,\bullet} \longrightarrow C^{\bullet,\bullet}(\mathcal{U})$$

induces isomorphisms in the cohomology of rows. Therefore, we conclude by Theorem 5.1

$$\epsilon^* : \mathrm{Tot}^\bullet(A) \longrightarrow \mathrm{Tot}^\bullet(C(\mathcal{U}))$$

induces isomorphisms in cohomology. But clearly $\mathrm{Tot}^n(A) = C^n_{\mathcal{U}}(M)$, which gives us that

$$\epsilon^* : C^\bullet_{\mathcal{U}}(M) \longrightarrow \mathrm{Tot}^\bullet(C(\mathcal{U}))$$

induces isomorphisms in cohomology.

We now show that r induces isomorphisms in cohomology. View $\Omega^\bullet(M)$ as a trivial double complex $A^{\bullet,\bullet}$ where

$$A^{p,q} := \begin{cases} \Omega^q(M) & p = 0 \\ 0 & p \geq 1 \end{cases}.$$

According to [1, Prop. 8.5] the sequence below is exact.

$$0 \longleftarrow \Omega^q(M) \xleftarrow{r} C^{0,q}(\mathcal{U},\Omega) \longleftarrow C^{1,q,\Omega} \longleftarrow \qquad \cdots$$

From this, we see directly that

$$r : A^{\bullet,\bullet} \longrightarrow C^{\bullet,\bullet}(\mathcal{U},\Omega)$$

induces isomorphisms in the cohomologies of rows. But then by Theorem 5.1,

$$r : \mathrm{Tot}^\bullet(A) \longrightarrow \mathrm{Tot}^\bullet(C(\mathcal{U},\Omega))$$

induces isomorphisms in cohomology. But clearly $\mathrm{Tot}^n(A) = \Omega^n(M)$, which gives us that

$$r : \Omega^\bullet(M) \longrightarrow \mathrm{Tot}^\bullet(C(\mathcal{U}, \Omega))$$

induces isomorphisms in cohomology.

It remains only to show that $\mathcal{P}$ induces isomorphisms in cohomology. We see from Lemma 5.1 that if we show

$$\mathcal{P} : C^{\bullet,\bullet}(\mathcal{U}, \Omega) \longrightarrow C^{\bullet,\bullet}(\mathcal{U})$$

induces isomorphisms in the cohomologies of columns, then we will be done. In particular, by passing through the direct sum, we see that it is enough to show that if $p \geq 0$, $\sigma \in N_p(\mathcal{U})$,

$$\mathcal{P} : \Omega^\bullet(U_\sigma) \longrightarrow C^\bullet(U_\sigma)$$

induces isomorphisms in cohomology. Since $\mathcal{U}$ is a good cover, we know that for all $\sigma \in N(\mathcal{U})$, U_σ is diffeomorphic to $\mathbb{R}^n$ for some n. In particular, we know by the Poincaré Lemma and by computation in section 4.1 that

$$H^q_{sing}(U_\sigma) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q \geq 1, \end{cases}$$

$$H^q(\Omega^\bullet(U_\sigma)) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q \geq 1 \end{cases}.$$

Hence

$$E_1^{p,q}\big(\, C^{\bullet,\bullet}(\mathcal{U}, \Omega)\,\big) = \begin{cases} 0 & q > 0 \\ \prod_{\sigma \in N_p(\mathcal{U})} \mathbb{R} & q = 0. \end{cases} = E_1\big(\, C^{\bullet,\bullet}(\mathcal{U})\,\big)$$

Recall that a singular 0-cocycle can be identified with a constant function on U_σ, and similarly a closed 0-form can also be identified with a constant function. This shows that the period map induces an isomorphism

$$\mathcal{P} : E_1^{p,q}\big(\, C^{\bullet,\bullet}(\mathcal{U}, \Omega)\,\big) \to E_1\big(\, C^{\bullet,\bullet}(\mathcal{U})\,\big)$$

and thus the map

$$\mathcal{P} : \mathrm{Tot}^\bullet(C(\mathcal{U}), \Omega) \longrightarrow \mathrm{Tot}^\bullet(C(\mathcal{U}))$$

induces isomorphisms in cohomology, and furthermore. From our commutative diagram (5.1) we finally conclude that

$$\mathcal{P}_M : \Omega^\bullet(M) \longrightarrow C^q(M)$$

induces isomorphisms in cohomology. □

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