On the curvature of singular complex hypersurfaces

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Abstract

We study the behavior of the Gauss-Bonnet integrand on the level sets of a holomorphic function in a neighborhood of an isolated critical point. This is a survey of some older results of Griffiths, Langevin, Lê and Teissier. It is a blend of classical integral geometry and complex Morse theory (a.k.a Picard-Lefschetz theory).

Motivation

Consider the family of plane complex curves

$$C_t = \bigl\{ (x,y)) \in \mathbb{C}^2;\;\; xy = t,\;\; |x|^2 + |y|^2 \leq 1 \bigr\},\;\; |t| \ll 1.$$

C_t is non-singular for $t \neq 0$, while for $t = 0$ the complex curve C_0 consists of the two plane disks

$$D_x = \bigl\{ (x,0);\;\; |x| = 1 \bigr\},\;\; D_y = \bigl\{ (0,y);\;\; |y| \leq 1 \bigr\}.$$

Denote by g_t the metric on C_t induces by the Euclidean metric on $\mathbb{C}^2$. The boundary of C_t is

$$\partial C_t := C_t \cap S_1(0),$$

where $S_r(p)$ denotes the sphere of radius r centered at $p \in \mathbb{C}^2$. Observe that ∂C_t consists of two boundary components corresponding to the two solutions of the equation (see Figure 1)

$$\sqrt{\rho^2 + \frac{|t|^2}{\rho^2}} = 1 \Longleftrightarrow \rho^4 + |t|^2 = \rho^2,\;\; \rho > 0.$$

For $t \neq 0$ the Riemann surface C_t is homotopy equivalent with the *vanishing circle* (see Figure 1)

$$\delta_t = \bigl\{ (x,y) \in C_t;\;\; |x| = |y| = \sqrt{|t|} \bigr\}.$$

Thus

$$\chi(C_t) = \chi(\delta_t) = 0.$$

Clearly $\chi(C_0) = \chi(\mathrm{pt}) = 1$ so that

$$\lim_{t \to 0} \chi(C_t) \neq \chi(C_0).$$

Denote by K_t the sectional curvature of the Riemann surface (C_t, g_t) and by κ_t the geodesic curvature of $\partial C_t \hookrightarrow C_t$ (see [9, vol 3, Chap. 4] or [10, §4.1] for a definition of

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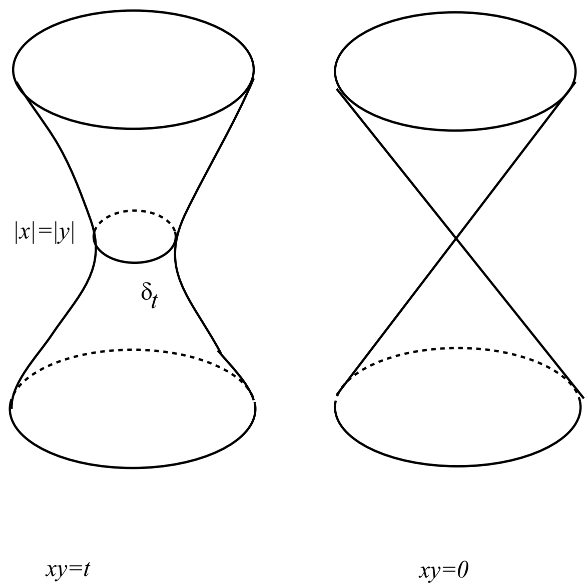

Figure 1: *A family of degenerating plane curves*

the geodesic curvature). Note that $K_0 \equiv 0$ and $\kappa_0 \equiv 1$. The Gauss-Bonnet theorem states that

$$\frac{1}{2\pi}\int_{C_t} K_t dV_t + \frac{1}{2\pi}\int_{\partial C_t} \kappa_t ds = \chi(V_t), \;\; \forall t \neq 0$$

For every $r > 0$ set

$$C_t(r) := \{(x,y) \in C_t); \;\; |x|^2 + |y|^2 \leq r\}.$$

For fixed $\varepsilon > 0$ we have

$$\lim_{t\to 0}\int_{C_t\setminus C_t(\varepsilon)} K_t dV_{g_t} = \int_{C_0\setminus C_0(\varepsilon)} K_0 dV_0 = 0.$$

On the other hand

$$\lim_{t\to 0}\int_{\partial C_t} \kappa_t ds = \int_{\partial C_0} \kappa_0 ds = \text{length}\,(\partial C_t) = 4\pi.$$

We have

$$0 = 2\pi\chi(C_t) = \int_{C_t(\varepsilon)} K_t dV_t + \int_{C_t\setminus C_t(\varepsilon)} K_t dV_t + \int_{\partial C_t} \kappa_t ds.$$

If we let $t \to 0$ we deduce that

$$\frac{1}{2\pi}\lim_{t\to 0}\int_{C_t(\varepsilon)} K_t dV_t = -2, \;\; \forall \varepsilon > 0. \qquad (*)$$

The above observations show that the curvature of C_t concentrates in the region $C_t(\varepsilon)$ whose area is of the order $2\pi\varepsilon^2$. Thus the average of the curvature over this region is $\approx -\frac{1}{\pi\varepsilon^2}$.

To put the equality $(*)$ in some perspective let us rewrite it as

$$\lim_{t\to 0}\int_{C_t} K_t dV_t - \int_{C_0\setminus 0} K_0 dV_0 = \lim_{\varepsilon\to 0}\lim_{t\to 0}\int_{C_t(\varepsilon)} K_t dV_t = -2. \qquad (**)$$

This shows two things. First, the integral of the curvature on C_t does not converge to the integral of the curvature on C_0 as one would expect to be the case if C_0 were smooth. Second, the difference between the limit and the actual integral over the singular level set is an integer.

We see that the limit has a topological meaning! To explain it consider the restriction of the polynomial $f(x, y) = xy$ to a generic line $y = mx$. It is $f^{(1)}(x) = mx^2$. Observe that

$$\chi := \chi(f = t) = 0, \quad \chi^{(1)} := \chi(f^{(1)} = s) = 2,$$

and we can rephrase $(**)$ as

$$\lim_{t \to 0} \int_{\{f=t\}} K_t dV_t - \int_{\{f=0\}} K_0 dV_0 = \lim_{\varepsilon \to 0} \lim_{t \to 0} \int_{\{f=t\} \cap B_\varepsilon(0)} K_t dV_t = \chi - \chi^{(1)}. \qquad (\dagger)$$

We want to show that a similar result holds for any polynomial f in any number of variables with an isolated singularity at the origin, where K_t is replaced by the Gauss-Bonnet integrand and $f^{(1)}$ is replaced by the restriction of f to a generic hyperplane.

Contents

1 A Crofton type formula

For $0 \leq k \leq n$ denote by $G_k := G_k(\mathbb{P}^n)$ the grassmanian of k-dimensional projective subspaces of $\mathbb{P}^n$. We have

$$G_k(\mathbb{P}^n) \cong G_{k+1}(\mathbb{C}^{n+1}).$$

The $U(n+1)$ acts on G_k is a symmetric space with isometry group isomorphic to $U(n+1)$. Denote by dS the unique $U(n+1)$-invariant measure on G_k with total volume 1.

Denote by Ω_H the Fubini-Study form on $\mathbb{P}^n$ normalized as in [4, p.30-31]. Observe that for every k-dimensional projective subspace $S \subset \mathbb{P}^n$ we have

$$\int_S \Omega_H^k = 1.$$

Theorem 1.1 (Crofton formula). *Suppose V is a bounded open subset of $\mathbb{C}^{n-k}$ and $F : V \to \mathbb{P}^n$ is a holomorphic map. Then*

$$\int_V F^* \Omega_H^{n-k} = \int_{G_k(\mathbb{P}^n)} \#(F(V) \cap S) dS. \qquad (1.1)$$

Remark 1.2. The above identity can be loosely rephrased as saying that Ω_H^{n-k} computes the average of intersection of k-planes per unit of $(n-k)$-dimensional volume. If we choose local holomorphic coordinates $(z_1, \cdots, z_n)$ and we choose V to be the very small piece of surface $z^n = \cdots = z_{n-k+1} = 0$ of size dz_i, $1 \leq i \leq n-k$, then we deduce that the quantity

$$\Omega_H^{n-k}(\partial_{z_1}, \partial_{\bar{z}_1}, \cdots, \partial_{z_{n-k}}, \partial_{\bar{z}_{n-k}}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{n-k} \wedge d\bar{z}_{n-k}$$

is the average number of intersections of the projective k-planes with the $(n-k)$-dimensional patch

$$[z_1, z_1 + dz_1] \times \cdots [z_{n-k}, z_{n-k} + dz_{n-k}] \times \{z_{n=k+1} = \cdots = z_n = 0\}$$

□

Proof of Crofton's formula We will follow the approach in [3, p. 475-478].

Fix a k-plane $S_0 \hookrightarrow \mathbb{P}^n$. Then the cohomology class dual to S_0 is Ω_H^{n-k} that is

$$\int_{S_0} \alpha = \int_{\mathbb{P}^n} \alpha \wedge \Omega_H^{n-k}, \;\; \forall \alpha \in \Omega^k(\mathbb{P}^n), \;\; d\alpha = 0.$$

Denote by G_0 the stabilizer of S_0 with respect to the $U(n+1)$ action on $\mathbb{P}^n$. Note that

$$G_0 \cong U(k+1) \times U(n-k) \hookrightarrow U(n+1).$$

The region $\mathbb{P}^n \setminus S_0$ is G_0-invariant and so is the form Ω_H^{n-k}. Denote by $N_r(S_0)$ the tube of radius r around S, i.e.

$$N_r(LS_0) = \{p \in \mathbb{P}^n; \;\; \text{dist}\,(p, S_0) < r\},$$

where the distance is measured with respect to the Fubini-Study metric. For $r \ll 1$ this tube is G_0-invariant. Choose a closed $2(n-k)$-form δ_0^r supported in $N_r(L_0)$ representing the Poincaré dual of L_0. For the existence of such forms we refer to [8, Lemma 7.3.10]. Averaging over G_0 we can assume that δ_0^r is also G_0-invariant. We can thus find a smooth $(2n-3)$-form η_0^r such that

$$d\eta_0^r = \Omega_H^{n-k} - \delta_0^r.$$

Averaging the above equality over G_0 we can assume that η_0^r is G_0-invariant as well.

Observe that if S is another k-plane in $\mathbb{P}^n$ and $g, h \in U(n+1)$ are such that $gS = hS = S_0$ then $gh^{-1} \in G_0$ so that

$$g^*\delta_0^r = h^*\delta_0^r, \;\; g^*\eta_0^r = h^*\eta_0^r,$$

so that these differential forms depend only on the plane S. We denote them by δ_S^r and η_S^r. The resulting correspondences $S \mapsto \delta_S^r, \eta_S^r$ a $U(n+1)$-equivariant, i.e.

$$\delta_{g^{-1}S} := g^*\delta_S, \;\; \eta_{g^{-1}S} := g^*\eta_S, \;\; \forall g \in U(n+1).$$

If we denote by G_S the stabilizer of the k-lane S we deduce that for every S and every $r \ll 1$ the form η_S^r is G_S-invariant, it is supported inside $N_r(S)$ and satisfies the equality

$$d\eta_S^r = \Omega_H^{n-k} - \delta_S^r. \tag{1.2}$$

Consider now manifold

$$\mathcal{Z} := V \times \mathbb{P}^n \times G_k(\mathbb{P}^n)$$

and the submanifolds

$$\Gamma_F := \{(v, F(v), S) \in \mathcal{Z}; \;\; v \in V\}, \;\; I := \{(v, p, S) \in \mathcal{Z}; \; p \in S\}.$$

Then $\Gamma_F \pitchfork I$ and we denote their intersection by $\mathcal{Y}$. We set $\pi_0 := \pi\mid_{\mathcal{Y}}$ where $\pi : \mathcal{Z} \to G_k(\mathbb{P}^n)$ is the natural projection. Then for $S \in G_k(\mathbb{P}^n)$

$$\pi_0^{-1}(S) := \{(v,p,S) \in \mathcal{Z};\ \ p = F(v),\ \ p \in S\} \cong F^{-1}(S)$$

We deduce that $\#\pi_0^{-1}(S) = \#F(V) \cap S$. From Sard's theorem we deduce that the critical set of π_0 has zero measure, that is

$F(V)$ *intersects almost all k-planes transversally.*

Observe next that both sides of (1.1) are additive with respect to partitions of V so that upon subdividing we may assume it is a complex submanifold with boundary. Suppose S is a k-plane which intersects $F(V)$ transversally. Fix r sufficiently small such that

$$V \cap F^{-1}(N_r(S)) \subset \mathrm{int}\,(V).$$

Since δ_S^r represents the Poincaré dual of S and is supported in a very thin neighborhood of S we deduce from [8, Lemma 7.3.12] that

$$\int_V F^*\delta_S^r = \#(F(V) \cap S).$$

Integrating (1.2) over V we deduce

$$\int_V F^*\Omega_H^{n-k} = \int_{\partial V} F^*\eta_S^r + \#(V \cap S).$$

The above equality is valid for almost all k-planes S. Hence

$$\int_{G_k(\mathbb{P}^n)} \Big(\int_V F^*\Omega_H^{n-k}\Big) dS = \int_{G_k(\mathbb{P}^n)} \#(F(V) \cap S) dS + \int_{G_k(\mathbb{P}^n)} \Big(\int_{\partial V} F^*\eta_S^r\Big) dS$$

so that

$$\int_V F^*\Omega_H^{n-k} = \int_{G_k(\mathbb{P}^n)} \#(F(V) \cap S) dS + \int_{G_k(\mathbb{P}^n)} \Big(\int_{\partial V} F^*\eta_S^r\Big) dS. \tag{1.3}$$

Now observe that up to a multiplicative constant C we have

$$\int_{G_k(\mathbb{P}^n)} \Big(\int_{\partial V} F^*\eta_S^r\Big) dS = \int_{\partial V} F^*\Big(\int_{G_k(\mathbb{P}^n)} \eta_S^r dS\Big)$$

$$= C\int_{\partial V} F^* \int_{U(n+1)} \Big(\int_{\partial V} g^*\eta_0^r\Big) dg = C\int_{\partial V} F^* \underbrace{\Big(\int_{U(n+1)} g^*\eta_0^r dg\Big)}_{:=\langle \eta^r \rangle}.$$

$\langle \eta^r \rangle$ is a smooth, odd-degree, $U(n+1)$-invariant form on the symmetric space $\mathbb{P}^n = U(n+1)/(U(1) \times U(n))$. The space of invariant forms on a compact symmetric space is isomorphic to the deRham cohomology of the space (see [8, §7.4]). On our symmetric space the deRham cohomology vanishes in odd degrees so that

$$\langle \eta^r \rangle \equiv 0.$$

Using this information in (1.3) we obtain Crofton formula.

□

Let us say a few words about integration on analytic subvarieties. Suppose V is an n-dimensional complex subvariety defined in an open subset $U \subset \mathbb{C}^N$. Denote by V^* the smooth part of V and by V_{sing} its singular part. We denote by $\Omega_c^k(U)$ the vector space of compactly supported, complex valued k-forms on U. We have the following result. For a proof we refer to [4, p. 31-33].

Theorem 1.3 (Lelong). *V defines a closed (n,n)-current, i.e the following hold.*
(i) For any $2n$-form $\alpha \in \Omega_c^{2n}(U)$ the integral $\int_{V^*} \alpha$ is absolutely convergent.
(ii) For any $\alpha \in \Omega_c^{2n-1}(U)$ we have

$$\int_{V^*} d\alpha = 0.$$

(iii) If $\alpha \in \Omega_c^{p.2n-p}(U)$, $p \neq n$ then

$$\int_{V^*} \alpha = 0.$$

We will denote by $[V]$ the current of integration defined in the above theorem.

2 Chern forms of smooth submanifolds in $\mathbb{C}^N$

Suppose $M^n \hookrightarrow \mathbb{C}^N$ is a smooth complex submanifold in $\mathbb{C}^N$. As such it is equipped with a natural Kähler metric. Let $F_M \in \Omega^{1,1}(T^{1,0}M)$ denote the curvature of the associated Chern connection on the holomorphic tangent bundle $T^{1,0}M$. The Chern forms $c_k(M)$ are then defined by the equality

$$c_t(M) := \sum_{k=0}^{n} c_k(M)t^k = \det(1 + \frac{t\mathbf{i}}{2\pi}F_M), \ \ \mathbf{i} := \sqrt{-1}.$$

In particular $c_n(M)$ coincides with the Euler form of M with the induced Riemann metric.

On the other hand we have a *Gauss map*

$$\mathcal{G}_M : M \to G_n(\mathbb{C}^N) = \text{the grassmanian of } n\text{-dimensional subspaces in } \mathbb{C}^N.$$

We denote by $E_n \to G_n(\mathbb{C}^N)$ the tautological vector bundle. It is equipped with a natural hermitian metric, and we denote by F_n its curvature. Define the Chern forms as before

$$\sum_{k=1}^{n} c_k(G_n)t^k = \det(1 + \frac{t\mathbf{i}}{2\pi}F_n).$$

We have the following result,[3, §3].

Theorem 2.1 (Theorema Egregium - The complex case).

$$\mathcal{G}_M^* c_t(G_n) = c_t(M). \tag{2.1}$$

The above theorem has one interesting consequence.

Proposition 2.2. *Suppose $V^n \hookrightarrow \mathbb{C}^N$ is a pure n-dimensional complex subvariety of $\mathbb{C}^N$. Denote by V_{reg} the regular part of V and by $c_n(V_{reg})$ the n-th Chern class (with respect to the induced Kähler metric. Then for any open set $U \subset V_{reg}$ which is bounded in $\mathbb{C}^N$ we have*

$$\int_U c_n(V_{reg}) < \infty.$$

Remark 2.3. Observe that the conclusion of the above proposition does not follow from Lelong's theorem since $c_n(V_{reg})$ is defined only on V_{reg}. To apply Lelong's theorem we need to know that $c_n(V_{reg})$ is the restriction to V_{reg} of an (n,n)-form defined in some open neighborhood of V in $\mathbb{C}^N$. It is not at all obvious that this is indeed the case.

□

Proof Consider the Gauss map

$$\mathcal{G} : V_{reg} \to G_n(\mathbb{C}^N).$$

Its graph is an n-dimensional subvariety $\Gamma \subset \mathbb{C}^N \times G_n(\mathbb{C}^N)$

$$\Gamma = \big\{(p, E) \in \mathbb{C}^N \times G_n(\mathbb{C}^N);\ \ p \in V_{reg},\ \ T_p V_{reg} = E\big\}.$$

Denote by $\imath$ the obvious inclusion $\imath : V_{reg} \hookrightarrow \Gamma$ and by π the obvious projection

$$\pi : \mathbb{C}^N \times G_n(\mathbb{C}^N) \twoheadrightarrow G_n(\mathbb{C}^N).$$

Observe that $\mathcal{G} = \pi \circ \imath$ so that $\mathcal{G}^* = \imath^* \circ \pi^*$. The form $\pi^* c_n(G_n) \in \Omega^{n,n}(\mathbb{C}^N \times G_n(\mathbb{C}^N))$ is locally integrable along Γ by Lelong's theorem.

Let $U \subset V_{reg}$ be a bounded open subset and set $\hat{U} = \imath(U) \subset \Gamma$. Then

$$\int_U c_n(V_{reg}) \stackrel{(2.1)}{=} \int_U \mathcal{G}^* c_n(G_n) = \int_U \imath^*(\pi^* c_n(G_n))$$

$$= \int_{\hat{U}} \pi^* c_n(G_n) < \infty.$$

□

We conclude this section with a discussion of a rather confusing issue. Consider a complex $n+1$-dimensional vector space V and denote by $G_k(V)$ the grassmanian of k-dimensional subspaces of V. We have a natural biholomorphic map

$$G_k(V) \to G_{n+1-k}(V^*),\ \ V \supset E \mapsto E^0 := \{v^* \in V^*;\ \ v^*(e) = 0,\ \ \forall e \in E\} \subset V^*.$$

In particular we have a natural biholomorphic map

$$\delta : \mathbb{P}(V^*) \cong G_1(V^*) \to G_n(V).$$

We denote by $E_n = E_n(V)$ the tautological vector bundle over $G_n(V)$. E_n is a subbundle of the trivial bundle $\underline{V} \cong \underline{\mathbb{C}}^{n+1}$ and we set $Q_n = Q_n(V) := \underline{V}/E_n$. Denote by $E_1 = E_1(V^*)$ the tautological line bundle over $\mathbb{P}(V^*)$.

Lemma 2.4.

$$\delta^* Q_n \cong E_1^*.$$

Proof The map δ has the form $\ell \mapsto \ell^0$ for any line in V^*. We need to produce a map $\Delta : E_1^* \to Q_n$ such that the diagram below is commutative

$$\begin{array}{ccc} E_1^* & \stackrel{\Delta}{\longrightarrow} & Q_n \\ \downarrow & & \downarrow \\ \mathbb{P}(V^*) & \stackrel{\delta}{\longrightarrow} & G_n(V) \end{array}$$

and which is linear along the fibers.

More precisely, for any line $\ell \subset V^*$ the restriction on Δ to the fiber of E_1^* over ℓ coincides with the tautological isomorphism

$$\Delta_\ell : \ell^* \to V/\ell^0$$

defined by the natural pairing

$$\ell \times V/\ell^0 \to \mathbb{C}.$$

□

From the short exact sequence of vector bundles over $G_n(V)$

$$0 \to E_n \to \underline{V} \to Q_n \to 0$$

we deduce

$$1 = c(\underline{V}) = c(E_n)c(Q_n).$$

Denote by H the image of the hyperplane class in $H^2(\mathbb{P}(V^*))$ via $(\delta^{-1})^*$. Lemma 2.4 now implies

$$1 = c(E_n)(1 + H^*) \Longrightarrow c(E_n) = \sum_{k \geq 0} (-1)^k H^k.$$

In particular

$$c_n(E_n) = (-1)^n H^n \tag{2.2}$$

3 The Langevin formula

To formulate and prove our promised generalization of (†) we need to review some facts concerning the isolated singularities of complex hypersurfaces.

Suppose $f = f(z_0, z_1, \cdots, z_n)$ is a holomorphic function of $n+1$-variables defined in a neighborhood of the origin such that the origin $0 \in \mathbb{C}^{n+1}$ is an isolated critical point. Denote by $\mathcal{O}_0 = \mathcal{O}_{\mathbb{C}^{n+1},0}$ the ring of germs at $0 \in \mathbb{C}^{n+1}$ of holomorphic functions and by $\mathfrak{m}$ its maximal ideal.

The origin is an isolated point of the analytic set

$$\Delta := \{\frac{\partial f}{\partial z_0} = \frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_n} = 0\}.$$

If we denote by J_f the ideal in $\mathcal{O}_0$ generated by $\{\frac{\partial f}{\partial z_i};\ \ 0 \leq i \leq n\}$ we deduce from the analytic Nullstelensatz, that

$$\sqrt{J_f} = \mathfrak{m}$$

so that $\mathfrak{m}^\nu \subset J_f$ for some integer ν and thus

$$\dim_{\mathbb{C}} \mathcal{O}_0/J_f < \infty.$$

This integer is called the *Milnor number* of f at 0 or the Milnor number of the hypersurface X_0 at 0. It is denoted by $\mu(f, 0)$ or $\mu(X_0, 0)$.

For $t \in \mathbb{C}$, $|t| \ll 1$, and any open set U we set

$$X_t = \{\vec{z} \in \mathbb{C}^{n+1};\ \ f(\vec{z}) = t\},\ \ X_t^U := X_t \cap U.$$

We have the following fundamental result.

Theorem 3.1 (Milnor). *For every neighborhood U of $0 \in \mathbb{C}^{n+1}$ there exists $\tau = \tau(U) > 0$ such that for all $0 < |t| < \tau$ the hypersurface X_t^U is homotopic to a wedge of μ spheres of dimension n,*

$$X_t^U \simeq \underbrace{S^n \vee \cdots \vee S^n}_{\mu}.$$

We have

$$X_t^U \cong_{C^\infty} X_t^V, \ \ \forall 0 < |t| \min(\tau(U), \tau(V)).$$

The manifold X_t^U is called the Milnor fiber of f.

For a proof we refer to the beautiful monograph [7].

Example 3.2. Consider polynomial $f = f_{p,q}(x, y) = x^p - y^q$, $p > q$, $\gcd(p, q) = 1$. Then $J_f = (x^{p-1}, y^{p-1})$ and we see that any germ at 0 of holomorphic function is congruent modulo J_f to a unique polynomial of the form

$$\sum_{0 \leq i < p-1,\ 0 \leq j < q-1} a_{ij} x^i y^j.$$

this shows

$$\mu(f, 0) = (-1)(q-1).$$

□

Denote by G the grassmanian $G_n(\mathbb{C}^{n+1})$ of hyperplanes of $\mathbb{C}^{n+1}$ containing the origin. For every neighborhood U of the origin we now construct a parameterized Gauss map

$$\mathcal{G} = \mathcal{G}_U : U^* := U \setminus 0 \to G, \ \ p \mapsto T_p X_{f(p)},$$

and denote by $\Gamma = \Gamma_U$ its graph

$$\Gamma = \Big\{ (p, H) \in U \times G;\ \ p \neq 0, \ \ H = T_p X_{f(p)} \Big\}.$$

The function f induces a natural map

$$\pi_f : U \times G \to \mathbb{C}, \ \ (p, H) \mapsto f(p)$$

Denote by $\hat{\Gamma}$ the closure of Γ in $\mathcal{Y}$. It is an analytic subspace (see [12, §16]). Set $\Gamma_t := \Gamma \cap \pi_f^{-1}(t)$, $\hat{\Gamma}_t := \pi_f^{-1}(t) \cap \hat{\Gamma}$. More precisely Γ_t is the graph of the Gauss map

$$X_t \setminus 0 \to G.$$

Denote by $\bar{\Gamma}_0$ the closure of Γ_0 in $U \times G$. $\bar{\Gamma}_0$ is called the *Nash blowup of* X_0. The set

$$\mathcal{L} := \bar{\Gamma}_0 \setminus \Gamma_0 \subset \{0\} \times \mathbb{P}^n$$

is called the *space of limits of tangents planes*.

Remark 3.3. More accurately, we denote by $\pi_f : \mathbf{Bl}_{J_f}(\mathbb{C}^{n+1}) \to \mathbb{C}^{n+1}$ the blowup of $\mathbb{C}^{n+1}$ along the scheme defined by the Jacobian ideal J_f (see [2, Prop. IV.22, p.169]). Then $\hat{\Gamma}_0$ which is the *total transform* of X_0, and $\bar{\Gamma}_0$ is the *strict transform*. The exceptional divisor of this blowup is the Cartier divisor defined as the preimage of the zero dimensional scheme described by the Jacobian ideal. It is called the *Plücker defect*.

□

Example 3.4. Consider the polynomial $f(x,y) = x^p - y^q$, $p > q$ of Example 3.2. Observe that the tangent spaces to the level sets of f are the kernels of df so the Gauss map can be given the description

$$\mathcal{G} : U \setminus 0 \to \mathbb{P}^1, \ \ (x,y) \mapsto [px^{p-1}, qy^{q-1}].$$

With graph

$$\Gamma = \Big\{ (x,y;[px^{p-1}, qy^{q-1}]); \ \ (x,y) \neq (0,0) \Big\}.$$

Since x^{p-1}, y^{p-1} is a regular sequence in $\mathbb{C}[x,y]$ we deduce from [2, Prop. IV.25] that closure in $\mathbb{C}^2 \times \mathbb{P}^1$ is the subvariety described by the equation

$$\bar{\Gamma} = \Big\{ (x,y;[a,b]); \ \ qy^{q-1}a = px^{p-1}b \Big\}.$$

$\mathbb{C}^2 \times \mathbb{P}^1$ is covered by two coordinate charts $U_a := \{a \neq 0\}$, $U_b := \{b \neq 0\}$, $a = 1/b$ on $U_a \cap U_b$. On U_a we have coordinates (x,y,b) and $\bar{\Gamma}$ is described by

$$\bar{\Gamma}_a = \bar{\Gamma} \cap U_a = \{qy^{q-1} = px^{p-1}b\}$$

On U_b we have the coordinates (x,y,a) and

$$\bar{\Gamma}_b = \bar{\Gamma} \cap U_b = \{qy^{q-1}a = px^{p-1}\}.$$

The Nash blowup of $f = 0$ is the subvariety of $\bar{\Gamma}$ described by

$$\bar{\Gamma}_0 = \Big\{ (x,y,[a,b]); \ \ qy^{q-1}a - px^{p-1}b = x^p - y^q = 0 \}.$$

Let us point out that the blowup of the Jacobian ideal need not be normal. Consider for example the case $p = 5$, $q = 3$. In this case $\bar{\Gamma}$ is described on U_a by the equation

$$x^4 b = y^2.$$

The ring $R = \mathbb{C}[x,y,b]/I$, $I = (x^4 b - y^2)$ is an integral domain. The element $t = y/x^2$ satisfies the equation $t^2 - b = 0$ so that it is integral over R. On the other hand, it does not belong to R. The real part of this surface is depicted[1] in Figure 2.

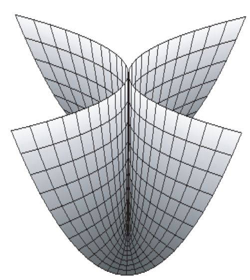

Figure 2: *Blowing up the plane at the Jacobian ideal* (x^4, y^2).

□

[1]We generated Figure 2 using *MAPLE* and the normalization map $\mathbb{C}^2 \to \bar{\Gamma}_a$ described by $x = s$, $y = ts^2$, $b = t^2$.

Theorem 3.5 (Teissier, [11]). *Fix a small neighborhood U of the origin $0 \in \mathbb{C}^{n+1}$ and set $X_t^U := X_t \cap U$. Then the following hold.*

(a) $H \in \mathbb{P}^n \setminus \mathfrak{L}$ (i.e. H is not a limit of tangent planes) if and only if

• *$X_0^U \cap H$ has an isolated singularity at the origin $0 \in H$. Denote by $\mu(X_0 \cap H)$ the Milnor number of the hypersurface $X_0 \cap H \subset H$.*
• *For any $H \in \Omega$ the Milnor number $\mu(X_0 \cap H)$ is minimum amongst the Milnor numbers of hyperplane sections $X_0 \cap H'$, $H' \in G$, with isolated singularities at the origin.*

□

Definition 3.6. *Define f' to be the restriction of f to a generic hyperplane $H \subset \mathbb{C}^{n+1}$, $H \in \Omega$, and $\mu'(f,0)$ as the Milnor number of f'. Iterating we define*

$$f^{(k+1)} := (f^{(k)})', \quad \mu^{(k+1)}(f,0) := \mu'(f^{(k)},0).$$

The integers $\mu^{(k)}(f,0)$, $0 \leq k \leq n$ are called the Milnor-Teissier numbers *of the hypersurface germ $(f = 0)$.*

□

We will denote by $\chi^{(k)}(f,0)$ the Euler characteristic of the Milnor fiber of $f^{(k)}$. Using Milnor's theorem 3.1 we deduce that

$$\chi^{(k)} = 1 + (-1)^{(n-k)}\mu^{(k)}(f,0)$$

so that

$$\chi(f,0) - \chi'(f,0) = (-1)^n\big(\mu(f,0) + \mu'(f,0)\big). \tag{3.1}$$

Example 3.7. Consider polynomial $f = f_{p,q}(x,y) = x^p - y^q$, $p > q$, $\gcd(p,q) = 1$. We know that $\mu(f,0) = (p-1)(q-1)$. Then

$$f^{(1)}(t) = f\,|_{x=t,y=mt} = t^p - m^q t^q,$$

so that

$$\mu'(f,0) = q - 1.$$

□

For every linear functional $u : \mathbb{C}^{n+1} \to \mathbb{C}$ and any open neighborhood V of the origin define

$$F_u : V \to \mathbb{C}^2, \quad \vec{z} \mapsto (f(\vec{z}), u(\vec{z})).$$

Observe that the fiber of F_u over (t_0, u_0) is the hyperplane section

$$X_{t_0,u_0}^V = X_{t_0}^V \cap \{u = u_0\}.$$

Observe that $\vec{z}$ is a critical point of F_0 iff df and du are linearly independent, i.e. if the tangent space at $\vec{z}$ to the hypersurface $X_{f(\vec{z})}$ is parallel to the hyperplane $u = 0$. We denote by C_u the *critical locus* of F_u, which is the scheme defined by the vanishing of the 2-form $df \wedge du$.

For u in the generic set Ω of Theorem 3.5 the map F_u is flat, which in this case is equivalent to the fact that the fibers are complete intersections. We deduce from [6, Thm. 2.8 (iii)] that $\dim C_u = 1$. We have the following nontrivial genericity result.

Theorem 3.8 (Hamm-Lê, [5]). *There exists a Zariski open subset $\Omega' \in \Omega$ with the following property.*

For any linear functional $u : \mathbb{C}^{n+1} \to \mathbb{C}$ which describes a hyperplane $H \in \Omega'$ there exists an open neighborhood $U = U_H$ of the origin and $\tau > 0$ such that for every $0 \leq |t| < \tau$ the restriction of u to $X_t^U \setminus 0$ has only nondegenerate critical points.

Pick u as above. From the theory of discriminants we deduce that in a punctured neighborhood of the origin the critical locus C_u is reduced and smooth away from the origin. (see [6, §4.5]).

Example 3.9. Consider again the polynomial $f_{p,q}(x,y) = x^p - y^q$. We assume $p > q$, $\gcd(p,q) = 1$. We denote by $X_{p,q}$ the germ at zero of the plane curve $x^p - y^q = 0$.

For each $m \in \mathbb{C}$ consider a line H_m in $\mathbb{C}^2$ of slope m, $y = mx$ and consider the associated linear functional $u_m : \mathbb{C}^2 \to \mathbb{C}$, $u_m(x,y) = mx - y$. We make the change in coordinates

$$v = y, \;\; u = mx - y \Longleftrightarrow y = v, \;\; x = \frac{u+v}{m}.$$

In these coordinates H_m is given by $u = 0$. Define

$$F_m : \mathbb{C}^2 \to \mathbb{C}^2, \;\; (u,v) \mapsto (t,s) = \Big(\frac{(u+v)^p}{m^p} - v^q, u\Big).$$

Finally we introduce new coordinates $w = u + v$, $v = v$ so that $u = w - v$ and

$$F_m : \mathbb{C}^2 \to \mathbb{C}^2, \;\; (v,w) \mapsto (t,s) = \Big(\frac{w^p}{m^p} - v^q, w - v\Big).$$

The polar curve (critical locus) corresponding to this slope is given by

$$C_m = \{F_m^*(dt \wedge ds) = 0\} = \Big\{\frac{p}{m^p} w^{p-1} - qv^{q-1} = 0\Big\}.$$

One can prove that the function $qv^{q-1} - \frac{p}{m^p} w^{p-1}$ defines an irreducible germ in $\mathcal{O}_{\mathbb{C}^2,0}$. Its zero locus is smooth in a punctured neighborhood of the origin. Away from 0 it admits the parametrization

$$v = \tau^{p-1}, \quad w = (c\tau)^{q-1}, \;\; \text{where} \;\; c^{(p-1)(q-1)} = \frac{qm^p}{p}.$$

Observe that along this curve we have

$$f(x,y) = f(v,w) = f(\tau^{p-1}, (c\tau)^{q-1}) = \frac{(c\tau)^{p(q-1)}}{m^p} - \tau^{q(p-1)}.$$

Thus the order of f at zero along this curve is

$$p(q-1) = (p-1)(q-1) + (q-1) = \mu(f_{p,q}, 0) + \mu'(f_{p,q}, 0).$$

Equivalently, the multiplicity of the intersection $X_{p,q} \cap C_m$ at zero is

$$(X_{p,q}, C_m)_0 = \mu(f_{p,q}, 0) + \mu'(f_{p,q}, 0).$$

This implies that for generic t_0 there are exactly $\mu(f_{p,q}, 0) + \mu'(f_{p,q}, 0)$ points on the Milnor fiber $f = t_0$ in a small neighborhood of the origin where the tangent line has slope m.

□

Theorem 3.10 (Langevin, [4]). *Suppose $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ has an isolated singularity at the origin. Then for every sufficiently small neighborhood V of $0 \in \mathbb{C}^{n+1}$ we have*

$$\lim_{t \to 0} \int_{X_t^V} c_n(X_t) - \int_{X_0^V} c_n(X_0) = \lim_{W} \lim_{t \to 0} \int_{X_t^U} c_n(X_t) = \chi(f,0) - \chi'(f,0), \tag{3.2}$$

where W is the filter of neighborhoods of $0 \in \mathbb{C}^{n+1}$.

Proof Fix a small neighborhood V of the origin. Denote by G the grassmanian of hyperplanes in $\mathbb{C}^{n+1}$ through the origin, by $E \to G$ the tautological rank n bundle over G and by $c_n(E)$ the n-th Chern form of E equipped with the natural hermitian metric. Using the identification $G \cong \mathbb{P}((\mathbb{C}^{n+1})^*)$ we deduce that

$$c_n(E) = (-1)^n \Omega^n,$$

where Ω is the Fubini-Study form normalized so that its integral over each projective line is 1. In particular Ω^n defines a volume form on G which we denote by dH. For every $|t| \ll 1$ we gave a Gauss map

$$\mathcal{G}_t : X_t^V \setminus 0 \to G.$$

Using Theorema Egregium we deduce

$$\int_{X_t^V} c_n(X_t) = \int_{X_t^V} \mathcal{G}_t^* c_n(E) = (-1)^n \int_{X_t^V} \mathcal{G}_t^* \Omega^n.$$

For any open set $\mathcal{O}$ we set

$$c_H(t, \mathcal{O}) = \#(\mathcal{G}_t^{-1}(H) \cap \mathcal{O} \setminus 0).$$

Crofton's formula now implies that

$$\int_{X_t^V} c_n(X_t) = (-1)^n \int_G c_H(t, V) dH. \tag{3.3}$$

We want to prove that for any $H \in \Omega'$ there exists a small neighborhood W of $0 \in \mathbb{C}^{n+1}$ such that

$$\lim_{t \to 0} c_H(t, W) = \mu(f, 0) + \mu'(f, 0). \tag{3.4}$$

This is equivalent to the condition that for every $H \in \Omega'$ we have

$$\lim_{W} \lim_{t \to 0} c_H(t, W) = \mu(f, 0) + \mu'(f, 0). \tag{3.5}$$

Fix a hyperplane $H \in \Omega'$ and a linear function $u : V \to \mathbb{C}$ defining it. For every t_0, u_0 and any open set $\mathcal{O}$ we define

$$X_{t_0, u_0} = (X_{t_0} \cap \{u = u_0\}), \;\; X_{t_0, u_0}^{\mathcal{O}} = X_{t_0, u_0} \cap \mathcal{O}, \;\; c(t, \mathcal{O}) = c_H(t, \mathcal{O}).$$

Then $c(t, \mathcal{O})$ is the number of points in $X_t \cap \mathcal{O}$ where the tangent plane is parallel to H. Equivalently, it is the number of critical points of the restriction of the linear function u to $X_t \cap \mathcal{O}$. In terms of the critical curve C_u we have

$$c(t, \mathcal{O}) = \#(C_u \cap X_t^{\mathcal{O}}).$$

Due to our choice of H all these critical points inside a small neighborhood $\mathcal{O}$ of the origin are nondegenerate.

Remark 3.11. The definition of the intersection numbers in analytic geometry implies that there exists a small neighborhood W of the origin such that

$$\lim_{t \to 0} \#(C_u \cap X_t^W) = (C_u \cdot X_0)_0 = \text{the mutiplicity at 0 of the intersection } C_u \cap X_0.$$

Thus we can rephrase (3.5) as saying that

$$(C_u \cdot X_0)_0 = \mu(f, 0) + \mu'(f, 0).$$

When $n = 1$ so that X_0 is a plane curve this is an old result going back to the Plücker in the 19th century (see the beautiful discussion in [1, III.9.1]). The general case is more recent and is due to Teissier, [11] who proved it by algebraic means. Below we present a topological proof. □

We will need the following technical result.

Lemma 3.12. *There exists a small neighborhood W of the origin such that $u(X_t \cap W) \subset \mathbb{C}$ is a disk centered at the origin for all $0 < |t| \ll 1$.*

Proof Consider the holomorphic map

$$F_u : \mathbb{C}^{n+1} \to \mathbb{C}^2, \ \ \vec{z} \mapsto (f(\vec{z}), u(\vec{z})).$$

Due to our generic choice of u the map F_u *is flat* in a neighborhood U' of the origin and thus it is open on U'. Choose a polydisk

$$\mathbb{D} = \{|f| < r_1\} \times \{|u| < r_2\} \subset F_u(U') \subset \mathbb{C}^2.$$

Now set

$$W = U' \cap F_u^{-1}(\mathbb{D}).$$

Then for every $|t| < r_1$ we have $u(X_t \cap W) = \{|z_0| < r_2\}$. □

Proof of (3.5). For $0 < |t| \ll 1$ the hypersurface X_t^W is diffeomorphic to the Milnor fiber so that

$$\chi(X_t^W) = \chi(f, 0) = 1 + (-1)^n \mu(f, 0).$$

Set $U_t = u(X_t^W)$. We know that U_t is a disk in $\mathbb{C}$. The choice of u shows that for t sufficiently small the plane section $X_{t,0}^W$ is smooth and is diffeomorphic to the Milnor fiber of the hyperplane section $f\,|_H$ so that

$$\chi(X_{t,0}^W) = \chi'(f, 0).$$

Denote by $C_t \subset U_t$ the set of critical values of $u : X_t^W \to U_t$. C_t is a finite set. Then

$$\chi(f, 0) = \chi(X_{t,0}^W) \cdot \chi(U_t \setminus C_t) + \sum_{v \in C_t} \chi(X_{t,v}^\varepsilon)$$

Since the critical points are nondegenerate we deduce that each of them corresponds to a vanishing $(n-1)$-sphere in $X_{t,0}^W$. Hence

$$\sum_{v \in C_t} \chi(X_{t,v}^W) = \sum_{v \in C_t} \chi(X_{t,0}^W) - (-1)^{n-1} c(X_t, 0, \varepsilon) = |C_t| \chi'(f, 0) + (-1)^n c(X_t, 0, \varepsilon).$$

Lemma 3.12 implies that $\chi(U_t) = 1$ so that $\chi(U_t \setminus C_t) = 1 - |C_t|$ and thus

$$\chi(f, 0) = \chi'(f, 0) + (-1)^n c(X_t, 0, \varepsilon), \ \ \forall 0 < |t| \ll 1.$$

This proves (3.5). □

Langevin's formula now follows by passing to the limit[2] in (3.3) and using (3.4). □

[2]We need to invoke the dominated convergence theorem to conclude

$$\lim_{W} \lim_{t \to 0} \int_G c_H(t, W) dH = \int_G \lim_{W} \lim_{t \to 0} c_H(t, W).$$

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