# Orientation transport

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## 1 $S^1$-bundles over 3-manifolds: homological properties

Let $(Y, g)$ denote a compact, oriented Riemann 3-manifold without boundary. Denote by $\pi : X \to Y$ a principal $S^1$-bundle over $Y$, and by $Z \to Y$ the associated 2-disk bundle. Set

$$c := c_1(Z) \in H^2(Y, \mathbb{Z}).$$

Denote by $\mathfrak{t}_Z \in H^2(Z, X; \mathbb{Z})$ the Thom class of $Z \to Y$, by $j$ the inclusion $X \hookrightarrow Z$ and by $\zeta : Y \hookrightarrow Z$ the natural inclusion. Using the Thom isomorphism

$$H^\bullet(Z) \stackrel{\cup \mathfrak{t}_Z}{\longrightarrow} H^{\bullet+2}(Z, X; \mathbb{Z}), \;\; c = \zeta^* \mathfrak{t}_Z,$$

and the long exact cohomological sequence of the pair $(Z, X)$ we obtain the Gysin sequence

$$\cdots \stackrel{\pi_!}{\longrightarrow} H^{k-2}(Y, \mathbb{Z}) \stackrel{\cup c}{\longrightarrow} H^k(Y, \mathbb{Z}) \stackrel{\pi^*}{\longrightarrow} H^k(X, \mathbb{Z}) \stackrel{\pi_!}{\longrightarrow} H^{k-1}(Y, \mathbb{Z}) \stackrel{\cup c}{\longrightarrow} \cdots$$

If $c$ is a torsion class we denote by $\mathbf{ord}(c)$ its order. Otherwise we set $\mathbf{ord}(c) = 0$. The kernel of the map $\cup c : H^0(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is $\mathbf{ord}(c) \cdot \mathbb{Z}$ so for $k = 1$ we obtain an isomorphism

$$H^1(X, \mathbb{Z}) \cong \pi^* H^1(Y, \mathbb{Z}) \oplus \mathbf{ord}(c)\mathbb{Z}.$$

For $k = 2$ we obtain a short exact sequence

$$0 \to H^2(Y, \mathbb{Z})/\langle c \rangle \to H^2(X, \mathbb{Z}) \to \ker\Big( H^1(Y, \mathbb{Z}) \stackrel{\cup c}{\longrightarrow} H^3(Y, \mathbb{Z}) \Big) \to 0.$$

The last group is free so the sequence is split. The image of the morphism

$$H^1(Y, \mathbb{Z}) \stackrel{\cup c}{\longrightarrow} H^3(Y, \mathbb{Z})$$

is a subgroup of $H^3(Y, \mathbb{Z}) \cong \mathbb{Z}$ so it has the form $n\mathbb{Z}$ for some nonnegative integer $n$. We set $\deg c := n$. Observe that

$$\deg c = 0 \Longleftrightarrow c \text{ is a torsion class} \Longleftrightarrow \mathbf{ord}(c) > 0.$$

For $k=3$ we obtain a short exact sequence

$$0 \to \mathbb{Z}/\deg c \to H^3(X,\mathbb{Z}) \xrightarrow{\pi_!} H^2(Y,\mathbb{Z}) \to 0.$$

Homologically, the Thom isomorphism is described by

$$\zeta^! : H_\bullet(Z,X;\mathbb{Z}) \to H_{\bullet-2}(Y,\mathbb{Z}), \;\; H_\bullet(Z,X;\mathbb{Z}) \ni \sigma \mapsto \sigma \cap [Y] \in H_{\bullet-2}(Z,\mathbb{Z}) \cong H_{\bullet-2}(Y,\mathbb{Z}).$$

We obtain the homological Gysin sequence

$$\cdots \to H_k(X,\mathbb{Z}) \xrightarrow{j_*} H_k(Z,\mathbb{Z}) \xrightarrow{\zeta^!} H_{k-2}(Y,\mathbb{Z}) \xrightarrow{\pi^!} H_{k-1}(X,\mathbb{Z}) \to \cdots$$

The morphism $\pi^!$, also known as the *tube map* is described geometrically as follows. Represent $\sigma \in H_m(Y,\mathbb{Z})$ by an embedded oriented submanifold $S$. The total space of the restriction of the $S^1$-bundle $X \to Y$ to $S$ is a $(m+1)$-dimensional submanifold of $X$ representing $\pi^!\sigma$ .

If we use the isomorphism $\pi_* : H_\bullet(Z,\mathbb{Z}) \to H_\bullet(Y,\mathbb{Z})$ and we represent the Poincaré dual of $c \in H^2(Y,\mathbb{Z})$ by a link $\mathfrak{L} \hookrightarrow Y$ then we can describe the Gysin sequence as

$$\cdots \to H_k(X,\mathbb{Z}) \xrightarrow{\pi_*} H_k(Y,\mathbb{Z}) \xrightarrow{\cap \mathfrak{L}} H_{k-2}(Y) \xrightarrow{\pi^!} H_{k-1}(X,\mathbb{Z}) \to \cdots$$

## 2 $S^1$-bundles over 3-manifolds: geometric properties

Denote by $\hat{d}$ the exterior derivative on $X$. Denote by $\Theta \in \Omega^2(Y)$ the $g$-harmonic 2-form on $Y$ representing the first Chern class of the disk bundle $Z \to Y$. We denote by $\partial_\varphi \in \mathrm{Vect}(X)$ the infinitesimal generator of the $S^1$-action on $X$

$$(\partial_\varphi f)(x) := \frac{d}{dt} f(e^{\mathbf{i}t} \cdot x), \;\; \forall x \in X.$$

We identify $\underline{u}(1)$-the Lie algebra of $U(1)$-with $\mathbf{i}\mathbb{R}$. Now choose a $\underline{u}(1)$-valued connection 1-form $\mathbf{i}\varphi \in \mathbf{i}\Omega^1(X)$ such that

$$\partial_\varphi \lrcorner \varphi = 1, \;\; \pi^*\Theta = \frac{\mathbf{i}}{2\pi}\hat{d}(\mathbf{i}\varphi) \Longleftrightarrow \pi^*\Theta = -\frac{1}{2\pi}\hat{d}\varphi.$$

For every $r \geq 0$ we set $\varphi_r := r\varphi$ and define a metric $\hat{g}_r$ on $X$ by

$$\hat{g}_r = \varphi_r^2 + \pi^* g.$$

With respect to this metric the fibers of $\pi : X \to Y$ have length $2\pi r$.

Choose an oriented orthonormal frame $\{e_1, e_2, e_3\}$ $TY$ defined on an open subset $U \subset Y$ and we denote by $\{e^1, e^2, e^3\}$ the dual coframe. We denote by

$$\Gamma_g = \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix} \in \Omega^1(U) \otimes \underline{so}(3)$$

the 1-form describing the Levi-Civita connection with respect to the frame $\{e_1, e_2, e_3\}$. From Cartan's structural equations we deduce

$$d\begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix} = \Gamma_g \wedge \begin{bmatrix} e^1 \\ e^2 \\ e^3 \end{bmatrix}. \tag{2.1}$$

Set $f^0 = f^0(r) = \varphi_r$, $f^i = \pi^* e^i$, $i = 1, 2, 3$, so that $\{f^0, f^1, f^2, f^3\}$ is a $\hat{g}_r$-orthonormal co-frame. We denote by $\{f_0 = f_0(r), f_1, f_2, f_3\}$ the dual frame and by $\hat{\Gamma}_r$ the connection 1-form describing the Levi-Civita connection $\hat{\nabla}^r$ of the metric $\hat{g}_r$. $\hat{\Gamma}_r$ is also characterized by Cartan's structural equations

$$\hat{d}\begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{bmatrix} = \hat{\Gamma}_r \wedge \begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{bmatrix}.$$

Using (2.1) and the equality $\hat{d}f^0 = \hat{d}\varphi_r = -2\pi r\Theta$ we deduce

$$\hat{d}\begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{bmatrix} = \begin{bmatrix} -2\pi r\Theta \\ -A_3 \wedge f^2 + A_2 \wedge f^3 \\ A_3 \wedge f^1 - A_1 \wedge f^3 \\ -A_2 \wedge f^1 + A_1 \wedge f^2 \end{bmatrix} = \hat{\Gamma}_r \wedge \begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{bmatrix}. \tag{2.2}$$

We set

$$\Theta = \Theta_{23} e^2 \wedge e^3 + \Theta_{31} e^3 \wedge e^1 + \Theta_{12} e^1 \wedge e^2, \;\; \Theta_{ij} = -\Theta_{ji},$$

and we write

$$\hat{\Gamma}_r = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \pi^*\Gamma \end{bmatrix}}_{:=\hat{\Gamma}_0} + \underbrace{\begin{bmatrix} 0 & {}_r\Xi^0_1 & {}_r\Xi^0_2 & {}_r\Xi^0_3 \\ {}_r\Xi^1_0 & 0 & {}_r\Xi^1_2 & {}_r\Xi^1_3 \\ {}_r\Xi^2_0 & {}_r\Xi^2_1 & 0 & {}_r\Xi^2_3 \\ {}_r\Xi^3_0 & {}_r\Xi^2_1 & {}_r\Xi^3_2 & 0 \end{bmatrix}}_{:={}_r\Xi}, \;\; {}_r\Xi^\alpha_\beta = -{}_r\Xi^\beta_\alpha.$$

The bundle $TX$ admits a $\hat{g}_r$-orthogonal decomposition $TX \cong \langle f_0 \rangle \oplus \pi^* TY$ and as such it is equipped with a metric connection

$$\hat{\nabla}^0 = f^0 \otimes \partial_{f_0} \oplus \pi^* \nabla^g.$$

The 1-form describing this connection with respect to the frame $\{f_\alpha\}$ is $\hat{\Gamma}_0$. Then

$$\hat{\nabla}^r = \hat{\nabla}^0 + {}_r\Xi.$$

Using (2.2) we deduce

$${}_r\Xi \wedge \begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{bmatrix} = \begin{bmatrix} -2\pi r\Theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \left\{ \begin{array}{rcr} {}_r\Xi^0_1 \wedge f^1 + {}_r\Xi^0_2 \wedge f^2 + {}_r\Xi^0_3 \wedge f^3 & = & -2\pi r\Theta =: \Psi^0 \\ {}_r\Xi^1_0 \wedge f^0 + {}_r\Xi^1_2 \wedge f^2 + {}_r\Xi^1_2 \wedge f^3 & = & 0 =: \Psi^1 \\ {}_r\Xi^2_0 \wedge f^0 + {}_r\Xi^2_1 \wedge f^1 + {}_r\Xi^2_3 \wedge f^3 & = & 0 =: \Psi^2 \\ {}_r\Xi^3_0 \wedge f^0 + {}_r\Xi^3_1 \wedge f^1 + {}_r\Xi^3_2 \wedge f^2 & = & 0 =: \Psi^3 \end{array} \right. \tag{2.3}$$

Set

$$ {}_r\Xi^\alpha_\beta = {}_r\Xi^\alpha_{\beta\gamma} f^\gamma, \;\; \Psi^\alpha = \frac{1}{2}\sum_{\beta,\gamma} \Psi^\alpha_{\beta\gamma} f^\beta \wedge f^\gamma, \;\; \Psi^\alpha_{\beta\gamma} = -\Psi^\alpha_{\gamma\beta}. $$

Arguing as in [1, §4.2.3] we deduce

$$ {}_r\Xi^\alpha_{\beta\gamma} = \frac{1}{2}\Big( \Psi^\alpha_{\beta\gamma} + \Psi^\beta_{\gamma\alpha} - \Psi^\gamma_{\alpha\beta} \Big) $$

We deduce

$$ {}_r\Xi^0_{ij} = -\pi r \Theta_{ij}, \;\; \forall 1 \leq i,j \leq 3, $$

so that

$$ {}_r\Xi^0_i = -\pi r \sum_j \Theta_{ij} f^j = -\pi r f_i \lrcorner \Theta. $$

Next, observe that for $1 \leq i,j,k \leq 3$ we have ${}_r\Xi^i_{jk} = 0$ so that

$$ {}_r\Xi^i_j = {}_r\Xi^i_{j0} f^0 = \frac{1}{2}\Psi^0_{ij} f^0 = \pi r \Theta_{ij} f^0 $$

Hence

$$ {}_r\Xi = \pi r \left[ \begin{array}{cccc} 0 & -f_1 \lrcorner \Theta & -f_2 \lrcorner \Theta & -f_3 \lrcorner \Theta \\ f_1 \lrcorner \Theta & 0 & \Theta_{12} f^0 & \Theta_{13} f^0 \\ f_2 \lrcorner \Theta & \Theta_{21} f^0 & 0 & \Theta_{23} f^0 \\ f_3 \lrcorner \Theta & \Theta_{31} f^0 & \Theta_{32} f^0 & 0 \end{array} \right], \;\; f^0 = r\varphi. $$

Consider the isometry

$$ L_r : (TX, \hat{g}_r) \to (TX, \hat{g}_1), \;\; \partial_\varphi \mapsto r\partial_\varphi, \;\; f_i \mapsto f_i, \;\; i = 1,2,3. $$

Now set

$$ \tilde{\nabla}^r := L_r \hat{\nabla}^r L_r^{-1}, \;\; r \in [0,1]. $$

This is a connection on $TX$, compatible with the metric $\hat{g}_1$. Its torsion is *nontrivial.*

**Lemma 2.1.** *With respect to the $\hat{g}_1$-orthonormal frame $\partial_\varphi, f_1, f_2, f_3$ we have decomposition*

$$ \tilde{\nabla}^r = \hat{\nabla}^0 + {}_r\Xi, $$

*that is, if $V = \sum_{\alpha=0}^3 V^\alpha f_\alpha \in \mathrm{Vect}(X)$, $f_0 = \partial_\varphi$ we have*

$$ \tilde{\nabla}^r V = \hat{\nabla}^0 V + \sum_{\alpha,\beta=0}^3 {}_r\Xi^\beta_\alpha V^\alpha f_\beta. $$

*In particular,*

$$ \lim_{r\searrow 0} \tilde{\nabla}^r = \hat{\nabla}^0. $$

**Proof.** For $\alpha > 0$ and $V \in \mathrm{Vect}(X)$ we have

$$L_r \hat{\nabla}^r_V L_r^{-1} f_\alpha = L_r \hat{\nabla}^r_V f_\alpha = L_r \hat{\nabla}^0_V f_\alpha + L_r \sum_{\beta=0}^{3} r\Xi^\beta_\alpha(V) f_\beta$$

$$= L_r \hat{\nabla}^0_V f_\alpha + L_r(\frac{1}{r} r\Xi^0_\alpha(V)\partial_\varphi) + \sum_{\beta=1}^{3} r\Xi^\beta_\alpha(V) f_\beta$$

$$= \hat{\nabla}^0_V f_\alpha - \pi r(V \lrcorner f_\alpha \lrcorner \Theta)\partial_\varphi \longrightarrow \hat{\nabla}^0_V f_\alpha \text{ as } r \searrow 0.$$

$$L_r \hat{\nabla}^r_V L_r^{-1} \partial_\varphi = L_r \hat{\nabla}^r_V f_0 = L_r \hat{\nabla}^0 f_0 + \pi r \sum_{i=1}^{3} (V \lrcorner f_i \lrcorner \Theta) f_i \to \hat{\nabla}^0_V \partial_\varphi \text{ as } r \searrow 0.$$

□

Recall (see [1, §4.1.5]) that the exterior derivative $\hat{d} : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ can be described as the composition

$$C^\infty(\Lambda^\bullet T^*X) \xrightarrow{\hat{\nabla}^1} C^\infty(T^*X \otimes \Lambda^\bullet T^*X) \xrightarrow{\varepsilon} C^\infty(\Lambda^{\bullet+1} T^*X), \tag{2.4}$$

where $\varepsilon : T^*X \otimes \Lambda^\bullet T^*X \to \Lambda^{\bullet+1} T^*X$ denotes the exterior multiplication. Denote by $\tilde{d}_r$ the operator obtained by replacing in (2.4) the connection $\hat{\nabla}^1$ with the connection $\tilde{\nabla}^r$.

## 3 The $ASD$ operator on $S^1$-bundles over 3-manifolds

Denote $\hat{*}$ the Hodge $*$-operator on $(X, \hat{g}_1)$ and by $*$ the Hodge operator on $Y$. The $ASD$ operator on $(X, \hat{g}_r)$ is the first order elliptic operator

$$ASD = \sqrt{2}\hat{d}^+ \oplus \hat{d}^* : \Omega^1(X) \to \Omega^2_+(X) \oplus \Omega^0(X).$$

Set

$$E := \underline{\mathbb{R}} \oplus \pi^* T^*Y \cong \mathbb{R}\langle f^0 \rangle \pi^* T^*Y,$$

We identify as above $\Lambda^1 T^*X$ and $(\Lambda^0 \oplus \Lambda^2_+) T^*X$ with $E$ as follows.

As in [2, Ex. 4.1.24] we have an $\hat{g}_1$-isometry

$$T^*X \longrightarrow E = \mathbb{R}\langle f^0 \rangle \oplus \pi^* T^*Y, \;\; a \longmapsto a_0 \oplus a_H, \;\; a_0 := f_0 \lrcorner a, \;\; a_H = a - a_0 f^0.$$

To produce an identification of $(\Lambda^0 \oplus \Lambda^2_+) T^*X$ with $E$ we use the $\hat{g}_1$-isometry

$$\sqrt{2} f_0 \lrcorner : \Lambda^2_+ T^*X \longrightarrow \pi^* T^*Y.$$

If $\omega$ is a 2-form on $X$, so that

$$\omega = f^0 \wedge \eta + \theta, \;\; \lrcorner_r \theta = 0$$

then

$$\hat{*}\omega = f^0 \wedge *\theta + *\eta, \;\; \omega^+ = \frac{1}{2}\Big( f^0 \wedge (\eta + *\theta) + (\theta + *\eta) \Big)$$

$$\sqrt{2}f_0 \lrcorner\, \omega^+ = \frac{1}{\sqrt{2}}(\eta + *\theta)$$

Via the above identifications we can regard the ASD operator with a differential operator

$$C^\infty(E) \longrightarrow C^\infty(E).$$

We will locally represent the sections of $E$ as linear combinations

$$a_0 f^0 + \underbrace{a_1 f^1 + a_2 f^2 + a_3 f^3}_{:=a_H}, \quad f^0 = \varphi.$$

$$\tilde{d}_0[a^0, a_1, a_2, a_3] = \sum_{\beta=0}^{3} \hat{d} a_\beta \wedge f^\beta + \sum_{j=1}^{3} a_j \pi^* \Gamma^j_k \wedge f^k$$

where $\Gamma^1_2 = -A_3$, $\Gamma^3_1 = -A_2$, $\Gamma^2_3 = -A_1$ and $\Gamma^i_j = -\Gamma^j_i$. Set for simplicity

$$\tilde{d}_H = \sum_{j=1}^{3} f^j \tilde{\nabla}^0_{f_j} : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X), \ \ \partial_\varphi a_H = \sum_{j=1}^{3} (\partial_\varphi a_j) f^j.$$

Observe that

$$\tilde{d}_H(\pi^*\omega) = \pi^* d\omega, \ \ \forall \omega \in \Omega^\bullet(Y).$$

Then

$$\tilde{d}_0(a_0 f^0 + a_1 f^1 + a_2 f^2 + a_3 f^3) = f^0 \wedge (-\tilde{d}_H a_0 + \partial_\varphi a_H) + \tilde{d}_H a_H$$
$$\sqrt{2} f_0 \lrcorner\, (\sqrt{2}\tilde{d}^+_0) = (-\tilde{d}_H a_0 + \partial_\varphi a_H) + *\tilde{d}_H a_H.$$

Next we look at the differential operator

$$\tilde{d}_0 : \Omega^0(X) \to \Omega^1(X) = \varphi \wedge \partial_\varphi + d_H.$$

Since $\partial_\varphi$ generates a 1-parameter group of $\hat{g}_1$-isometries we deduce $\div_{\hat{g}_1} \partial_\varphi = 0$ so that $\partial^*_\varphi = -\partial_\varphi$ and

$$\tilde{d}^*_0(a_0\varphi + a_H) = -\partial_\varphi a_0 + d^*_H a_H.$$

If we define

$$\mathbf{ASD}_0 := \tilde{d}^*_0 \oplus \sqrt{2}\tilde{d}^+_0 : C^\infty(E) \longrightarrow C^\infty(E)$$

$$\begin{bmatrix} a_0 \\ a_H \end{bmatrix} \longmapsto \begin{bmatrix} -\partial_\varphi a_0 + \tilde{d}^*_H a_H \\ -\tilde{d}_H a_0 + \partial_\varphi a_H + *\tilde{d}_H a_H \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \partial_\varphi \begin{bmatrix} a_0 \\ a_H \end{bmatrix} + \begin{bmatrix} 0 & \tilde{d}^*_H \\ -\tilde{d}_H & *\tilde{d}_H \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_H \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left( \partial_\varphi + \underbrace{\begin{bmatrix} 0 & -\tilde{d}^*_H \\ -\tilde{d}_H & *\tilde{d}_H \end{bmatrix}}_{:=\mathcal{S}} \right) \cdot \begin{bmatrix} a_0 \\ a_H \end{bmatrix}.$$

Similarly, if $W$ is metric vector bundle on $Y$ and $A$ is a metric connection on $W$ then we get a differential operator

$$d_A : \Omega^\bullet(W) \to \Omega^{\bullet+1}.$$

We can pull back the bundle $W$ and the connection $A$ on $X$. Denote by $\tilde{\nabla}^{r,A}$ the connection on $TX \otimes W$ obtained by twisting $\tilde{\nabla}^0$ with $\pi^*A$ and then similarly

$$d_{H,A} = \sum_{j=1}^{3} f^j \wedge \tilde{\nabla}^{A,0}.$$

We obtain twisted $ASD$-operators

$$\mathbf{ASD}_{A,r} : \Omega^1(\pi^*W) \to \Omega^0(W) \oplus \Omega^2_+(W).$$

and as above we deduce

$$\mathbf{ASD}_{A,0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left( \partial_\varphi + \underbrace{\begin{bmatrix} 0 & -\tilde{d}^*_{H,A} \\ -\tilde{d}_{H,A} & *\tilde{d}_{H,A} \end{bmatrix}}_{:=\mathcal{S}_A} \right)$$

We set

$$\mathcal{P}_A := \partial_\varphi + \mathcal{S}_A.$$

Then

$$\ker \mathbf{ASD}_{A,0} = \ker \mathcal{P}_A = \ker \mathcal{A}^*_A \mathcal{A}_A, \;\; \operatorname{ind} \mathcal{A}_W = \operatorname{ind} \mathbf{ASD}_{A,0}.$$

The operators $\partial_\varphi$ and $\mathcal{S}_W$ commute so that

$$\mathcal{P}^*_A \mathcal{P}_A = -\partial^2_\varphi + \mathcal{S}^2_A.$$

We deduce that if $a = a_0 + a_H \in \ker \mathcal{P}_A$ then

$$\partial_\varphi a = 0, \;\; \mathcal{S}_A a = 0.$$

This shows that the pullback by $\pi$ induces an isomorphism

$$\pi^* : \ker \mathcal{S}_A \to \ker \mathcal{P}_A.$$

A similar argument shows that if $A_t$ is a path of metric connections on $W$ then the orientation transport along the path $\mathcal{P}_{A_t}$ is equal to

$$(-1)^{SF(\mathcal{S}_{A_t})}$$

Now observe that the difference

$$D_{A,r} := \mathbf{ASD}_{A,r} - \mathbf{ASD}_{A,0}$$

is a zeroth order operator which converges to zero in in any $C^k$-norm. We denote by $OT_{A,r}$ the orientation transport along the path

$$t \mapsto \mathbf{ASD}_{A,(1-t)r}$$

which connects $\mathbf{ASD}_{A,r}$ to $\mathbf{ASD}_{A,0}$. Since

$$\operatorname{ind}\mathbf{ASD}_{A,r} = \operatorname{ind}\mathbf{ASD}_{A,0}$$

we deduce that if $\ker \mathcal{S}_A = 0$ then $\ker \mathbf{ASD}_{A,r} = 0$ for all $0 \leq r \ll 1$. In particular

$$OT_{A,r} = 0, \;\; \forall 0 < r \ll 1.$$

Suppose $\{A_t;\;\; t \in [0,1]\}$ is a path of connections on $W$ such that $\ker \mathcal{S}_{A_j} = 0$ for $j = 0, 1$. Then for every $r > 0$ we have

$$OT(\mathbf{ASD}_{A_t,r}) = OT_{A_0,r}(-1)^{SF(\mathcal{S}_{A_t})}OT_{A_1,r}.$$

For $r$ sufficiently small we deduce

$$OT(\mathbf{ASD}_{A_t,r}) = (-1)^{SF(\mathcal{S}_{A_t})}$$

Now it remains to see that the operator $\mathbf{ASD}_{A_t,r}$ is conjugate (via $L_r$ with the usual $ASD$-operator defined using the metric $\hat{g}_r$ of radius $r$ and the twisting connection $A_t$).

## References


[1] L.I. Nicolaescu: *Lectures on the Geometry of Manifolds*, World Sci. Pub. Co. 1996.

[2] L.I. Nicolaescu: *Notes on Seiberg-Witten theory*, Graduate Studies in Math, vol. 28, Amer. Math. Soc., 2000.