RANDOM MORSE FUNCTIONS AND SPECTRAL GEOMETRY

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Abstract. We study random Morse functions on a Riemann manifold (M^m, g) defined as random Fourier series of eigenfunctions of the Laplacian of the metric g. The randomness is determined by a fixed Schwartz function w and a small parameter $\varepsilon > 0$. We first prove that, as $\varepsilon \to 0$, the expected distribution of critical values of this random function approaches a universal measure on $\mathbb{R}$, independent of g, that can be explicitly described in terms of the statistic of the Wigner ensemble of random $(m+1)\times(m+1)$ symmetric matrices. Moreover, we prove that the metric g and its curvature are determined by the statistics of the Hessians of the random function for small ε.

Contents

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1. Overview

1.1. **The setup.** Suppose that (M, g) is a smooth, compact, connected Riemann manifold of dimension $m > 1$. We denote by $|dV_g|$ the volume density on M induced by g. We assume that the metric is normalized so that

$$\mathrm{vol}_g(M) = 1. \tag{$*$}$$

For any $\boldsymbol{u}, \boldsymbol{v} \in C^\infty(M)$ we denote by $(\boldsymbol{u}, \boldsymbol{v})_g$ their L^2 inner product defined by the metric g. The L^2-norm of a smooth function $\boldsymbol{u}$ is denoted by $\|\boldsymbol{u}\|$.

Let $\Delta_g : C^\infty(M) \to C^\infty(M)$ denote the scalar Laplacian defined by the metric g. Fix an orthonormal Hilbert basis $(\Psi_k)_{k\geq 0}$ of $L^2(M)$ consisting of eigenfunctions of Δ_g,

$$\Delta_g \Psi_k = \lambda_k \Psi_k, \;\; \|\Psi_k\| = 1, \;\; k_0 < k_1 \Rightarrow \lambda_{k_0} \leq \lambda_{k_1}.$$

Fix an even measurable function $w : \mathbb{R} \to [0, \infty)$ such that

$$\lim_{t\to\infty} t^n w(t) = 0, \;\; \forall n \in \mathbb{Z}_{>0}.$$

For $\varepsilon > 0$ and $k \geq 0$ we set

$$w_\varepsilon(t) := w(\varepsilon t), \;\; \forall t \in \mathbb{R}, \;\; v_k^\varepsilon := w_\varepsilon\big(\sqrt{\lambda_k}\,\big). \tag{1.1}$$

Consider random functions on M of the form

$$\boldsymbol{u}_\varepsilon = \sum_{k\geq 0} X_k \sqrt{v_k^\varepsilon}\,\Psi_k, \tag{1.2}$$

where the coefficients X_k are independent standard Gaussian random variables. Note that

$$\Delta^N \boldsymbol{u}_\varepsilon = \sum_{k\geq 0} \lambda_k^N X_k \sqrt{v_k^\varepsilon} \Psi_k, \;\; \forall N > 0.$$

The fast decay of w, the Weyl asymptotic formula, [10, VI.4], coupled with the Borel-Cantelli lemma imply that for any $N > 0$ the function $\Delta^N u^\varepsilon$ is almost surely (a.s.) in L^2. In particular, this shows that $\boldsymbol{u}_\varepsilon$ is a.s. smooth.

The covariance kernel of the Gaussian random function $\boldsymbol{u}^\varepsilon$ is given by the function

$$\mathscr{E}^\varepsilon : M \times M \to \mathbb{R}, \;\; \mathscr{E}^\varepsilon(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{E}\big(\,\boldsymbol{u}_\varepsilon(p)\boldsymbol{u}_\varepsilon(q)\,\big) = \sum_{k\geq 0} w_\varepsilon\big(\sqrt{\lambda_k}\,\big)\Psi_k(\boldsymbol{p})\Psi_k(\boldsymbol{q}).$$

Since w_ε is rapidly decreasing, the kernel $\mathscr{E}^\varepsilon$ is a smooth function. More precisely, $\mathscr{E}^\varepsilon$ is the Schwartz kernel of the smoothing operator

$$w\big(\varepsilon\sqrt{\Delta}\,\big) : C^\infty(M) \to C^\infty(M).$$

Remark 1.1. Let us observe that if $w(0) = 1$, then as $\varepsilon \searrow 0$ the function w_ε converges uniformly on compacts to the constant function $w_0(t) \equiv 1$ and $w_\varepsilon(\sqrt{\Delta})$ converges weakly to the identity operator. The Schwartz kernel of this limiting operator is the δ-function on $M \times M$ supported along the diagonal. It defines a generalized random function in the sense of [16] usually known as *white noise*. For this reason, we will refer to the $\varepsilon \to 0$ limit as *white noise limit*. □

In the papers [26, 27] we investigated the distribution of critical points and critical values of the random function $\boldsymbol{u}_\varepsilon$ in special case

$$w(t) = \boldsymbol{I}_{[-1,1]} := \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}$$

In this paper we investigate the same problem assuming that w is a *Schwartz function*. We will discuss later the similarities and the differences between these two situations.

The asymptotic estimates in Proposition 2.2 show that the random field $d\boldsymbol{u}^{\varepsilon}$ satisfies the hypotheses of [1, Cor. 11.2.2] for $\varepsilon \ll 1$. Invoking [1, Lemma 11.2.11] we obtain the following technical result.

Proposition 1.2. *The random function $\boldsymbol{u}_{\varepsilon}$ is almost surely Morse if $\varepsilon \ll 1$.* □

For any $\boldsymbol{u} \in C^1(M)$ we denote by $\mathbf{Cr}(\boldsymbol{u}) \subset M$ the set of critical points of $\boldsymbol{u}$ and by $\boldsymbol{D}(\boldsymbol{u})$ the set of critical values[1] of $\boldsymbol{u}$. To a Morse function $\boldsymbol{u}$ on M we associate a Borel measure $\mu_{\boldsymbol{u}}$ on M and a Borel measure $\boldsymbol{\sigma}_{\boldsymbol{u}}$ on $\mathbb{R}$ defined by the equalities

$$\mu_{\boldsymbol{u}} := \sum_{\boldsymbol{p} \in \mathbf{Cr}(\boldsymbol{u})} \delta_{\boldsymbol{p}}, \quad \boldsymbol{\sigma}_{\boldsymbol{u}} := \boldsymbol{u}_*(\mu_{\boldsymbol{u}}) = \sum_{t \in \mathbb{R}} |\boldsymbol{u}^{-1}(t) \cap \mathbf{Cr}(\boldsymbol{u})| \delta_t.$$

Observe that

$$\operatorname{supp} \mu_{\boldsymbol{u}} = \mathbf{Cr}(\boldsymbol{u}), \quad \operatorname{supp} \boldsymbol{\sigma}_{\boldsymbol{u}} = \boldsymbol{D}(\boldsymbol{u}).$$

When $\boldsymbol{u}$ is not Morse, we set

$$\mu_{\boldsymbol{u}} := |dV_g|, \quad \boldsymbol{\sigma}_{\boldsymbol{u}} = \delta_0 = \text{the Dirac measure on } \mathbb{R} \text{ concentrated at the origin.}$$

Observe that for any Morse function $\boldsymbol{u}$, and any Borel subset $B \subset \mathbb{R}$, the number $\boldsymbol{\sigma}_{\boldsymbol{u}}(B)$ is equal to the number of critical values of $\boldsymbol{u}$ in B counted with multiplicity. We will refer to $\boldsymbol{\sigma}_{\boldsymbol{u}}$ as the *variational complexity* or *variational spectrum* of $\boldsymbol{u}$.

To the random function $\boldsymbol{u}^{\varepsilon}$ we associate the random (or empirical) measure $\boldsymbol{\sigma}_{\boldsymbol{u}_{\varepsilon}}$. Its expectation

$$\boldsymbol{\sigma}^{\varepsilon} := \boldsymbol{E}(\sigma_{\boldsymbol{u}_{\varepsilon}})$$

is the measure on $\mathbb{R}$ uniquely determined by the equality

$$\int_{\mathbb{R}} f(t)\, \boldsymbol{\sigma}^{\varepsilon}(dt) = \boldsymbol{E}\left(\int_{\mathbb{R}} f(t)\, \boldsymbol{\sigma}_{\boldsymbol{u}^{\varepsilon}}(dt) \right),$$

for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$. In §2.1 we show that the measure $\boldsymbol{\sigma}^{\varepsilon}$ is well defined for $\varepsilon \ll 1$. We will refer to it as *the expected variational complexity* of the random function $\boldsymbol{u}_{\varepsilon}$. We are interested in two problems.

(i) Describe the white noise limit of $\boldsymbol{\sigma}^{\varepsilon}$.
(ii) Recover the geometry of (M, g) from white noise behavior the random function $\boldsymbol{u}_{\varepsilon}$.

Remark 1.3. Before we state precisely our main results we believe that it is instructive to discuss some elementary topologic and geometric features of the white noise behavior of $\boldsymbol{u}_{\varepsilon}$. For simplicity, we assume that $w(0) = 1$ so that $\boldsymbol{u}^{\varepsilon}$ does converge to the white noise on M.

(a) It is not hard to prove that, for any given Morse function $f : M \to \mathbb{R}$, and any $\hbar > 0$, the probability that $\|f - \boldsymbol{u}_{\varepsilon}\|_{C^3} < \hbar$ is positive for ε sufficiently small. If f happens to be a stable Morse function, i.e., it has at most one critical point per level set, then for $\hbar$ sufficiently small, any C^3-function $g : M \to \mathbb{R}$ satisfying $\|f - g\|_{C^3} < \hbar$ is topologically equivalent to f. Thus, as $\varepsilon \to 0$, the random function $\boldsymbol{u}_{\varepsilon}$ samples all the topological types of Morse functions.

(b) The rescaling $w \to w_{\varepsilon}$ can be alternatively implemented as as follows. Consider the rescaled metric $g_{\varepsilon} := \varepsilon^{-2} g$. As $\varepsilon \to 0$, the metric g_{ε} becomes flatter and flatter. The

[1] The set $\boldsymbol{D}(\boldsymbol{u})$ is sometime referred to as the *discriminant set* of $\boldsymbol{u}$.

Laplacian of g_ε is $\Delta_{g_\varepsilon} = \varepsilon^2\Delta_g$. Its eigenvalues are $\lambda_k^\varepsilon = \varepsilon^2\lambda_k$ and the collection $\Psi_k^\varepsilon = \varepsilon^{\frac{m}{2}}\Psi_k$ is an orthonormal eigen-basis of $L^2(M, |dV_{g_\varepsilon}|)$. For any $\varepsilon > 0$ we define the random function

$$\boldsymbol{v}_\varepsilon = \sum_{k\geq 0} X_k w\Big(\sqrt{\lambda_k^\varepsilon}\,\Big)^{\frac{1}{2}}\Psi_k^\varepsilon = \sum_{k\geq 0} X_k \sqrt{v_k^\varepsilon}\,\Psi_k^\varepsilon,$$

where the coefficients X_k are independent standard Gaussian random variables. Observe that $\boldsymbol{v}_\varepsilon = \varepsilon^{\frac{m}{2}}\boldsymbol{u}_\varepsilon$. This shows that the expected distribution $\boldsymbol{\sigma}^\varepsilon(\boldsymbol{v})$ of critical values of $\boldsymbol{v}_\varepsilon$ is a rescaling of $\boldsymbol{\sigma}^\varepsilon$. □

1.2. **Statements of the main results.** Observe that if $\boldsymbol{u} : M \to \mathbb{R}$ is a fixed Morse function and c is a constant, then

$$\mathbf{Cr}(c + \boldsymbol{u}) = \mathbf{Cr}(\boldsymbol{u}),\ \ \mu_{c+\boldsymbol{u}} = \mu_{\boldsymbol{u}},$$

but

$$\boldsymbol{D}(\boldsymbol{u} + c) = c + \boldsymbol{D}(\boldsymbol{u}),\ \ \boldsymbol{\sigma}_{\boldsymbol{u}+c} = \delta_c * \boldsymbol{\sigma}_{\boldsymbol{u}},$$

where $*$ denotes the convolution of two finite measures on $\mathbb{R}$.

More generally, if X is a scalar random variable with probability distribution ν_X, then the expected variational complexity of the random function $X + \boldsymbol{u}$ is the measure $\boldsymbol{E}(\,\boldsymbol{\sigma}_{X+\boldsymbol{u}}\,) = \nu_X * \boldsymbol{\sigma}_{\boldsymbol{u}}$. If $\boldsymbol{u}$ itself is a random function, and X is independent of $\boldsymbol{u}$, then the above equality can be rephrased as

$$\boldsymbol{E}(\,\boldsymbol{\sigma}_{X+\boldsymbol{u}}\,) = \nu_X * \boldsymbol{E}(\boldsymbol{\sigma}_u).$$

In particular, if the distribution ν_X is Gaussian, then the measure $\boldsymbol{E}(\boldsymbol{\sigma}_{\boldsymbol{u}})$ is uniquely determined by the measure $\boldsymbol{E}(\boldsymbol{\sigma}_{X+\boldsymbol{u}})$ since the convolution with a Gaussian is an injective operation. It turns out that, in certain cases, it is easier to understand the statistics of the variational complexity of a perturbation of $\boldsymbol{u}_\varepsilon$ with an independent Gaussian variable of cleverly chosen variance.

To explain this perturbation we need to introduce several quantities that will play a crucial role throughout this paper. We define

$$\begin{aligned} s_m &:= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} w(|x|)dx, \quad d_m := \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} x_1^2 w(|x|)dx,\\ h_m &:= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} x_1^2x_2^2 w(|x|)dx. \end{aligned} \tag{1.3}$$

The statistical relevance of these quantities is explained in Proposition 2.2. If we set

$$I_k(w) := \int_0^\infty w(r)r^k dr, \tag{1.4}$$

then we deduce from [25, Lemma 9.3.10]

$$(2\pi)^m s_m = \left(\int_{|x|=1} dA(x)\right) I_{m-1}(w) = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} I_{m-1}(w),$$

$$(2\pi)^m d_m = \left(\int_{|x|=1} x_1^2 dA(x)\right) I_{m+1}(w) = \frac{\pi^{\frac{m}{2}}}{\Gamma(1+\frac{m}{2})} I_{m+1}(w) = \frac{2\pi^{\frac{m}{2}}}{m\Gamma(\frac{m}{2})} I_{m+1}(w),$$

$$(2\pi)^m h_m = \left(\int_{|x|=1} x_1^2x_2^2 dA(x)\right) I_{m+1}(w)$$

$$= \frac{\pi^{\frac{m}{2}}}{2\Gamma(2+\frac{m}{2})} I_{m+3}(w) = \frac{2\pi^{\frac{m}{2}}}{m(m+2)\Gamma(\frac{m}{2})} I_{m+3}(w).$$

We set

$$q_m := \frac{s_m h_m}{d_m^2} = \frac{m}{m+2} \frac{I_{m-1}(w) I_{m+3}(w)}{I_{m+1}(w)^2}. \tag{1.5}$$

The Cauchy inequality implies that $I_{m+1}(w)^2 \leq I_{m-1}(w) I_{m+3}(w)$ so that

$$q_m \geq \frac{m}{m+2}. \tag{1.6}$$

The sequence $(q_m)_{m\geq 1}$ can be interpreted as a measure of the tail of w, the heavier the tail, the faster the growth of q_m as $m \to \infty$; see Section 3 for more details. We set

$$r_n := \max(1, q_n), \tag{1.7}$$

and define $\boldsymbol{\omega}_m \geq 0$ via the equality

$$r_n = \frac{(s_m + \boldsymbol{\omega}_m) h_m}{d_m^2}. \tag{1.8}$$

Set $\check{s}_m := s_m + \boldsymbol{\omega}_m$ so that (compare with (1.5))

$$r_m = \frac{\check{s}_m h_m}{d_m^2}. \tag{1.9}$$

Observe that

$$\boldsymbol{\omega}_m = 0 \Longleftrightarrow q_m = r_m \geq 1 \Longleftrightarrow \check{s}_m = s_m, \tag{1.10}$$

while the inequality (1.6) implies that

$$\lim_{m\to\infty} \frac{\boldsymbol{\omega}_m}{s_m} = 0, \quad \lim_{m\to\infty} \frac{r_m}{q_m} = 1. \tag{1.11}$$

Choose a scalar Gaussian random variable $X_{\boldsymbol{\omega}(\varepsilon)}$ with mean 0 and variance $\boldsymbol{\omega}(\varepsilon) := \boldsymbol{\omega}_m \varepsilon^{-m}$ independent of $\boldsymbol{u}_\varepsilon$ and form the new random function

$$\check{\boldsymbol{u}}_\varepsilon := X_{\boldsymbol{\omega}(\varepsilon)} + \boldsymbol{u}_\varepsilon.$$

We denote by $\check{\boldsymbol{\sigma}}^\varepsilon$ the expected variational complexity of $\check{\boldsymbol{u}}_\varepsilon$. We have the equality

$$\check{\boldsymbol{\sigma}}^\varepsilon = \boldsymbol{\gamma}_{\boldsymbol{\omega}(\varepsilon)} * \boldsymbol{\sigma}^\varepsilon, \quad \boldsymbol{\omega}(\varepsilon) := \boldsymbol{\omega}_m \varepsilon^{-m}, \tag{1.12}$$

Note that

$$\boldsymbol{N}^\varepsilon = \int_{\mathbb{R}} \check{\boldsymbol{\sigma}}^\varepsilon(dt) = \int_{\mathbb{R}} \boldsymbol{\sigma}^\varepsilon(dt)$$

is the expected number of critical points of the random function $\boldsymbol{u}^\varepsilon$.

To formulate our main results we need to recall some terminology from random matrix theory.

For $v \in (0,\infty)$ and N a positive integer we denote[2] by GOE_N^v the space Sym_N of real, symmetric $N \times N$ matrices A equipped with a Gaussian measure such that the entries a_{ij} are independent, zero-mean, normal random variables with variances

$$\boldsymbol{var}(a_{ii}) = 2v, \quad \boldsymbol{var}(a_{ij}) = v, \quad \forall 1 \leq i < j \leq N.$$

Let $\rho_{N,v} : \mathbb{R} \to \mathbb{R}$ be the *normalized correlation* function of GOE_N^v. It is uniquely determined by the equality

$$\int_{\mathbb{R}} f(\lambda) \rho_{N,v}(\lambda) d\lambda = \frac{1}{N} \boldsymbol{E}_{\mathrm{GOE}_N^v}\big(\operatorname{tr} f(A)\big),$$

[2] GOE = Gaussian Orthogonal Ensemble

for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$. The function $\rho_{N,v}(\lambda)$ also has a probabilistic interpretation: for any Borel set $B \subset \mathbb{R}$ the expected number of eigenvalues of a random $A \in \mathrm{GOE}_N^v$ that are located in B is equal to

$$N \int_B \rho_{N,v}(\lambda) d\lambda.$$

For any $t > 0$ we denote by $\mathcal{R}_t : \mathbb{R} \to \mathbb{R}$ the rescaling map $\mathbb{R} \ni x \mapsto tx \in \mathbb{R}$. If μ is a Borel measure on $\mathbb{R}$ we denote by $(\mathcal{R}_t)_*\mu$ its pushforward via the rescaling map $\mathcal{R}_t$.

The celebrated Wigner semicircle theorem, [3, 24], states that, as $N \to \infty$, the rescaled probability measures

$$(\mathcal{R}_{\frac{1}{\sqrt{N}}})_*\big(\rho_{N,v}(\lambda) d\lambda \big)$$

converge weakly to the semicircle measure given by the density

$$\rho_{\infty,v}(\lambda) := \frac{1}{2\pi v} \times \begin{cases} \sqrt{4v - \lambda^2}, & |\lambda| \leq \sqrt{4v} \\ 0, & |\lambda| > \sqrt{4v}. \end{cases}$$

We can now state the main results of this paper.

Theorem 1.4. *For $v > 0$ and $N \in \mathbb{Z}_{>0}$ we set*

$$\theta^{\pm}_{N,v}(x) := \rho_{N,v}(x) e^{\pm \frac{x^2}{4v}}.$$

(a) There exists a constant $C = C_m(w)$ that depends only on the dimension m and the weight w such that

$$\boldsymbol{N}^{\varepsilon} \sim C_m(w) \varepsilon^{-m} \big(1 + O(\varepsilon) \big) \ \text{ as } \varepsilon \to 0. \tag{1.13}$$

More precisely

$$C_m(w) = 2^{\frac{m+4}{2}} r_m^{\frac{1}{2}} \left(\frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \Gamma\left(\frac{m+3}{2} \right) \int_{\mathbb{R}} (\gamma_{r_m - 1} * \theta^{+}_{m+1, r_m})(y) \gamma_1(y) dy. \tag{1.14}$$

(b) As $\varepsilon \searrow 0$ the rescaled probability measures

$$\frac{1}{\boldsymbol{N}^{\varepsilon}} \Big(\mathcal{R}_{\frac{1}{\sqrt{\check{s}_m \varepsilon^{-m}}}} \Big)_* \check{\boldsymbol{\sigma}}^{\varepsilon}$$

converge weakly to a probability measure $\check{\boldsymbol{\sigma}}_m$ on $\mathbb{R}$ uniquely determined by the proportionalities

$$\check{\boldsymbol{\sigma}}_m \propto \big(\gamma_{r_m - 1} * \theta^{+}_{m+1, r_m}(y) \big) \gamma_1(y) dy \tag{1.15a}$$

$$\propto \theta^{-}_{m+1, \frac{1}{r_m}} * \gamma_{\frac{r_m - 1}{r_m}}(y) dy. \tag{1.15b}$$

When $r_m = q_m \geq 1$, we have (see (1.10)) $\boldsymbol{\omega}_m = 0$ and $\check{\boldsymbol{\sigma}}^{\varepsilon} = \boldsymbol{\sigma}^{\varepsilon}$. In general, Theorem 1.4 implies the following universality result.

Corollary 1.5 (**Universality**). *As $\varepsilon \to 0$, the rescaled probability measures*

$$\frac{1}{\boldsymbol{N}^{\varepsilon}} \Big(\mathcal{R}_{\frac{1}{\sqrt{\check{s}_m \varepsilon^{-m}}}} \Big)_* \boldsymbol{\sigma}^{\varepsilon}$$

converge weakly to a probability measure $\boldsymbol{\sigma}_m$ uniquely determined by the convolution equation

$$\gamma_{\frac{\boldsymbol{\omega}_m}{\check{s}_m}} * \boldsymbol{\sigma}_m = \check{\boldsymbol{\sigma}}_m.$$

Wigner's semicircle theorem [3, Thm. 2.1.1] allows us extract a bit more about the measure $\boldsymbol{\sigma}_m$ for m large, provided that the behavior of w at ∞ is not too chaotic.

Theorem 1.6 (**Central limit theorem**). *Suppose that the weight w is **regular**, i.e., the sequence r_m defined in (1.7) has a limit $r \in [1,\infty]$ as $m\to\infty$. Then*

$$\lim_{m\to\infty}\boldsymbol{\sigma}_m = \gamma_{\frac{r+1}{r}}.$$

The above regularity assumption on w is a constraint on the behavior of its tail. In Section 3 we describe many classes of regular weights.

Corollary 1.7. *As $m\to\infty$ we have*

$$\begin{aligned} C_m(w) \sim &\frac{8}{\sqrt{\pi m}}\Gamma\left(\frac{m+3}{2}\right)\left(\frac{h_m}{\pi d_m}\right)^{\frac{m}{2}} \\ \sim &\frac{8}{\sqrt{\pi m}}\Gamma\left(\frac{m+3}{2}\right)\left(\frac{2I_{m+3}(w)}{\pi(m+2)I_{m+1}(w)}\right)^{\frac{m}{2}}. \end{aligned} \tag{1.16}$$

Following [1, §12.2] we define the symmetric $(0,2)$-tensor h^ε on M

$$h^\varepsilon(X,Y) := \frac{\varepsilon^{m+2}}{d_m}\boldsymbol{E}\big(X\boldsymbol{u}_\varepsilon(\boldsymbol{p}), Y\boldsymbol{u}_\varepsilon(\boldsymbol{p})\big),\ \ \forall \boldsymbol{p}\in M,\ \ X,Y\in \mathrm{Vect}(M), \tag{1.17}$$

where Xu denotes the derivative of the smooth function u along the vector field X.

Theorem 1.8 (**Probabilistic reconstruction of the geometry**). *(a) For $\varepsilon>0$ sufficiently small the tensor h^ε defines a Riemann metric on M.*
(b) For any vector fields X,Y on M the function $h^\varepsilon(X,Y)$ converges uniformly to $g(X,Y)$ as $\varepsilon\searrow 0$.
(c) The sectional curvatures of h^ε converge to the corresponding sectional curvatures of g as $\varepsilon\searrow 0$.

Remark 1.9. The C^0-convergence of h^ε towards the original metric was observed earlier by S. Zelditch [35]. The main novelty of the above theorem is part (c) which, as detailed below, implies the C^∞ convergence of h^ε to g. However, the qualitative jump from C^0 to C^∞-converges requires considerable extra effort.

The construction of the metrics h^ε generalizes the construction in [6]. Note that for any $\varepsilon>0$ we have a smooth map $\boldsymbol{\Xi}_\varepsilon : M\to L^2(M,g)$

$$M\ni p\mapsto \boldsymbol{\Xi}_\varepsilon(\boldsymbol{p}) := \left(\frac{\varepsilon^{m+2}}{d_m}\right)^{\frac{1}{2}}\sum_{k\geq 0} w_\varepsilon\big(\sqrt{\lambda_k}\,\big)^{\frac{1}{2}}\Psi_k(\boldsymbol{p})\Psi_k\in L^2(M,g). \tag{1.18}$$

For small $\varepsilon>0$ this map is an immersion and h^ε is the pullback by $\boldsymbol{\Xi}_\varepsilon$ of the Euclidean metric on $L^2(M,g)$. Let us point out that [6, Thm.5] is a special case of Theorem 1.8 corresponding to the weight $w(t)=e^{-t^2}$.

Theorem 1.8 coupled with the results in [32] imply that the metrics h^ε converge $C^{1,\alpha}$ to g as $\varepsilon\searrow 0$. The convergence of sectional curvatures coupled with the technique of harmonic coordinates in [2, 32] can be used to bootstrap this convergence to a C^∞ convergence.

Suppose that w has compact support, say $\operatorname{supp} w\subset[-1,1]$ and $w(0)\neq 0$. In this case the map $\boldsymbol{\Xi}_\varepsilon$ is actually a map to the *finite dimensional* Euclidean space

$$\boldsymbol{U}_\varepsilon := \mathrm{span}\big\{\Psi_k;\ \ \lambda_k\leq\varepsilon^{-2}\,\big\}\subset L^2(M,g).$$

For small $\varepsilon > 0$ it is an embedding and Theorem 1.8 implies that for $\varepsilon > 0$ sufficiently small the map $\boldsymbol{\Xi}_\varepsilon$ is a near-isometric embedding of M in a *finite dimensional* space. It is conceivable that this near-isometric embedding could be deformed to an actual isometry by using the strategy of X. Wang and K. Zhu [34]. □

Remark 1.10. (a) We want to say a few words about the analytic facts hiding behind Theorem 1.8. Fix a point $\boldsymbol{p} \in M$ and normal coordinates (x^i) at $\boldsymbol{p}$. The techniques pioneered by L. Hörmander [20], [22, §17.4] (see Proposition 2.2) show that, as $\varepsilon \searrow 0$, we have the 1-term asymptotic expansions

$$\boldsymbol{E}\Big(\,\partial^2_{x^ix^i}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\cdot\,\partial^2_{x^jx^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\Big) = h_m\varepsilon^{-(m+4)}\Big(\,1+O(\varepsilon^2)\,\Big), \tag{1.19a}$$

$$\boldsymbol{E}\Big(\,\partial^2_{x^ix^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\cdot\,\partial^2_{x^ix^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\Big) = h_m\varepsilon^{-(m+4)}\Big(\,1+O(\varepsilon^2)\,\Big). \tag{1.19b}$$

All these 1-term expansions are *independent* of the background metric g. Note that (1.19a) and (1.19b) imply the estimate

$$\boldsymbol{E}\Big(\,\partial^2_{x^ix^i}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\cdot\,\partial^2_{x^jx^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\Big) - \boldsymbol{E}\Big(\,\partial^2_{x^ix^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\cdot\,\partial^2_{x^ix^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\Big) = O\big(\,\varepsilon^{-(m+2)}\,\big). \tag{1.20}$$

Theorem 1.8 is equivalent with the following *sharper* estimate

$$\boldsymbol{E}\Big(\,\partial^2_{x^ix^i}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\cdot\,\partial^2_{x^jx^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\Big) - \boldsymbol{E}\Big(\,\partial^2_{x^ix^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\cdot\,\partial^2_{x^ix^j}\boldsymbol{u}^\varepsilon(\boldsymbol{p})\,\Big) \sim d_mK^g_{ij}(\boldsymbol{p})\varepsilon^{-(m+2)},$$

where $K^g_{ij}(\boldsymbol{p})$ denotes the sectional curvature of g at $\boldsymbol{p}$ along the 2-plane spanned by $\partial_{x^i}, \partial_{x^j}$.

To prove Theorem 1.8 it would help if we could extend (1.19a) and (1.19b) to *explicit, two-term* asymptotic expansions. Unfortunately, in general it is very hard, if not even impossible, to produce *explicit* descriptions of the second order terms.

We can however extract enough partial information and, miraculously, the terms over which we have no *explicit* control cancel each other out when considering the asymptotics (1.20). To extract even this partial information we had to burrow deep into Hörmander's proof [22, §17.4] of the short times asymptotic expansion of the wave kernel.

(b) In [30] we described another approach to the probabilistic reconstruction of the geometry of M using certain Gaussian ensembles of random 1-forms. They are defined as follows.

Consider the covariant Laplacian

$$\Delta^{T^*M} : (\nabla^g)^*\nabla^g : C^\infty(T^*M) \to C^\infty(T^*M),$$

with spectrum $0 \leq \mu_0 \leq \mu_1 \leq \cdots$. Fixing an orthonormal eigenbasis $(\eta_k)_{k\geq 0}$ of $L^2(T^*M)$ we define the family of random 1-forms

$$\eta^\varepsilon = \sum_{k\geq 0} X_k\sqrt{w^\varepsilon_k}\,\eta_k,\;\; w^\varepsilon_k = w\big(\,\varepsilon\sqrt{\mu_k}\,\big),\;\; \varepsilon > 0,$$

where $(X_k)_{k\geq 0}$ are independent standard normal random variables.

The random 1-forms $d\boldsymbol{u}^\varepsilon$ employed for geometric reconstruction in this paper are obviously closed. If $w(0) \neq 0$, then, for small $\varepsilon > 0$, the probability that η^ε is not closed is positive. (We believe that this probability is 1.) This shows that the ensembles η^ε and $d\boldsymbol{u}^\varepsilon$ are qualitatively very different. When $w(0) \neq 0$, the ensemble η^ε samples the entire space $C^\infty(T^*M)$ as $\varepsilon \searrow 0$, whereas the ensemble $d\boldsymbol{u}^\varepsilon$ samples a rather "thin" subspace, that consisting of exact 1-forms.

This suggests that the random forms $d\boldsymbol{u}_\varepsilon$ contain a lot less information than the ensembles η^ε. It is thus natural to expect that it is harder to extract precise information from the "thin" ensemble $d\boldsymbol{u}_\varepsilon$. This is what we have accomplished in Theorem 1.8.

(c) When $w = \boldsymbol{I}_{[-1,1]}$, the second order expansions of the Schwartz kernel of $w(\varepsilon\sqrt{\Delta})$ are very difficult to obtain for an arbitrary metric g since they tend to depend on global properties of the metric. $\square$

The convergence of the metrics h^ε leads to a cute probabilistic proof of the Gauss-Bonnet theorem for the original metric g (and thus for any metric on M). Here is the simple principle behind this proof.

Assume for simplicity that M is oriented and $m = \dim M$ is even. To a Morse function f we associate the signed measure

$$\nu_f = \sum_{df(\boldsymbol{p})=0} (-1)^{\operatorname{ind}(f,p)} \delta_{\boldsymbol{p}},$$

where $\operatorname{ind}(f, \boldsymbol{p})$ denotes the Morse index of the critical point of the Morse function f. The Poincaré-Hopf theorem implies that

$$\int_M \nu_f = \chi(M). \tag{1.21}$$

We can also think of ν_f as a degree 0-current. Thus, the random function $\boldsymbol{u}^\varepsilon$ determines a random 0-current $\nu_{\boldsymbol{u}^\varepsilon}$. It turns out (see Section 4) that the expectation of this current is a current represented by a rather canonical top degree form. More precisely, we prove that,

$$\boldsymbol{E}\big(\,\nu_{\boldsymbol{u}^\varepsilon}\,\big) = \boldsymbol{e}_{h^\varepsilon}(M), \tag{1.22}$$

where $\boldsymbol{e}_{h^\varepsilon}(M)$ is the Euler form defined by the metric h^ε which appears in the Gauss-Bonnet theorem. Using (1.21) we conclude that

$$\int_M \boldsymbol{e}_{h^\varepsilon}(M) = \int_M \boldsymbol{E}\big(\,\nu_{\boldsymbol{u}^\varepsilon}\,\big) = \boldsymbol{E}\left(\int_M \nu_{\boldsymbol{u}^\varepsilon}\right) = \chi(M),$$

and as a bonus we obtain the Gauss-Bonnet theorem for the metric h^ε. Letting $\varepsilon \to 0$ we obtain the Gauss-Bonnet theorem for g since $h^\varepsilon \to g$ and $\boldsymbol{e}_{h^\varepsilon}(M) \to \boldsymbol{e}_g(M)$. In particular, this shows that $\boldsymbol{E}(\,\nu_{\boldsymbol{u}^\varepsilon}\,)$ converges in the sense of currents to $\boldsymbol{e}_g(M)$, the Euler form determined by the metric g.

Remark 1.11. In [29] we have extended these ideas to arbitrary Gaussian ensembles of random sections of arbitrary real oriented vector bundles and we have given a geometric description of the expectation of the random zero-locus current determined by such a random section. $\square$

1.3. **A bit of perspective.** In [26] we proved the counterparts of Theorem 1.4, Corollary 1.5 and Theorem 1.6 in the case of the singular weight $w = \boldsymbol{I}_{[-1,1]}$. In this case the random function $\boldsymbol{u}^\varepsilon$ could be loosely interpreted as a random polynomial of large degree because since this is the case when (M, g) is the round sphere.

The fact that the results in the singular case $w = \boldsymbol{I}_{[-1,1]}$ are very similar to the results in the smooth case when w is Schwartz function could be erroneously interpreted as an indication that there are no qualitative differences between these two situations. This is not the case.

There is one subtle and meaningful qualitative difference buried in the proofs of Theorem 1.4 and Theorem 1.6. It has to do with the size of the tail of w as encoded by the quantity $q_m = q_m(w)$ defined in (1.5). Loosely speaking, a large q_m is an indication of a heavy tail.

The proof of Theorem 1.4 requires different arguments depending on whether $q_m \geq 1$ or $q_m < 1$; see Case 1 and Case 2 in the proof of Theorem 1.4. Since $q_m(w) \geq \frac{m}{m+2}$ for any w, we see that, for m large, the situation $q_m < 1$ is rather atypical. The case of the singular weight $w = \boldsymbol{I}_{[0,1]}$ is atypical because in this case $q_m(w) = \frac{m+2}{m+4} < 1$.

The size of the tail plays an even more fundamental role in the proof of the the Central Limit Theorem 1.6. The large m-limit of $\boldsymbol{\sigma}_m$ exists because of two facts: Wigner's semicircle theorem and the fact limit $\lim_m q_m = r = r(w)$ exists. However, the proof depends heavily on the size of the tail and there are two dramatically different cases, $r < \infty$ and $r = \infty$. The fact that the central limit theorem has a similar statement in both cases is a bit miraculous because different forces are at play in these two cases.

In Section 3 we show that the two behaviors, $r < \infty$ and $r = \infty$ are not just theoretically possible, they can actually happen for various choices of w. The quantity $r(w)$ also affects the size of the constant $C_m(w)$ in (1.13) which states that the expected number of critical points of $\boldsymbol{u}^\varepsilon$ is asymptotic to $C_m(w)\varepsilon^{-m}$ as $\varepsilon \to 0$.

For example, if $w(t) \sim t^{-\log\log(t)}$ as $t \to \infty$ (w has a very heavy tail), then

$$r(w) = \infty, \;\; \log C_m(w) \overset{(3.3)}{\sim} \frac{m}{2} e^{m+2}(e^2 - 1) \;\text{ as }\; m \to \infty.$$

If $w(t) \sim e^{-c(\log t)^2}$ as $t \to \infty$, then

$$r(w) = e^{8/c}, \;\; \log C_m(w) \overset{(3.5)}{\sim} \frac{1}{2c} m^2 \;\text{ as }\; m \to \infty.$$

If $w(t) \sim e^{-t^2}$ as $t \to \infty$ (w has a very light tail), then

$$r(w) = 1, \;\; \log C_m(w) \overset{(3.1)}{\sim} \frac{1}{2} m \log m \;\text{ as }\; m \to \infty.$$

These examples indicate the existence of three phases $r = 1$, $1 < r < \infty$, $r = \infty$. The transition from one phase to another manifests itself as a dramatic increase in the expected number of critical points. A similar phase transition phenomenon was observed by Y. Fyodorov [14] in a different context.

It is well known that if w is a Schwartz function, then the Schwartz kernel of $w_\varepsilon(\sqrt{\Delta})$ has a complete asymptotic expansion as $\varepsilon \searrow 0$; see e.g. [33, Chap. XII]. While the leading term of this expansion is well understood, the higher order terms are more nebulous. In Theorem B.5 we describe an explicit relationship between the second order term of this expansion and geometric invariants of the Riemann manifold (M, g).

Theorem B.5 is a new result and we have delegated it to an appendix, not to diminish its importance, but to help the reader separate the two conceptually different facts responsible for Theorem 1.8.

1.4. **The organization of the paper.** The remainder of the paper is organized as follows. Section 2 contains the proofs of the main results. In Section 3 we describe many classes of regular weights w. In particular, these examples show that the limit $r = \lim_{m\to\infty} r_m$ that appears in the statement of Theorem 1.6 can have any value in $[1, \infty]$. Section 4 contains the details of the probabilistic proof of the Gauss-Bonnet theorem outlined above.

To smooth the flow of the presentation we gathered in Appendices various technical results used in the proofs of the mains results. In Appendix A we describe the jets of order ≤ 4 along the diagonal of the square of the distance function $\mathrm{dist}_g : M \times M \to \mathbb{R}$ which are needed in the two-step asymptotics of the correlation kernel. This feels like a classical problem, but since precise references are hard to find we decided to include a complete proof. Our approach, based on the Hamilton-Jacobi equation satisfied by the distance function, is similar to the one sketched in [12, p.281-282].

In Appendix B we describe the small ε asymptotics of the Schwartz kernel of $w(\varepsilon\sqrt{\Delta})$ by relating them to the short time asymptotics for the wave kernel described in L. Hörmander [22, §17.4]. The central result in this appendix is Theorem B.5. It essentially states that the Riemann curvature tensor can be recovered from the *second order* terms of the $\varepsilon \to 0$ asymptotics of the fourth order jets along the diagonal of the Schwartz kernel of $w(\varepsilon\sqrt{\Delta})$.

In Appendix C we describe a few facts about Gaussian measures in a coordinate free form suitable for our geometric purposes. Finally, in Appendix D we have collected some facts about a family of Gaussian random symmetric matrices that appear in our investigation.

1.5. Notations.

(i) For any set S we denote by $|S| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ its cardinality. For any subset A of a set S we denote by $\boldsymbol{I}_A$ its characteristic function

$$\boldsymbol{I}_A : S \to \{0, 1\}, \;\; \boldsymbol{I}_A(s) = \begin{cases} 1, & s \in A \\ 0, & s \in S \setminus A. \end{cases}.$$

(ii) For any point x in a smooth manifold X we denote by δ_x the Dirac measure on X concentrated at x.

(iii) For any smooth manifold M we denote by $\mathrm{Vect}(M)$ the vector space of smooth vector fields on M.

(iv) For any random variable ξ we denote by $\boldsymbol{E}(\xi)$ and respectively $\boldsymbol{var}(\xi)$ its expectation and respectively its variance.

(v) For any finite dimensional real vector space $\boldsymbol{V}$ we denote by $\boldsymbol{V}^\vee$ its dual, $\boldsymbol{V}^\vee := \mathrm{Hom}(\boldsymbol{V}, \mathbb{R})$.

(vi) For any Euclidean space $\boldsymbol{V}$ we denote by $\mathrm{Sym}(\boldsymbol{V})$ the space of symmetric linear operators $\boldsymbol{V} \to \boldsymbol{V}$. When $\boldsymbol{V}$ is the Euclidean space $\mathbb{R}^m$ we set $\mathrm{Sym}_m := \mathrm{Sym}(\mathbb{R}^m)$. We denote by $\mathbb{1}_m$ the identity map $\mathbb{R}^m \to \mathbb{R}^m$.

(vii) We denote by $\mathcal{S}(\mathbb{R}^m)$ the space of Schwartz functions on $\mathbb{R}^m$.

(viii) For $v > 0$ we denote by γ_v the centered Gaussian measure on $\mathbb{R}$ with variance v,

$$\gamma_v(x)dx = \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}} |dx|.$$

Since $\lim_{v \searrow 0} \gamma_v = \delta_0$, we set $\gamma_0 := \delta_0$. For a real valued random variable X we write $X \in \boldsymbol{N}(0, v)$ if the probability distribution of X is γ_v.

(ix) If μ and ν are two finite measures on a common space X, then the notation $\mu \propto \nu$ means that

$$\frac{1}{\mu(X)}\mu = \frac{1}{\nu(X)}\nu.$$

2. Proofs

2.1. **A Kac-Rice type formula.** The key result behind Theorem 1.4 is a Kac-Rice type result which we intend to discuss in some detail in this section. This result gives an explicit, yet quite complicated description of the measure $\check{\boldsymbol{\sigma}}^\varepsilon$. More precisely, for any Borel subset $B \subset \mathbb{R}$, the Kac-Rice formula provides an integral representation of $\check{\boldsymbol{\sigma}}^\varepsilon(B)$ of the form

$$\check{\boldsymbol{\sigma}}^\varepsilon(B) = \int_M f_{\varepsilon,B}(\boldsymbol{p}) \, |dV_g(\boldsymbol{p})|,$$

for some integrable function $f_{\varepsilon,B} : M \to \mathbb{R}$. The core of the Kac-Rice formula is an explicit probabilistic description of the density $f_{\varepsilon,B}$.

Fix a point $\boldsymbol{p}\in M$. This determines three Gaussian random variables

$$\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\in\mathbb{R},\;\; d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\in T^*_{\boldsymbol{p}}M,\;\; \operatorname{Hess}_{\boldsymbol{p}}(\check{\boldsymbol{u}}_\varepsilon)\in \operatorname{Sym}(T_{\boldsymbol{p}}M), \tag{RV}$$

where $\operatorname{Hess}_{\boldsymbol{p}}(\check{\boldsymbol{u}}_\varepsilon):T_{\boldsymbol{p}}M\times T_{\boldsymbol{p}}M\to\mathbb{R}$ is the Hessian of $\boldsymbol{u}_{\boldsymbol{\omega}}$ at $\boldsymbol{p}$ defined in terms of the Levi-Civita connection of g and then identified with a symmetric endomorphism of $T_{\boldsymbol{p}}M$ using again the metric g. More concretely, if $(x^i)_{1\le i\le m}$ are g-normal coordinates at $\boldsymbol{p}$, then

$$\operatorname{Hess}_{\boldsymbol{p}}(\check{\boldsymbol{u}}_\varepsilon)\partial_{x^j}=\sum_{i=1}^m\partial^2_{x^ix^j}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\partial_{x^i}.$$

For $\varepsilon>0$ sufficiently small the covariance form of the Gaussian random vector $d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})$ is positive definite; see (2.3). We can identify it with a symmetric, positive definite linear operator

$$\boldsymbol{S}\big(\,d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big):T_{\boldsymbol{p}}M\to T_{\boldsymbol{p}}M.$$

More concretely, if $(x^i)_{1\le i\le m}$ are g-normal coordinates at $\boldsymbol{p}$, then we identify $\boldsymbol{S}\big(\,d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big)$ with a $m\times m$ real symmetric matrix whose (i,j)-entry is given by

$$\boldsymbol{S}_{ij}\big(\,d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big)=\boldsymbol{E}\big(\,\partial_{x_i}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\cdot\partial_{x^j}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big).$$

Theorem 2.1. *Fix a Borel subset $B\subset\mathbb{R}$. For any $\boldsymbol{p}\in M$ define*

$$f_{\varepsilon,B}(\boldsymbol{p}):=\big(\det\big(\,2\pi\boldsymbol{S}(\,\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,)\,\big)^{-\frac{1}{2}}\boldsymbol{E}\Big(\,|\det\operatorname{Hess}_{\boldsymbol{p}}(\check{\boldsymbol{u}}_\varepsilon)|\cdot\boldsymbol{I}_B(\,\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,)\;\Big|\;d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})=0\,\Big),$$

*where $\boldsymbol{E}\big(\;$**var** $\;|\,$**cons**$\;\big)$ stands for the conditional expectation of the variable* **var** *given the constraint* **cons**. *Then*

$$\check{\boldsymbol{\sigma}}^\varepsilon(B)=\int_M f_{\varepsilon,B}(\boldsymbol{p})\,|dV_g(\boldsymbol{p})|. \tag{2.1}$$

□

This theorem is a special case of a general result of Adler-Taylor, [1, Cor. 11.2.2]. Proposition 2.2 below shows that the technical assumptions in [1, Cor. 11.2.2] are satisfied if $\varepsilon\ll 1$.

For the above theorem to be of any use we need to have some concrete information about the Gaussian random variables (RV). All the relevant statistical invariants of these variables can be extracted from the covariance kernel of the random function $\check{\boldsymbol{u}}_\varepsilon$.

2.2. **Proof of Theorem 1.4.** Fix $\varepsilon>0$. For any $\boldsymbol{p}\in M$, we have the centered Gaussian random vector

$$\big(\,\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p}),d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p}),\operatorname{Hess}_{\boldsymbol{p}}(\check{\boldsymbol{u}}_\varepsilon)\,\big)\in\mathbb{R}\oplus T^*_{\boldsymbol{p}}M\oplus\operatorname{Sym}(T_{\boldsymbol{p}}M).$$

We fix normal coordinates $(x^i)_{1\le i\le m}$ at $\boldsymbol{p}$ and we can identify the above Gaussian vector with the centered Gaussian vector

$$\big(\,\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p}),\;(\partial_{x^i}\check{\boldsymbol{u}}_\varepsilon(p)\,)_{1\le i\le m},\;\partial^2_{x^ix^j}(\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,)_{1\le i,j\le m}\,\big)\in\mathbb{R}\oplus\mathbb{R}^m\oplus\operatorname{Sym}_m.$$

The next result is the key reason the Kac-Rice formula can be applied successfully to the problem at hand.

Proposition 2.2. *For any $1\le i,j,k,\ell\le m$ we have the uniform in $\boldsymbol{p}$ asymptotic estimates as $\varepsilon\searrow 0$*

$$\boldsymbol{E}\big(\,\check{\boldsymbol{u}}_\varepsilon(p)^2\,\big)=\check{s}_m\varepsilon^{-m}\big(1+O(\varepsilon^2)\,\big), \tag{2.2a}$$

$$\boldsymbol{E}\big(\,\partial_{x^i}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\partial_{x^j}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big)=d_m\varepsilon^{-(m+2)}\delta_{ij}\big(\,1+O(\varepsilon^2)\,\big), \tag{2.2b}$$

$$\boldsymbol{E}\big(\,\partial^2_{x^ix^j}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\partial^2_{x^kx^\ell}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big) = h_m\varepsilon^{-(m+4)}(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\big(\,1+O(\varepsilon^2)\,\big), \tag{2.2c}$$

$$\boldsymbol{E}\big(\,\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\partial^2_{x^ix^j}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big) = -d_m\varepsilon^{-(m+2)}\delta_{ij}\big(\,1+O(\varepsilon^2)\,\big), \tag{2.2d}$$

$$\boldsymbol{E}\big(\,\check{\boldsymbol{u}}_\varepsilon(p)\partial_{x^i}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big) = O(\varepsilon^{-m}),\;\; \boldsymbol{E}\big(\,\partial_{x^i}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\partial^2_{x^jx^j}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big) = O(\varepsilon^{-(m+2)}), \tag{2.2e}$$

where $\check{s}_m = s_m + \boldsymbol{\omega}_m$ and the constants s_m, d_m, h_m are defined by (1.3). □

Proof. Denote by $\check{\mathscr{E}}^\varepsilon$ the covariance kernel of the random function $\check{\boldsymbol{u}}_\varepsilon = X_{\boldsymbol{\omega}(\varepsilon)} + \boldsymbol{u}_\varepsilon$. Note that

$$\check{\mathscr{E}}^\varepsilon(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{\omega}(\varepsilon) + \mathscr{E}^\varepsilon(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{\omega}_m\varepsilon^{-m} + \mathscr{E}^\varepsilon(\boldsymbol{p},\boldsymbol{q}).$$

Fix a point $\boldsymbol{p}_0 \in M$ and normal coordinates at $\boldsymbol{p}_0$ defined in an open neighborhood $\mathcal{O}_0$ of $\boldsymbol{p}_0$. The restriction of $\mathscr{E}^\varepsilon$ to $\mathcal{O}_0\times\mathcal{O}_0$ can be viewed as a function $\mathscr{E}^\varepsilon(x,y)$ defined in an open neighborhood of $(0,0)$ in $\mathbb{R}^m\times\mathbb{R}^m$. For any $\alpha,\beta\in(\mathbb{Z}_{\geq 0})|^m$ we have

$$\boldsymbol{E}\big(\,\partial^\alpha_x\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p}_0)\partial^\beta_x(\check{\boldsymbol{u}}_\varepsilon)\,\big) = \partial^\alpha_x\partial^\beta_y\check{\mathscr{E}}^\varepsilon(x,y)_{x=y=0}.$$

Proposition 2.2 is now a consequence of the spectral estimates (B.1) in Appendix B. □

From the estimate (2.2b) we deduce that

$$\boldsymbol{S}(\,d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,) = d_m\varepsilon^{-(m+2)}\big(\,\mathbb{1}_m + O(\varepsilon^2)\,\big), \tag{2.3}$$

so that

$$\sqrt{|\det\boldsymbol{S}(\check{\boldsymbol{u}}_\varepsilon(p))|} = (d_m)^{\frac{m}{2}}\varepsilon^{-\frac{m(m+2)}{2}}\big(\,1+O(\varepsilon^2)\,\big)\;\text{ as }\varepsilon\to 0. \tag{2.4}$$

Consider the rescaled random vector

$$(\check{s}^\varepsilon, v^\varepsilon, H^\varepsilon) =:= \big(\varepsilon^{\frac{m}{2}}\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p}), \varepsilon^{\frac{m+2}{2}}d\check{\boldsymbol{u}}_\varepsilon(p),\;\; \varepsilon^{\frac{m+4}{2}}\nabla^2\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big).$$

From Proposition 2.2 we deduce the following (uniform in $\boldsymbol{p}$) estimates as $\varepsilon\searrow 0$.

$$\boldsymbol{E}(\,(\check{s}^\varepsilon)^2\,) = \check{s}_m\big(1+O(\varepsilon^2)\,\big), \tag{2.5a}$$

$$\boldsymbol{E}\big(\,v^\varepsilon_i v^\varepsilon_j\,\big) = d_m\delta_{ij}\big(\,1+O(\varepsilon^2)\,\big), \tag{2.5b}$$

$$\boldsymbol{E}\big(\,H^\varepsilon_{ij}H^\varepsilon_{kl}\,\big) = h_m(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\big(\,1+O(\varepsilon^2)\,\big), \tag{2.5c}$$

$$\boldsymbol{E}\big(\,\check{s}^\varepsilon H^\varepsilon_{ij}\,\big) = -d_m\delta_{ij}\big(\,1+O(\varepsilon^2)\,\big), \tag{2.5d}$$

$$\boldsymbol{E}\big(\,\check{s}^\varepsilon v^\varepsilon_i\,\big) = O(\varepsilon),\;\; \boldsymbol{E}\big(\,v^\varepsilon_i H^\varepsilon_{jk}\,\big) = O(\varepsilon). \tag{2.5e}$$

The probability distribution of the variable s^ε is

$$d\gamma_{\check{s}_m(\varepsilon)}(x) = \frac{1}{\sqrt{2\pi\check{s}_m(\varepsilon)}}e^{-\frac{x^2}{2\check{s}_m(\varepsilon)}}|dx|,$$

where $\check{s}_m(\varepsilon) = \check{s}_m + O(\varepsilon)$. Fix a Borel set $B\subset\mathbb{R}$. We have

$$\begin{gathered}\boldsymbol{E}\big(\,|\det\nabla^2\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})|\boldsymbol{I}_B\big(\,\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\,\big)\;\big|\;d\check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})=0\,\big) = \varepsilon^{-\frac{m(m+4)}{2}}\boldsymbol{E}\big(\,|\det H^\varepsilon|\boldsymbol{I}_{\varepsilon^{\frac{m}{2}}B}(\check{s}^\varepsilon)\;\big|\;v^\varepsilon=0\,\big)\\ = \varepsilon^{-\frac{m(m+4)}{2}}\underbrace{\int_{\varepsilon^{\frac{m}{2}}B}\boldsymbol{E}\big(\,|\det H^\varepsilon|\;\big|\;\check{s}^\varepsilon=x,\;v^\varepsilon=0\,\big)\frac{e^{-\frac{x^2}{2\check{s}_m(\varepsilon)}}}{\sqrt{2\pi\check{s}_m(\varepsilon)}}|dx|}_{=:q_{\varepsilon,\boldsymbol{p}}(\varepsilon^{\frac{m}{2}}B)}.\end{gathered} \tag{2.6}$$

Using (2.4) and (2.6) we deduce from Theorem 2.1 that

$$\check{\boldsymbol{\sigma}}^\varepsilon(B) = \varepsilon^{-m}\left(\frac{1}{2\pi d_m}\right)^{\frac{m}{2}}\int_M q_{\varepsilon,\boldsymbol{p}}(\varepsilon^{\frac{m}{2}}B)\rho_L(\boldsymbol{p})|dV_g(\boldsymbol{p})|,$$

where $\rho_\varepsilon : M \to \mathbb{R}$ is a function that satisfies the uniform in $\boldsymbol{p}$ estimate

$$\rho_\varepsilon(p) = 1 + O(\varepsilon) \ \text{ as } \varepsilon \to 0. \tag{2.7}$$

Hence

$$\varepsilon^m \left(\mathcal{R}_{\varepsilon^{\frac{m}{2}}}\right)_* \check{\boldsymbol{\sigma}}^\varepsilon(B) = \left(\frac{1}{2\pi d_m}\right)^{\frac{m}{2}} \int_M q_{\varepsilon,\boldsymbol{p}}(B)\rho_\varepsilon(\boldsymbol{p})|dV_g(\boldsymbol{p})|. \tag{2.8}$$

To continue the computation we need to investigate the behavior of $q_{\varepsilon,\boldsymbol{p}}(B)$ as ε. More concretely, we need to elucidate the nature of the Gaussian vector

$$\big(\, H^\varepsilon \;\big|\; \check{s}^\varepsilon = x,\ v^\varepsilon = 0\,\big).$$

We will achieve this via the regression formula (C.3). For simplicity we set

$$Y^\varepsilon := (\check{s}^\varepsilon, v^\varepsilon) \in \mathbb{R} \oplus \mathbb{R}^m.$$

The components of Y^ε are

$$Y_0^\varepsilon = \check{s}^\varepsilon,\ \ Y_i^\varepsilon = v_i^\varepsilon,\ \ 1 \le i \le m.$$

Using (2.5a), (2.5b) and (2.5e) we deduce that for any $1 \le i, j \le m$ we have

$$\boldsymbol{E}(Y_0^\varepsilon Y_i^\varepsilon) = \check{s}_m \delta_{0i} + O(\varepsilon),\ \ \boldsymbol{E}(Y_i^\varepsilon Y_j^\varepsilon) = d_m \delta_{ij} + O(\varepsilon^2).$$

If $\boldsymbol{S}(Y^\varepsilon)$ denotes the covariance operator of Y, then we deduce that

$$\boldsymbol{S}(Y^\varepsilon)^{-1}_{0,i} = \frac{1}{\check{s}_m}\delta_{0i} + O(\varepsilon),\ \ \boldsymbol{S}(Y^\varepsilon)^{-1}_{ij} = \frac{1}{d_m}\delta_{ij} + O(\varepsilon). \tag{2.9}$$

We now need to compute the covariance operator $\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)$. To do so, we equip Sym_m with the inner product

$$(A, B) = \operatorname{tr}(AB),\ \ A, B \in \mathrm{Sym}_m$$

The space Sym_m has a canonical orthonormal basis

$$\widehat{E}_{ij},\ \ 1 \le i \le j \le m,$$

where

$$\widehat{\boldsymbol{E}}_{ij} = \begin{cases} \boldsymbol{E}_{ij}, & i = j, \\ \frac{1}{\sqrt{2}}\boldsymbol{E}_{ij}, & i < j, \end{cases}$$

and $\boldsymbol{E}_{ij}$ denotes the symmetric matrix nonzero entries only at locations (i, j) and (j, i) and these entries are equal to 1. Thus a matrix $A \in \mathrm{Sym}_m$ can be written as

$$A = \sum_{i \le j} a_{ij}\boldsymbol{E}_{ij} = \sum_{i \le j} \hat{a}_{ij}\widehat{\boldsymbol{E}}_{ij},$$

where

$$\widehat{a}_{ij} = \begin{cases} a_{ij}, & i = j, \\ \sqrt{2}a_{ij}, & i < j. \end{cases}$$

The covariance operator $\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)$ is the linear map $\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon) : \mathbb{R} \oplus \mathbb{R}^m \to \mathrm{Sym}_m$ given by

$$\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)\left(\sum_{\alpha=0}^m y_\alpha \boldsymbol{e}_\alpha\right) = \sum_{i<j,\alpha} \boldsymbol{E}(\widehat{H}^\varepsilon_{ij} Y^\varepsilon_\alpha) y_\alpha \widehat{\boldsymbol{E}}_{ij} = \sum_{i<j,\alpha} \boldsymbol{E}(H^\varepsilon_{ij} Y^\varepsilon_\alpha) y_\alpha \boldsymbol{E}_{ij},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ denotes the canonical orthonormal basis in $\mathbb{R} \oplus \mathbb{R}^m$. Using (2.5d) and (2.5e) we deduce that

$$\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)\left(\sum_{\alpha=0}^{m} y_\alpha \boldsymbol{e}_\alpha\right) = -y_0 d_m \mathbb{1}_m + O(\varepsilon). \tag{2.10}$$

We deduce that the transpose $\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)^\vee$ satisfies

$$\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)^\vee\left(\sum_{i\leq j} \hat{a}_{ij} \widehat{\boldsymbol{E}}_{ij}\right) = -d_m \operatorname{tr}(A)\boldsymbol{e}_0 + O(\varepsilon). \tag{2.11}$$

Set

$$Z^\varepsilon := (H^\varepsilon | \check{s}^\varepsilon = x, v^\varepsilon = 0) - \boldsymbol{E}(H^\varepsilon | \check{s}^\varepsilon = x, v^\varepsilon = 0).$$

Above, Z^ε is a *centered* Gaussian random matrix with covariance operator

$$\boldsymbol{S}(Z^\varepsilon) = \boldsymbol{S}(H^\varepsilon) - \boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)\boldsymbol{S}(Y^\varepsilon)^{-1}\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)^\vee.$$

This means that

$$\boldsymbol{E}\big(\,\widehat{z}^\varepsilon_{ij}\widehat{z}^\varepsilon_{k\ell}\,\big) = (\widehat{\boldsymbol{E}}_{ij}, \boldsymbol{S}(Z^\varepsilon)\widehat{\boldsymbol{E}}_{k\ell}).$$

Using (2.9), (2.10) and (2.11) we deduce that

$$\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)\boldsymbol{S}(Y^\varepsilon)^{-1}\boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)^\vee\left(\sum_{i\leq j} \hat{a}_{ij}\widehat{\boldsymbol{E}}_{ij}\right) = \frac{d_m^2}{\check{s}_m}\operatorname{tr}(A)\mathbb{1}_m + O(\varepsilon)$$

$$\boldsymbol{E}\big(\,(z^\varepsilon_{ij})^2\,\big) = h_m + O(\varepsilon),\;\; \boldsymbol{E}(z^\varepsilon_{ii} z^\varepsilon_{jj}) = h_m - \frac{d_m^2}{\check{s}_m} + O(\varepsilon),\;\; \forall i<j,$$

$$\boldsymbol{E}\big(\,(z^\varepsilon_{ii})^2\,\big) = 3h_m - \frac{d_m^2}{\check{s}_m} + O(\varepsilon),\;\; \forall i,$$

and

$$\boldsymbol{E}(z^\varepsilon_{ij} z^\varepsilon_{k\ell}) = O(\varepsilon),\;\; \forall i<j,\;\; k\leq \ell,\;\; (i,j)\neq(k,\ell).$$

We can rewrite these equalities in the compact form

$$\boldsymbol{E}(z^\varepsilon_{ij} z^\varepsilon_{k\ell}) = \left(h_m - \frac{d_m^2}{s_m}\right)\delta_{ij}\delta_{k\ell} + h_m(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + O(\varepsilon).$$

Note that

$$h_m - \frac{d_m^2}{\check{s}_m} \overset{(1.9)}{=} \frac{r_m - 1}{r_m} h_m.$$

We set

$$\kappa_m := \frac{(r_m - 1)}{2r_m},$$

so that

$$\boldsymbol{E}(z^\varepsilon_{ij} z^\varepsilon_{k\ell}) = 2\kappa_m h_m \delta_{ij}\delta_{k\ell} + h_m(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + O(\varepsilon).$$

Using (C.4) we deduce that

$$\boldsymbol{E}(H^\varepsilon | \check{s}^\varepsilon = x, v^\varepsilon = 0) = \boldsymbol{Cov}(H^\varepsilon, Y^\varepsilon)\boldsymbol{S}(Y^\varepsilon)^{-1}(x\boldsymbol{e}_0) = -\frac{x d_m}{\check{s}_m}\mathbb{1}_m + O(\varepsilon). \tag{2.12}$$

We deduce that the Gaussian random matrix $(H^\varepsilon | \check{s}^\varepsilon = x, v^\varepsilon = 0)$ converges uniformly in $\boldsymbol{p}$ as $\varepsilon \to 0$ to the random matrix $A - \frac{x}{r_m(m+4)} \mathbb{1}_m$, where A belongs to the Gaussian ensemble $\mathrm{Sym}_m^{2\kappa_m h_m, h_m}$ described in Appendix D. Thus

$$\lim_{\varepsilon \to 0} q_{\varepsilon, \boldsymbol{p}}(B) = q_\infty(B) := \int_B \boldsymbol{E}_{\mathrm{Sym}_m^{2\kappa_m h_m, h_m}} \Big(\big| \det\big(A - \frac{x d_m}{\check{s}_m} \mathbb{1}_m \big) \big| \Big) \frac{e^{-\frac{x^2}{2\check{s}_m}}}{\sqrt{2\pi \check{s}_m}} dx$$

$$= (h_m)^{\frac{m}{2}} \int_B \boldsymbol{E}_{\mathrm{Sym}_m^{2\kappa_m, 1}} \Big(\big| \det\big(A - \frac{x d_m}{\check{s}_m \sqrt{h_m}} \mathbb{1}_m \big) \big| \Big) \frac{e^{-\frac{x^2}{2\check{s}_m}}}{\sqrt{2\pi \check{s}_m}} dx$$

$$= (h_m)^{\frac{m}{2}} \int_{(\check{s}_m)^{-\frac{1}{2}} B} \boldsymbol{E}_{\mathrm{Sym}_m^{2\kappa_m, 1}} \Big(\big| \det\big(A - \alpha_m y \mathbb{1}_m \big) \big| \Big) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dx,$$

where

$$\alpha_m = \frac{d_m}{\sqrt{\check{s}_m h_m}} \stackrel{(1.9)}{=} \frac{1}{\sqrt{r_m}}.$$

This proves that

$$\lim_{\varepsilon \searrow 0} \mathcal{R}_{(\check{s}_m)^{-\frac{1}{2}}} q_{\varepsilon, \boldsymbol{p}}(B) = (h_m)^{\frac{m}{2}} \underbrace{\int_B \boldsymbol{E}_{\mathbb{S}_m^{2\kappa_m, 1}} \Big(\Big| \det\Big(A - \frac{y}{\sqrt{r_m}} \mathbb{1}_m \Big) \Big| \Big) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy}_{=: \mu_m(B)}.$$

Using the last equality, the normalization assumption ($*$) and the estimate (2.7) in (2.8) we conclude

$$\big(\mathcal{R}_{(\check{s}_m \varepsilon^{-m})^{-\frac{1}{2}}} \big)_* \check{\boldsymbol{\sigma}}^\varepsilon(B) = \varepsilon^{-m} \left(\left(\frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \mu_m(B) + O(\varepsilon) \right) \ \text{ as } \varepsilon \to 0. \tag{2.13}$$

In particular

$$\boldsymbol{N}^\varepsilon = \varepsilon^{-m} \left(\left(\frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \mu_m(\mathbb{R}) + O(\varepsilon) \right) \ \text{ as } \varepsilon \to 0. \tag{2.14}$$

Observe that the density of μ_m is

$$\frac{d\mu_m}{dy} = \boldsymbol{E}_{\mathrm{Sym}_m^{2\kappa_m, 1}} \Big(\Big| \det\Big(A - \frac{y}{\sqrt{r_m}} \mathbb{1}_m \Big) \Big| \Big) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \tag{2.15}$$

$(\tilde{A} = \sqrt{r_m} A)$

$$= r_m^{-\frac{m}{2}} \boldsymbol{E}_{\mathrm{Sym}_m^{2\kappa_m r_m, r_m}} \Big(\Big| \det\Big(\tilde{A} - y \mathbb{1}_m \Big) \Big| \Big) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

$(2k_m r_m = r_m - 1)$

$$\stackrel{(D.7b)}{=} r_m^{-\frac{m}{2}} 2^{\frac{3}{2}} (2r_m)^{\frac{m+1}{2}} \Gamma\left(\frac{m+3}{2} \right) (\gamma_{r_m - 1} * \theta^+_{m+1, r_m})(y) \gamma_1(y).$$

$$= 2^{\frac{m+4}{2}} r_m^{\frac{1}{2}} \Gamma\left(\frac{m+3}{2} \right) (\gamma_{r_m - 1} * \theta^+_{m+1, r_m})(y) \gamma_1(y).$$

This proves part (a) and (1.15a) in Theorem 1.4. To prove (1.15b) we distinguish two cases.

Case 1. $r_m > 1$. From Lemma D.2 we deduce that

$$\begin{aligned} &\boldsymbol{E}_{\mathrm{Sym}_m^{2\kappa_m,1}}\Big(\Big|\det\Big(A - \frac{y}{\sqrt{r_m}}\mathbb{1}_m\Big)\Big|\Big) \\ &= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi\kappa_m}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4\tau_m^2}(\lambda - \frac{y(\tau_m^2+1)}{\sqrt{r_m}})^2 + \frac{(\tau_m^2+1)y^2}{4r_m}}d\lambda, \end{aligned} \tag{2.16}$$

where

$$\tau_m^2 := \frac{\kappa_m}{\kappa_m - 1} = \frac{r_m - 1}{r_m + 1}.$$

Thus

$$\begin{aligned} \frac{d\mu_m}{dy} &= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi\kappa_m}}e^{\frac{(\tau_m^2+1-2r_m)y^2}{4r_m}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4\tau_m^2}(\lambda - \frac{y(\tau_m^2+1)}{\sqrt{r_m}})^2}d\lambda \\ &= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi\kappa_m}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4\tau_m^2}(\lambda - \frac{y(\tau_m^2+1)}{\sqrt{r_m}})^2 - \frac{r_m y^2}{2(r_m+1)}}d\lambda. \end{aligned}$$

An elementary computation yields

$$-\frac{1}{4\tau_m^2}\left(\lambda - (\tau_m^2+1)\frac{y}{\sqrt{r_m}}\right)^2 - \frac{r_m y^2}{2(r_m+1)} = -\frac{1}{4}\lambda^2 - \left(\sqrt{\frac{1}{2(r_m-1)}}\lambda - y\sqrt{\frac{r_m}{2(r_m-1)}}\right)^2.$$

Now set

$$\beta_m := \frac{1}{(r_m - 1)}.$$

We deduce

$$\frac{d\mu_m}{dy} = 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{2\pi\sqrt{\kappa_m}}\int_{\mathbb{R}}\rho_{m+1,1}(\lambda)e^{-\frac{1}{4}\lambda^2}e^{-\frac{\beta_m}{2}(\lambda - \sqrt{r_m}y)^2}d\lambda.$$

$(\lambda := \sqrt{r}\lambda)$

$$\begin{aligned} &= 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{2\pi\kappa_m}}\int_{\mathbb{R}}\sqrt{r_m}\rho_{m+1,1}(\sqrt{r_m}\lambda)e^{-\frac{r_m}{4}\lambda^2}e^{-\frac{r_m\beta_m}{2}(\lambda - y)^2}d\lambda \\ &\overset{(D.6)}{=} 2^{\frac{m+3}{2}}\Gamma\left(\frac{m+3}{2}\right)\frac{1}{\sqrt{\kappa_m r_m\beta_m}}\int_{\mathbb{R}}\rho_{m+1,1/r_m}(\lambda)e^{-\frac{r_m}{4}\lambda^2}d\gamma_{\frac{1}{\beta_m r_m}}(y-\lambda)d\lambda. \end{aligned}$$

$(\kappa_m r_m\beta_m = \frac{1}{2})$

$$\begin{aligned} &= 2^{\frac{m+4}{2}}\Gamma\left(\frac{m+3}{2}\right)\int_{\mathbb{R}}\rho_{m+1,1/r_m}(\lambda)e^{-\frac{r_m}{4}\lambda^2}d\gamma_{\frac{1}{\beta_m r_m}}(y-\lambda)d\lambda \\ &= 2^{\frac{m+4}{2}}\Gamma\left(\frac{m+3}{2}\right)\int_{\mathbb{R}}\rho_{m+1,1/r_m}(\lambda)e^{-\frac{r_m}{4}\lambda^2}d\gamma_{\frac{r_m-1}{r_m}}(y-\lambda)d\lambda \end{aligned}$$

Using the last equality in (2.13) we obtain the case $r_m > 1$ (1.15b) of Theorem 1.4.

Case 2. $r_m = 1$. The proof of Theorem 1.4 in this case follows a similar pattern. Note first that in this case $\kappa_m = 0$ so invoking Lemma D.1 we obtain the following counterpart of (2.16)

$$\boldsymbol{E}_{\mathrm{GOE}_m^1}\Big(\Big|\det\Big(A - y\mathbb{1}_m\Big)\Big|\Big) = 2^{\frac{m+4}{2}}\Gamma\left(\frac{m+3}{2}\right)e^{\frac{y^2}{4}}\rho_{m+1,1}(y).$$

Using this in (2.15) we deduce

$$\frac{d\mu_m}{dy} = 2^{\frac{m+4}{2}}\Gamma\left(\frac{m+3}{2}\right)e^{\frac{-y^2}{4}}\rho_{m+1,1}(y),$$

which is (1.15b) in the case $r_m = 1$. This completes the proof of Theorem 1.4. $\square$

2.3. **Proof of Corollary 1.5.** According to (1.12) we have $\gamma_{\boldsymbol{\omega}_m \varepsilon^{-m}} * \boldsymbol{\sigma}^\varepsilon = \check{\boldsymbol{\sigma}}^\varepsilon$. Thus

$$\gamma_{\frac{\boldsymbol{\omega}_m}{\check{s}_m}} * \left(\mathcal{R}_{\frac{1}{\sqrt{\check{s}_m \varepsilon^{-m}}}}\right)_* \boldsymbol{\sigma}^\varepsilon = \left(\mathcal{R}_{\frac{1}{\sqrt{\check{s}_m \varepsilon^{-m}}}}\right)_* \check{\boldsymbol{\sigma}}^\varepsilon.$$

Hence

$$\lim_{\varepsilon \to 0} \frac{1}{\boldsymbol{N}^\varepsilon} \gamma_{\frac{\boldsymbol{\omega}_m}{\check{s}_m}} * \left(\mathcal{R}_{\frac{1}{\sqrt{\check{s}_m \varepsilon^{-m}}}}\right)_* \boldsymbol{\sigma}^\varepsilon = \check{\boldsymbol{\sigma}}_m.$$

We can now conclude by invoking Lévy's continuity theorem [23, Thm.15.23(ii)]. $\square$

2.4. **Proof of Theorem 1.6.** We have

$$\check{\boldsymbol{\sigma}}_m = \frac{1}{K_m} \theta^-_{m+1, \frac{1}{r_m}} * \gamma_{\frac{r_m - 1}{r_m}} dy, \tag{2.17}$$

where

$$\theta^-_{m+1, \frac{1}{r_m}}(\lambda) = \rho_{m+1, \frac{1}{r_m}}(\lambda) e^{-\frac{r_m \lambda^2}{4}},$$

and

$$K_m = \int_{\mathbb{R}} \theta^-_{m+1, \frac{1}{r_m}} * \gamma_{\frac{r_m - 1}{r_m}}(y) dy = \int_{\mathbb{R}} \theta^-_{m+1, \frac{1}{r_m}}(\lambda) d\lambda = \int_{\mathbb{R}} \rho_{m+1, \frac{1}{r_m}}(\lambda) e^{-\frac{r_m \lambda^2}{4}} d\lambda.$$

We set

$$R_m(\lambda) := \rho_{m+1, \frac{1}{m}}(\lambda), \quad R_\infty(x) := \frac{1}{2\pi} \boldsymbol{I}_{\{|x| \le 2\}} \sqrt{4 - x^2}.$$

Fix $c \in (0, 2)$. In [27, §4.2] we proved that

$$\lim_{m \to \infty} \sup_{|x| \le c} |\bar{R}_m(x) - R_\infty(x)| = 0, \tag{2.18a}$$

and

$$\sup_{|x| \ge c} |\bar{R}_m(x) - R_\infty(x)| = O(1) \ \text{ as } m \to \infty. \tag{2.18b}$$

Then

$$\rho_{m+1, \frac{1}{r_m}}(\lambda) = \sqrt{\frac{r_m}{m}} R_m\left(\sqrt{\frac{r_m}{m}} \lambda\right), \quad \theta^-_{m+1, \frac{1}{r_m}}(\lambda) = \sqrt{\frac{r_m}{m}} R_m\left(\sqrt{\frac{r_m}{m}} \lambda\right) e^{-\frac{r_m \lambda^2}{4}}.$$

We now distinguish two cases.

Case 1. $r = \lim_{m \to \infty} r_m < \infty$. In particular, $r \in [1, \infty)$. In this case we have

$$K_m = \sqrt{\frac{r_m}{m}} \int_{\mathbb{R}} R_m\left(\sqrt{\frac{r_m}{m}} \lambda\right) e^{-\frac{r_m \lambda^2}{4}} d\lambda,$$

and using (2.18a)-(2.18b) we deduce

$$\lim_{m \to \infty} \int_{\mathbb{R}} R_m\left(\sqrt{\frac{r_m}{m}} \lambda\right) e^{-\frac{r_m \lambda^2}{4}} d\lambda = R_\infty(0) \int_{\mathbb{R}} e^{-\frac{r \lambda^2}{4}} dr = R_\infty(0) \sqrt{\frac{4\pi}{r}}.$$

Hence

$$K_m \sim K'_m = R_\infty(0) \sqrt{\frac{4\pi}{m}} \ \text{ as } m \to \infty. \tag{2.19}$$

Now observe that

$$\frac{1}{K'_m} \theta^-_{m+1, \frac{1}{r_m}}(\lambda) d\lambda = \frac{1}{R_\infty(0)} R_m\left(\sqrt{\frac{r_m}{m}} \lambda\right) \frac{r_m}{\sqrt{4\pi}} e^{-\frac{r_m \lambda^2}{4}} d\lambda$$

$$= \frac{1}{R_\infty(0)} R_m \left(\sqrt{\frac{r_m}{m}} \lambda \right) \gamma_{\frac{2}{r_m}}(d\lambda)$$

Using (2.18a) and (2.18b) we conclude that the sequence of measures

$$\frac{1}{K'_m} \theta^-_{m+1, \frac{1}{r_m}}(\lambda) d\lambda$$

converges weakly to the Gaussian measure $\gamma_{\frac{2}{r}}$. Using this and the asymptotic equality (2.19) in (2.17) we deduce

$$\lim_{m\to\infty} \check{\boldsymbol{\sigma}}_m = \gamma_{\frac{2}{r}} * \gamma_{\frac{r-1}{r}} = \gamma_{\frac{r+1}{r}}.$$

This proves Theorem 1.6 in the case $r < \infty$ since

$$\gamma_{\frac{\boldsymbol{\omega}}{\check{s}_m}} * \boldsymbol{\sigma}_m = \check{\boldsymbol{\sigma}}_m \;\; \text{and} \;\; \lim_{m\to\infty} \frac{\boldsymbol{\omega}_m}{\check{s}_m} \stackrel{(1.11)}{=} 0.$$

Case 2. $\lim_{m\to\infty} r_m = \infty$. In this case we have

$$\theta^-_{m+1, \frac{1}{r_m}}(\lambda) d\lambda = \sqrt{\frac{4\pi}{m}} R_m \left(\sqrt{\frac{r_m}{m}} \lambda \right) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda.$$

Lemma 2.3. *The sequence of measures*

$$R_m \left(\sqrt{\frac{r_m}{m}} \lambda \right) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda$$

converges weakly to the measure $R_\infty(0)\delta_0$.

Proof. Fix a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$. Observe first that

$$\lim_{m\to\infty} \underbrace{\int_{\mathbb{R}} \left(R_m \left(\sqrt{\frac{r_m}{m}} \lambda \right) - R_\infty \left(\sqrt{\frac{r_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda}_{=D_m} = 0. \tag{2.20}$$

Indeed, we have

$$D_m = \underbrace{\int_{|\lambda| < c\frac{\sqrt{m}}{\sqrt{r_m}}} \left(R_m \left(\sqrt{\frac{r_m}{m}} \lambda \right) - R_\infty \left(\sqrt{\frac{r_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda}_{=:D'_m}$$

$$+ \underbrace{\int_{|\lambda| > c\frac{\sqrt{m}}{\sqrt{r_m}}} \left(R_m \left(\sqrt{\frac{r_m}{m}} \lambda \right) - R_\infty \left(\sqrt{\frac{r_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda}_{=:D''_m}.$$

Observe that

$$D'_m \leq \sup_{|x| \leq c} |R_m(x) - R_\infty(x)| \int_{|\lambda| < c\frac{\sqrt{m}}{\sqrt{r_m}}} f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda$$

and invoking (2.18a) we deduce

$$\lim_{m\to\infty} D'_m = 0.$$

Using (2.18b) we deduce that there exists a constant $S > 0$ such that

$$D'_m \leq S \int_{|\lambda| > c\frac{\sqrt{m}}{\sqrt{r_m}}} \gamma_{\frac{2}{r_m}}(\lambda) d\lambda.$$

On the other hand, Chebyshev's inequality shows that

$$\int_{|\lambda|>c\frac{\sqrt{m}}{\sqrt{r_m}}} \gamma_{\frac{2}{r_m}}(\lambda)d\lambda \leq \frac{2}{c^2 m}.$$

Hence

$$\lim_{m\to\infty} D''_m = 0.$$

This proves (2.20).

The sequence of measures $\gamma_{\frac{2}{r_m}}(\lambda)d\lambda$ converges to δ_0 so that

$$R_\infty(0)f(0) = \lim_{m\to\infty}\int_{\mathbb{R}} R_\infty(0)f(\lambda)\gamma_{\frac{2}{r_m}}(\lambda)d\lambda.$$

Using (2.20) and the above equality we deduce that the conclusion of the lemma is equivalent to

$$\lim_{m\to\infty}\underbrace{\int_{\mathbb{R}}\left(R_\infty(0) - R_\infty\left(\sqrt{\frac{r_m}{m}}\lambda\right)\right)f(\lambda)\gamma_{\frac{2}{r_m}}(\lambda)d\lambda}_{=F_m} = 0. \tag{2.21}$$

To prove this we decompose F_m as follows.

$$F_m = \underbrace{\int_{|\lambda|<m^{-\frac{1}{4}}\frac{\sqrt{m}}{\sqrt{r_m}}}\left(R_\infty(0) - R_\infty\left(\sqrt{\frac{r_m}{m}}\lambda\right)\right)f(\lambda)\gamma_{\frac{2}{r_m}}(\lambda)d\lambda}_{=:F'_m}$$

$$+\underbrace{\int_{|\lambda|>m^{-\frac{1}{4}}\frac{\sqrt{m}}{\sqrt{r_m}}}\left(R_\infty(0) - R_\infty\left(\sqrt{\frac{r_m}{m}}\lambda\right)\right)f(\lambda)\gamma_{\frac{2}{r_m}}(\lambda)d\lambda}_{=:F''_m}.$$

Observe that

$$F'_m \leq \sup_{|x|\leq m^{-\frac{1}{4}}}|R_\infty(0) - R_\infty(x)|\int_{|\lambda|<m^{-\frac{1}{4}}\frac{\sqrt{m}}{\sqrt{r_m}}} f(\lambda)\gamma_{\frac{2}{r_m}}(\lambda)d\lambda$$

and since R_∞ is continuous at 0 we deduce

$$\lim_{m\to\infty} F'_m = 0.$$

Since R_∞ and f are bounded we deduce that there exists a constant $S > 0$ such that

$$F''_m \leq S\int_{|\lambda|>m^{-\frac{1}{4}}\frac{\sqrt{m}}{\sqrt{r_m}}}\gamma_{\frac{2}{r_m}}(\lambda)d\lambda.$$

On the other hand, Chebyshev's inequality shows that

$$\int_{|\lambda|>m^{-\frac{1}{4}}\frac{\sqrt{m}}{\sqrt{r_m}}}\gamma_{\frac{2}{r_m}}(\lambda)d\lambda \leq \frac{2}{\sqrt{m}}.$$

Hence

$$\lim_{m\to\infty} F''_m = 0.$$

This proves (2.21) and the lemma. □

Lemma 2.3 shows that

$$K_m \sim K'_m = \sqrt{\frac{4\pi}{m}} R_\infty(0),$$

and

$$\lim_{m\to\infty} \frac{1}{K_m} \theta^-_{m+1,\frac{1}{r_m}}(\lambda) d\lambda = \delta_0.$$

On the other hand

$$\lim_{m\to\infty} \gamma_{\frac{r_m-1}{r_m}}(\lambda) d\lambda = \gamma_1(\lambda) d\lambda,$$

so that

$$\lim_{m\to\infty} \breve{\boldsymbol{\sigma}}_m = \delta_0 * \gamma_1 = \gamma_1.$$

This completes the proof of Theorem 1.6. □

2.5. **Proof of Corollary 1.7.** Using (2.14) we deduce

$$\begin{aligned}
\varepsilon^m \boldsymbol{N}^\varepsilon &= \left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}} \mu_m(\mathbb{R}) + O(\varepsilon) \\
&= 2^{\frac{m+4}{2}} \Gamma\left(\frac{m+3}{2}\right) \left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}} \int_{\mathbb{R}} \theta^-_{m+1,\frac{1}{r_m}} * \gamma_{\frac{r_m-1}{r_m}}(y) dy + O(\varepsilon) \\
&= 2^{\frac{m+4}{2}} \Gamma\left(\frac{m+3}{2}\right) \left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}} \int_{\mathbb{R}} \theta^-_{m+1,\frac{1}{r_m}}(\lambda) d\lambda + O(\varepsilon) \\
&= \underbrace{2^{\frac{m+4}{2}} \Gamma\left(\frac{m+3}{2}\right) \left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}} K_m}_{=C_m(w)} + O(\varepsilon).
\end{aligned}$$

Lemma 2.3 implies that, as $m \to \infty$, we have

$$K_m \sim \sqrt{\frac{4\pi}{m}} R_\infty(0) = \frac{2}{\sqrt{\pi m}}.$$

We deduce that

$$C_m(w) \sim \frac{2^{\frac{m+6}{2}}}{\sqrt{\pi m}} \Gamma\left(\frac{m+3}{2}\right) \left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}} \quad \text{as } m \to \infty.$$

□

2.6. **Proof of Theorem 1.8.** Fix a point $\boldsymbol{p} \in M$ and normal coordinates (x^i) near $\boldsymbol{p}$. The equality (2.2b) shows that as $\varepsilon \to 0$ we have the following estimate, uniform in $\boldsymbol{p}$.

$$\boldsymbol{E}\big(\partial_{x^i} \check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p}) \partial_{x^j} \check{\boldsymbol{u}}_\varepsilon(\boldsymbol{p})\big) = d_m \varepsilon^{-(m+2)} \big(\delta_{ij} + O(\varepsilon)^2\big)\big).$$

Hence

$$h^\varepsilon(\partial_{x^i}, \partial_{x^j}) = \delta_{ij} + O(\varepsilon^2) = g_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}) + O(\varepsilon^2). \tag{2.22}$$

This proves (a) and (b) of Theorem 1.4.

With $\boldsymbol{p}$ and (x^i) as above we set

$$\mathscr{E}^\varepsilon_{i_1,\dots,i_a;j_1,\dots,j_b} := \frac{\partial^{a+b} \mathscr{E}^\varepsilon(x,y)}{\partial x^{i_1} \cdots \partial x^{i_a} \partial y^{j_1} \cdots \partial y^{j_b}}\Big|_{x=y=0},$$

$$h^\varepsilon_{ij} := h^\varepsilon_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}), \;\; 1 \leq i, j \leq m.$$

We denote by K^ε_{ij} the sectional curvature of h^ε along the plane spanned by $\partial_{x^i}, \partial_{x^j}$. Using [1, Lemma 12.2.1] and that the sectional curvatures of a metric are inverse proportional to the metric we deduce as in [27, §3.3] that

$$K^\varepsilon_{ij} = \frac{d_m}{\varepsilon^{m+2}} \times \frac{\mathscr{E}^\varepsilon_{ii;jj} - \mathscr{E}^\varepsilon_{ij;ij}}{\mathscr{E}^\varepsilon_{i;i}\mathscr{E}^\varepsilon_{j,j} - (\mathscr{E}^\varepsilon_{i;j})^2}.$$

Using Theorem B.5 we deduce that there exists a universal constant $\mathcal{Z}_m$ that depends only on m and w such that

$$\mathscr{E}^\varepsilon_{ii;jj} - \mathscr{E}^\varepsilon_{ij;ij} = \varepsilon^{-(m+2)}\mathcal{Z}_m K_{ij}(\boldsymbol{p})\big(\,1 + O(\varepsilon^2)\,\big), \tag{2.23}$$

where $K_{ij}(\boldsymbol{p})$ denotes the sectional curvature of g at $\boldsymbol{p}$. The estimate (2.2b) implies that

$$\mathscr{E}^\varepsilon_{i;i}\mathscr{E}^\varepsilon_{j,j} - (\mathscr{E}^\varepsilon_{i;j})^2 = d_m^2\varepsilon^{-2(m+2)}\big(\,1 + O(\varepsilon^2)\,\big).$$

Thus

$$K^\varepsilon_{ij} = \frac{\mathcal{Z}_m}{d_m} K_{ij}(\boldsymbol{p})\big(\,1 + O(\varepsilon^2)\,\big).$$

To determine the constant $\frac{\mathcal{Z}_m}{d_m}$ it suffices to compute it on a special manifold. Assume that M is the unit sphere S^m equipped with the round metric. This is is a homogeneous space equipped with an invariant metric g with positive sectional curvatures. The metrics h^ε are also invariant so there exists a constant $C_\varepsilon > 0$ such that $h^\varepsilon = C_\varepsilon g$. The estimate (2.22) implies that $C_\varepsilon = 1$ and thus $K^\varepsilon_{ij} = K_{ij}(\boldsymbol{p})$ so that $\frac{\mathcal{Z}_m}{d_m} = 1$. □

3. Some examples

We want to discuss several examples of weights w satisfying the assumptions of the central limit theorem, Theorem 1.6. Observe first that

$$r_n(w) \sim R_m(w) = \frac{I_{m-1}(w)I_{m+3}(w)}{I_{m+1}(w)} \;\text{ as } m \to \infty.$$

Moreover

$$R_n(w_\varepsilon) = R_n(w).$$

Example 3.1. Suppose that $w(t) = e^{-t^2}$. In this case $\mathscr{E}^\varepsilon$ is the Schwartz kernel of the heat operator $e^{-\varepsilon\Delta}$ whose asymptotics as $\varepsilon \to 0$ have been thoroughly investigated. The momenta (1.4) are

$$I_k(w) = \int_0^\infty t^k e^{-t^2} dt = \frac{1}{2}\int_0^\infty s^{\frac{k-1}{2}} e^{-s} ds = \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right).$$

Hence

$$R_m(w) = \frac{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2}+2)}{\Gamma(\frac{m}{2}+1)^2} = \frac{m+4}{m+2} \geq 1,\;\; q_m = \frac{m(m+4)}{(m+2)^2} < 1,\;\; \forall m$$

so that $r_m = 1$ for all m. Moreover, in this case we have

$$\frac{I_{m+3}(w)}{I_{m+1}(w)} = m+2,$$

so that

$$C_m(w) \sim \frac{2^{\frac{m+6}{2}}}{\sqrt{m}\pi^{\frac{m+1}{2}}}\Gamma\left(\frac{m+3}{2}\right) \;\text{ as } m \to \infty,$$

and Stirling's formula implies

$$\log C_m(w) \sim \frac{m}{2}\log m \;\text{ as } m \to \infty. \tag{3.1}$$

□

Example 3.2. Suppose that
$$w(t)=\exp\big(-(\log t)\log(\log t)\big),\ \ \forall t\geq 1.$$
Observe that
$$I_k(w)=\int_0^1 r^k w(r)dr+\int_1^\infty r^k\exp\big(-(\log r)\log(\log r)\big)dr.$$
This proves that
$$I_k(w)\sim J_k:=\int_1^\infty r^k\exp\big(-(\log r)\log(\log r)\big)dr\ \text{ as } k\to\infty.$$
Using the substitution $r=e^t$ we deduce
$$J_k=\int_0^\infty e^{(k+1)t-t\log t}dt.$$
We want to investigate the large λ asymptotics of the integral
$$T_\lambda=\int_0^\infty e^{-\phi_\lambda(t)}dt,\ \ \phi_\lambda(t)=\lambda t-t\log t. \tag{3.2}$$
We will achieve this by relying on the Laplace method [9, Chap. 4]. Note that
$$\phi_\lambda'(t)=\lambda-\log t-1,\ \ \phi_\lambda''(t)=-\frac{1}{t}.$$
Thus $\phi_\lambda(t)$ has a unique critical point
$$\tau=\tau(\lambda):=e^{\lambda-1}.$$
We make the change in variables $t=\tau s$ in (3.2). Observe that
$$\lambda e^{\lambda-1}s-e^{\lambda-1}s\log(e^{\lambda-1}s)=e^{\lambda-1}s-(\lambda-1)e^{\lambda-1}s-e^{\lambda-1}\log s=e^{\lambda-1}s(1-\log s)$$
and we deduce
$$T_\lambda=\tau\int_0^\infty e^{-\tau h(s)}ds,\ \ h(s)=s(\log s-1).$$
The asymptotics of the last integral can be determined using the Laplace method and we have, [9, §4.1]
$$T_\lambda\sim\tau e^{-\tau h(1)}\sqrt{\frac{2\pi}{\tau h''(1)}}=\sqrt{2\pi\tau}e^\tau.$$
Hence
$$J_k=T_{k+1}\sim\sqrt{2\pi\tau(k+1)}e^{\tau(k+1)}=\sqrt{2\pi e^k}e^{e^k}\ \text{ as } k\to\infty.$$
In this case
$$R_m(w)\to\infty\ \text{ as } m\to\infty.$$
Note that
$$\frac{h_m}{d_m}=\frac{2I_{m+3}(w)}{(m+2)I_{m+1}(w)}.$$
We deduce that
$$\log\left(\frac{h_m}{d_m}\right)\sim e^{m+4}-e^{m+2}\ \text{ as } m\to\infty.$$

Hence

$$\log C_m(w) \sim \frac{m}{2}e^{m+2}(e^2-1) \;\text{ as } m\to\infty. \tag{3.3}$$

□

Example 3.3. Suppose that

$$w(r) = \exp\bigl(-C(\log r)^\alpha\bigr),\;\; C>0,\;\; r>1,\;\; \alpha>1.$$

Arguing as in Example 3.2 we deduce that as $k\to\infty$

$$I_k(w) \sim \int_1^\infty r^k \exp\bigl(\,-C(\log r)^\alpha\,\bigr)dr = \int_0^\infty e^{(k+1)t-Ct^\alpha}dt.$$

Again, set

$$T_\lambda := \int_0^\infty e^{-\phi_\lambda(t)}dt,\;\; \phi_\lambda(t) := Ct^\alpha - \lambda t.$$

We determine the asymptotics of T_λ as $\lambda\to\infty$ using the Laplace method. Note that

$$\phi_\lambda'(t) = \alpha C t^{\alpha-1} - \lambda.$$

The function ϕ_λ has a unique critical point

$$\tau = \tau(\lambda) = \left(\frac{\lambda}{\alpha C}\right)^{\frac{1}{\alpha-1}}.$$

Observe that

$$\phi_\lambda(\tau s) = a(s^\alpha - bs),\;\; a := \left(\frac{\lambda}{C^{1/\alpha}\alpha}\right)^{\frac{\alpha}{\alpha-1}},\;\; b := \alpha^{\frac{1}{\alpha-1}},$$

$$T_\lambda = \tau(\lambda)\int_0^\infty e^{-a(s^\alpha - bs)}ds.$$

We set $g(s) := s^\alpha - bs$. Using the Laplace method [9, §4.2] we deduce

$$T_\lambda \sim \tau(\lambda)e^{-ag(1)}\sqrt{\frac{2\pi}{ag''(1)}} = \sqrt{\frac{2\pi}{a\alpha(\alpha-1)}}e^{a(b-1)}.$$

Hence

$$\log T_\lambda \sim \left(\frac{\lambda^\alpha}{C}\right)^{\frac{1}{\alpha-1}}\frac{\alpha^{\frac{1}{\alpha-1}}-1}{\alpha^{\frac{\alpha}{\alpha-1}}} =: Z(\alpha,C)\lambda^{\frac{\alpha}{\alpha-1}}.$$

Hence

$$\begin{aligned}
\log R_m(w) &\sim \log T_m + \log T_{m+4} - 2\log T_{m+2}\\
&\sim Z(\alpha,C)\Bigl(\,m^{\frac{\alpha}{\alpha-1}} + (m+4)^{\frac{\alpha}{\alpha-1}} - 2(m+2)^{\frac{\alpha}{\alpha-1}}\,\Bigr)\\
&= Z(\alpha,C)m^{\frac{\alpha}{\alpha-1}}\left(1 + \Bigl(1+\frac{4}{m}\Bigr)^{\frac{\alpha}{\alpha-1}} - 2\Bigl(1+\frac{2}{m}\Bigr)^{\frac{\alpha}{\alpha-1}}\right)\\
&\sim Z(\alpha,C)m^{\frac{\alpha}{\alpha-1}} \times \frac{8}{m^2}\times\frac{\alpha}{\alpha-1}\Bigl(\frac{\alpha}{\alpha-1}-1\Bigr) = \frac{8\alpha Z(\alpha)}{(\alpha-1)^2}m^{\frac{2-\alpha}{\alpha-1}}.
\end{aligned}$$

Hence

$$r = \lim_{m\to\infty} r_m = \times\begin{cases}\infty, & \alpha<2,\\ e^{16Z(2,C)}, & \alpha=2,\\ 1, & \alpha>2.\end{cases} \tag{3.4}$$

which shows that r can have any value in $[1,\infty]$. Note that in this case

$$\log I_{m+3}(w) - \log I_{m+1}(w) \sim Z(\alpha,C) m^{\frac{\alpha}{\alpha-1}} \left(\left(1+\frac{4}{m}\right)^{\frac{\alpha}{\alpha-1}} - \left(1+\frac{2}{m}\right)^{\frac{\alpha}{\alpha-1}} \right)$$

$$\sim \frac{2Z(\alpha,C)}{\alpha-1} m^{\frac{1}{\alpha-1}}, \;\; m\to\infty,$$

so that

$$\log C_m(w) \sim \frac{Z(\alpha,C)}{\alpha-1} m^{\frac{\alpha}{\alpha-1}}, \;\; m\to\infty. \tag{3.5}$$

□

Example 3.4. Suppose now that w is a weight with compact support disjoint from the origin. For example, assume that on the positive semi-axis it is given by

$$w(x) = \begin{cases} e^{-\frac{1}{1-(x-c)^2}}, & |x-c|\leq 1 \\ 0, & |x-c|>1, \end{cases} , \;\; c>1.$$

Then

$$I_k(w) = \int_{c-1}^{c+1} t^k e^{-\frac{1}{1-(t-c)^2}}\,dt = \int_{-1}^{1} (t+c)^k e^{-\frac{1}{1-t^2}}\,dt$$

$$= \underbrace{\int_{-1}^{0} (t+c)^k e^{-\frac{1}{1-t^2}}\,dt}_{I_k^-} + \underbrace{\int_{0}^{1} (t+c)^k e^{-\frac{1}{1-t^2}}\,dt}_{I_k^+}.$$

Observe that

$$\lim_{k\to\infty} c^{-k} I_k^- = 0.$$

On the other hand

$$I_k^+ = \int_0^1 (c+1-t)^k e^{-\frac{1}{t^2}}\,dt,$$

and we deduce

$$c^k \int_0^1 e^{-\frac{1}{t^2}}\,dt \leq I_k^+ \leq (c+1)^k \int_0^1 e^{-\frac{1}{t^2}}\,dt.$$

Hence the asymptotic behavior of $I_k(w)$ is determined by I_k^+. We will determine the asymptotic behavior of I_k^+ by relying again on the Laplace method. Set $a:=(c+1)$ so that

$$I_k^+ = \int_0^1 (a-t)^k e^{-\frac{1}{t^2}}\,dt = a^k \int_0^{\frac{1}{a}} (1-s)^k e^{-\frac{1}{a^2s^2}}\,ds = a^k \int_a^\infty (u-1)^k u^{-(k+2)} e^{-\frac{u^2}{a^2}}\,du.$$

Consider the phase

$$\phi_\hbar(s) = \frac{1}{\hbar}\log(1-s) - \frac{1}{a^2s^2}, \;\; \hbar\searrow 0,$$

and set

$$P_\hbar = a^{\frac{1}{\hbar}} \int_0^{\frac{1}{a}} e^{\phi_\hbar(s)}$$

so that

$$I_k^+ = P_{1/k}.$$

We have

$$\phi_\hbar'(s) = -\frac{1}{\hbar(1-s)} + \frac{2}{a^2s^3}, \;\; \phi_\hbar''(t) = -\frac{1}{\hbar(1-s)^2} - \frac{6}{a^2s^4}.$$

The phase $\phi_\hbar$ as a unique critical point $\tau = \tau(\hbar) \in (0, 1/a)$ satisfying

$$\hbar = \frac{a^2\tau^3}{2(1-\tau)} = \frac{a^2\tau^3}{2}\big(1 + O(\tau)\big),$$

so that

$$\tau = \left(\frac{2\hbar}{a^2}\right)^{\frac{1}{3}} \Big(1 + O\big(\hbar^{\frac{1}{3}}\big)\Big) \text{ as } \hbar \searrow 0. \tag{3.6}$$

Set

$$v := v(\hbar) := -\frac{1}{\phi_k''(\tau)} \sim \frac{a^2\tau^4}{6} \sim \frac{(2\hbar)^{\frac{4}{3}}}{6a^{\frac{2}{3}}} = \frac{1}{6}\left(\frac{2\hbar^2}{a}\right)^{\frac{2}{3}}. \tag{3.7}$$

We make the change in variables $s = \tau + \sqrt{v}x$ and we deduce

$$P_\hbar = e^{\phi_\hbar(\tau)} a^{\frac{1}{\hbar}} \sqrt{v} \int_{J(\hbar)} e^{\phi_\hbar(\tau+\sqrt{v}x) - \phi_\hbar(\tau)} dx, \quad J(\hbar) = \Big[-\frac{\tau}{\sqrt{v}}, \frac{1/a - \tau}{\sqrt{v}}\Big].$$

We claim that

$$\lim_{\hbar\to 0} \int_{J(\hbar)} e^{\phi_\hbar(\tau+\sqrt{v}x) - \phi_\hbar(\tau)} dx = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}. \tag{3.8}$$

It is convenient to think of τ as the small parameter and then redefine

$$\hbar = \hbar(\tau) = \frac{a^2\tau^3}{2(1-\tau)}$$

and think of v as a function of τ. Finally set $\sigma := \sqrt{v}$ and

$$\begin{aligned}
\varphi_\tau(x) := \phi_{\hbar(\tau)}(\tau + \sigma x) - \phi_{\hbar(\tau)}(\tau) = \frac{2(1-\tau)}{a^2\tau^3}\log(1-s) - \frac{1}{a^2 s^2} \\
= \frac{2(1-\tau)}{a^2\tau^3}\Big(\log(1-\tau-\sigma x) - \log(1-\tau)\Big) - \frac{1}{a^2}\left(\frac{1}{(\tau+\sigma x)^2} - \frac{1}{\tau^2}\right) \\
= \frac{2(1-\tau)}{a^2\tau^3}\log\left(1 - \frac{\sigma}{1-\tau}x\right) - \frac{1}{a^2\tau^2}\left(\frac{1}{(1+\frac{\sigma}{\tau}x)^2} - 1\right) \\
= \frac{1}{a^2\tau^2}\left(\frac{2(1-\tau)}{\tau}\log\left(1 - \frac{\sigma}{1-\tau}x\right) - \left(\frac{1}{(1+\frac{\sigma}{\tau}x)^2} - 1\right)\right).
\end{aligned}$$

The equality (3.8) is equivalent to

$$\lim_{\tau\to\infty} \int_{J(\hbar)} e^{\varphi_\tau(x)} = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx. \tag{3.9}$$

By construction, we have

$$\varphi_\tau(0) = \varphi_\tau'(0) = 0, \;\; \varphi_\tau''(0) = -1, \;\; \varphi_\tau(x) \le 0, \;\; \forall x \in J(\hbar).$$

Let us observe that

$$\lim_{\tau\to 0} \varphi_\tau(x) = \frac{1}{2}\varphi_\tau''(0)x^2 = -\frac{x^2}{2}, \;\; \forall x \in \mathbb{R}. \tag{3.10}$$

Indeed, fix $x \in \mathbb{R}$ and assume τ is small enough so that

$$\tau|x| < \frac{1}{2}. \tag{3.11}$$

Observe that

$$\varphi_\tau^{(j)}(0) = \frac{1}{a^2\tau^2}\left(\frac{2(1-\tau)}{\tau}\frac{d^j}{dx^j}|_{x=0}\log\left(1 - \frac{\sigma}{1-\tau}x\right) - \frac{d^j}{dx^j}|_{x=0}\left(\frac{1}{(1+\frac{\sigma}{\tau}x)^2} - 1\right)\right)$$

$$= \frac{1}{a^2\tau^2}\left(-\frac{2(1-\tau)}{\tau}\left(\frac{\sigma}{1-\tau}\right)^j + (-1)^{j+1}(j+1)!\left(\frac{\sigma}{\tau}\right)^j\right).$$

Using the estimate $\sigma = O(\tau^2)$ as $\tau \to 0$ we deduce that there exists $C > 0$ such that, for any $j \geq 0$ we have

$$\left|\varphi_\tau^{(j)}(0)\right| \leq C(j+1)!\tau^{j-2}.$$

Hence

$$\frac{1}{j!}\left|\varphi_\tau^{(j)}(0)x^j\right| \leq Cj|\tau x|^{j-2}x^2, \ \ \forall j \geq 2.$$

Thus if τ satisfies (3.11), we have

$$\varphi_\tau(x) + \frac{x^2}{2} = \varphi_\tau(x) - \varphi_\tau'(0)x - \frac{1}{2}\varphi_\tau''(0)x^2 = \sum_{j\geq 3}\frac{1}{j!}\varphi_\tau^{(j)}(0)x^j,$$

where the series in the right-hand side is absolutely convergent. Hence

$$\left|\,\varphi_\tau(x) + \frac{x^2}{2}\,\right| \leq Cx^2|\tau x|\sum_{j\geq 3} j|\tau x|^{j-3} \leq C|\tau x|x^2\sum_{j\geq 3} j2^{j-3}.$$

This proves (3.10).

Next we want to prove that there exists a constant $A > 0$ such that

$$\varphi_\tau(x) \leq A(1-|x|), \ \ \forall x \in J(\hbar), \ \ \forall \tau \ll 1. \tag{3.12}$$

We will achieve this by relying on the concavity of φ_τ over the interval $J(\hbar)$. The graph of φ_τ is situated below either of the lines tangent to the graph at $x = \pm 1$. Thus

$$\varphi_\tau(x) \leq \varphi_\tau(1) + \varphi_\tau'(1)(x-1) \leq -\varphi_\tau'(1) + \varphi_\tau'(1)x,$$
$$\varphi_\tau(x) \leq \varphi_\tau(-1) + \varphi_\tau'(-1)(x+1) \leq \varphi_\tau'(-1) + \varphi_\tau'(-1).$$

Now observe that

$$\frac{d}{dx}\varphi_\tau(x) = \frac{1}{a^2\tau^2}\left(-\frac{2\sigma}{\tau}\frac{1}{1-\frac{\sigma}{1-\tau}x} + \frac{2\sigma}{\tau}\frac{1}{(1+\frac{\sigma}{\tau}x)^3}\right) = \frac{2\sigma}{a^2\tau^3}\left(\frac{1}{(1+\frac{\sigma}{\tau}x)^3} - \frac{1}{1-\frac{\sigma}{1-\tau}x}\right).$$

Using the fact that $\sigma = O(\tau^2)$ we deduce from the above equality that

$$|\varphi_\tau'(\pm 1)| = O(1), \ \ \text{as } \tau \to 0.$$

This proves (3.12). Using (3.10), (3.12) and the dominated convergence theorem we deduce

$$\lim_{\tau\to\infty}\int_{J(\hbar)} e^{\varphi_\tau(x)}dx = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}.$$

We conclude that

$$P_\hbar \sim e^{\phi_\hbar(\tau)}a^{\frac{1}{\hbar}}\sqrt{2\pi v} \ \ \text{as } \hbar \to 0 \tag{3.13}$$

Now observe that

$$\phi_\hbar(\tau) = \frac{1}{\hbar}\log(1-\tau) - \frac{1}{a^2\tau^2} = \frac{2(1-\tau)}{a^2\tau^2}\frac{\log(1-\tau)}{\tau} - \frac{1}{a^2\tau^2} \sim -\frac{3}{a^2\tau^2}.$$

Using (3.6) we deduce

$$\phi_\hbar(\tau) \sim -\frac{3}{a^2}\left(\frac{a^2}{2\hbar}\right)^{\frac{2}{3}} = -\frac{3}{(2a\hbar)^{\frac{2}{3}}} = -3\left(\frac{k}{2a}\right)^{\frac{2}{3}}, \ \ k = \frac{1}{\hbar}.$$

Also

$$e^{\phi_\hbar(\tau)} = (1-\tau)^{\frac{2(1-\tau)}{a^2\tau^3}}e^{-\frac{1}{a^2\tau^2}}.$$

In any case, using (3.6), (3.7) and (3.13) we deduce that

$$\log I_k(w) \sim k \log a = k \log(c+1) \;\text{ as } k \to \infty. \tag{3.14}$$

Thus

$$\log r_m(w) = \log\left(\frac{I_{m-1}(w) I_{m+3}(w)}{I_{m+1}(w)}\right) = 0,$$

so that

$$\lim_{m\to\infty} q_m = \lim_{m\to\infty} r_m = 1. \qquad \square$$

Example 3.5. If we let $c=0$ in the above example , then we deduce that

$$I_k(w) = \int_0^1 t^k e^{-\frac{1}{1-t^2}}\, dt \sim e^{\phi_\hbar(\tau)} \sqrt{2\pi v(\hbar)}$$

where

$$\phi_\hbar(\tau) \sim -3\left(\frac{k}{2}\right)^{\frac{2}{3}}, \quad v(\hbar) \sim \frac{1}{6}\left(\frac{2}{k^2}\right)^{\frac{2}{3}}.$$

Hence

$$\log I_k(w) \sim -3\left(\frac{k}{2}\right)^{\frac{2}{3}},$$

$$\log r_m(w) \sim -\frac{3}{2^{\frac{2}{3}}}\Big((m-1)^{\frac{2}{3}} + (m+3)^{\frac{2}{3}} - (m+1)^{\frac{2}{3}}\Big) \to 0,$$

so that

$$\lim_{m\to\infty} q_m = \lim_{m\to\infty} r_m = 1. \qquad \square$$

4. A probabilistic proof of the Gauss-Bonnet theorem

Suppose that M is a smooth, compact, connected *oriented* manifold of even dimension m. For any Riemann metric g we can view the Riemann curvature tensor R_g as a symmetric bundle morphism $R_g : \Lambda^2 TM \to \Lambda^2 TM$. Equivalently, using the metric identification $T^*M \cong TM$ we can view R_g as a section of $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$.

We will denote by $\Omega^{p,q}(M)$ the sections of $\Lambda^p T^*M \otimes \Lambda^q T^*M$ and we will refer to them of *double forms* of type (p,q). Thus $R_g \in \Omega^{2,2}(M)$. We have a natural product

$$\bullet : \Omega^{p,q}(M) \times \Omega^{p',q'}(M) \to \Omega^{p+p',q+q'}(M)$$

defined in a natural way; see [1, Eq. (7.2.3)] for a precise definition.

Using the metric g we can identify a double-form in $\Omega^{k,k}(M)$ with a section of $\Lambda^k T^*M \otimes \Lambda^k TM$, i.e., with a bundle morphism $\Lambda^k TM \to \Lambda^k TM$ and thus we have a linear map

$$\operatorname{tr} : \Omega^{k,k}(M) \to C^\infty(M).$$

For $1 \le k \le \frac{m}{2}$ we have a double form

$$R_g^{\bullet k} = \underbrace{R_g \bullet \cdots \bullet R_g}_{k} \in \Omega^{2k,2k}(M).$$

We denote by $dV_g \in \Omega^m(M)$ the volume *form* on M defined by the metric g and the orientation on M. We set

$$\boldsymbol{e}_g(M) := \frac{1}{(2\pi)^{\frac{m}{2}}(\frac{m}{2})!} \operatorname{tr}\Big(-R_g^{\bullet\frac{m}{2}}\Big) dV_g \in \Omega^m(M).$$

The form $\boldsymbol{e}_g(M)$ is called the *Euler form* of the metric g and the classical Gauss-Bonnet theorem states that

$$\int_M \boldsymbol{e}_g(M) = \chi(M) =: \text{the Euler characteristic of } M. \tag{4.1}$$

In this section we will show that the Gauss-Bonnet theorem for any metric g is an immediate consequence of the Kac-Rice formula coupled with the approximation theorem Thm. 1.8.

Fix a metric g. For simplicity we assume that $\mathrm{vol}_g(M) = 1$. This does not affect the generality since $\boldsymbol{e}_{cg}(M) = \boldsymbol{e}_g(M)$ for any constant $c > 0$. Consider the random function $\boldsymbol{u}_\varepsilon$ on M defined by (1.2, 1.1). Set

$$\boldsymbol{v}_\varepsilon = \left(\frac{\varepsilon^{m+2}}{d_m}\right)^{\frac{1}{2}} \boldsymbol{u}^\varepsilon.$$

Observe that for $\varepsilon > 0$ sufficiently small, any $X, Y \in \mathrm{Vect}(M)$ and any $\boldsymbol{p} \in M$ we have

$$h^\varepsilon(X(\boldsymbol{p}), Y(\boldsymbol{p})) = \boldsymbol{E}\big(\, X\boldsymbol{v}_\varepsilon(\boldsymbol{p}), Y\boldsymbol{v}_\varepsilon(\boldsymbol{p})\,\big)$$

where h^ε is the metric on M that appears in the approximation theorem, Theorem 1.8.

For any smooth function $f : M \to \mathbb{R}$ and any $\boldsymbol{p} \in M$ we denote by $\mathrm{Hess}^\varepsilon_{\boldsymbol{p}}(f)$ the Hessian of f at $\boldsymbol{p}$ defined in terms of the metric h^ε. More precisely

$$\mathrm{Hess}^\varepsilon_{\boldsymbol{p}}(f) = XYf(\boldsymbol{p}) - (\nabla^\varepsilon_X Y)f(\boldsymbol{p}),\ \ \forall X, Y \in \mathrm{Vect}(M),$$

where ∇^ε denotes the Levi-Civita connection of the metric h^ε. Using the metric h^ε we can identify this Hessian with a symmetric linear operator

$$\mathrm{Hess}^\varepsilon_{\boldsymbol{p}}(f) : (T_{\boldsymbol{p}}M, h^\varepsilon) \to (T_{\boldsymbol{p}}M, h^\varepsilon).$$

For any $\boldsymbol{p} \in M$ we have a random vector $d\boldsymbol{v}_\varepsilon(\boldsymbol{p}) \in T^*_{\boldsymbol{p}}M$. Its covariance form $S(d\boldsymbol{v}_\varepsilon(\boldsymbol{p}))$ is precisely the metric h^ε, and if we use the metric h^ε to identify this form with an operator we deduce that $S(d\boldsymbol{v}_\varepsilon(\boldsymbol{p}))$ is identified with the identity operator.

For every smooth Morse function f on M and any integer $0 \le k \le m$ we have a measure $\nu_{f,k}$ on M

$$\nu_{f,k} = \sum_{df(\boldsymbol{p})=0,\ \mathrm{ind}(f,\boldsymbol{p})=k} \delta_{\boldsymbol{p}},$$

where $\mathrm{ind}(f,\boldsymbol{p})$ denotes the *Morse index* of the critical point $\boldsymbol{p}$ of the Morse function f. We set

$$\nu_f = \sum_{k=0}^m (-1)^k \nu_{f,k}$$

The Poincaré-Hopf theorem implies that for any Morse function we have

$$\int_M \nu_f(d\boldsymbol{p}) = \chi(M). \tag{4.2}$$

Using the random Morse function $\boldsymbol{v}_\varepsilon$ we obtain the random measures $\nu_{\boldsymbol{v}_\varepsilon,\boldsymbol{p}}$, $\nu_{\boldsymbol{v}_\varepsilon}$. We denote by ν^ε_k and respectively ν^ε their expectations. The Kac-Rice formula implies that

$$\nu_k = \frac{1}{(2\pi)^{\frac{m}{2}}} \rho^\varepsilon_k(\boldsymbol{p}) |dV_{h^\varepsilon}(\boldsymbol{p})|,$$

where

$$\rho^\varepsilon_k(\boldsymbol{p}) = \frac{1}{\sqrt{\det S(\boldsymbol{v}_\varepsilon(\boldsymbol{p}))}} \boldsymbol{E}\Big(\, |\det \mathrm{Hess}^\varepsilon_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon)|\, \big|\, d\boldsymbol{v}_\varepsilon(\boldsymbol{p}) = 0,\ \ \mathrm{ind}\, \mathrm{Hess}^\varepsilon_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) = k \Big)$$

$$= (-1)^k \boldsymbol{E}\Big(\det \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) \,\big|\, d\boldsymbol{v}_\varepsilon(\boldsymbol{p}) = 0, \;\; \mathrm{ind}\, \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) = k \Big).$$

As shown in [1, Eq. (12. 2.11)], the Gaussian random variables $\mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon)$ and $d\boldsymbol{v}_\varepsilon(\boldsymbol{p})$ are independent so that

$$\rho^{\varepsilon}_k(\boldsymbol{p}) = (-1)^k \boldsymbol{E}\Big(\det \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) \,\big|\, \mathrm{ind}\, \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) = k \Big).$$

Thus

$$\nu^{\varepsilon} = \frac{1}{(2\pi)^{\frac{m}{2}}} \sum_{k=0}^{m} (-1)^k \rho^{\varepsilon}_k(\boldsymbol{p}) |dV_{h^\varepsilon}(\boldsymbol{p})|,$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \sum_{k=0}^{m} \boldsymbol{E}\Big(\det \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) \,\big|\, \mathrm{ind}\, \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) = k \Big) |dV_{h^\varepsilon}(\boldsymbol{p})|$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \boldsymbol{E}\Big(\det \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) \Big) |dV_{h^\varepsilon}(\boldsymbol{p})|.$$

From the Poincaré-Hopf equality (4.2) we deduce

$$\chi(M) = \int_M \nu^{\varepsilon}(d\boldsymbol{p}) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M \boldsymbol{E}\Big(\det \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) \Big) |dV_{h^\varepsilon}(\boldsymbol{p})|. \tag{4.3}$$

Observe that Hessian $\mathrm{Hess}^{\varepsilon}(f)$ of a function f can also be viewed as a double form

$$\mathrm{Hess}^{\varepsilon}(f) \in \Omega^{1,1}(M).$$

In particular, $\mathrm{Hess}^{\varepsilon}(\boldsymbol{v}_\varepsilon)$ is a random $(1,1)$ double form and we have the following equality, [1, Lemma 12.2.1]

$$-2R_{h^\varepsilon} = \boldsymbol{E}\big(\mathrm{Hess}^{\varepsilon}(\boldsymbol{v}_\varepsilon)^{\bullet 2} \big), \tag{4.4}$$

where R_{h^ε} denotes the Riemann curvature tensor of the metric h^ε. On the other hand we have the equality [1, Eq. (12.3.1)]

$$\det \mathrm{Hess}^{\varepsilon}(\boldsymbol{v}_\varepsilon) = \frac{1}{m!} \,\mathrm{tr}\, \mathrm{Hess}^{\varepsilon}(\boldsymbol{v}_\varepsilon)^{\bullet m} \tag{4.5}$$

Using (4.4), (4.5) and the algebraic identities in [1, Lemma 12.3.1] we conclude that

$$\frac{1}{(2\pi)^{\frac{m}{2}}} \boldsymbol{E}\Big(\det \mathrm{Hess}^{\varepsilon}_{\boldsymbol{p}}(\boldsymbol{v}_\varepsilon) \Big) = \frac{1}{(2\pi)^{\frac{m}{2}} (\frac{m}{2})!} \,\mathrm{tr}\Big(-R_{h^\varepsilon}^{\bullet \frac{m}{2}} \Big).$$

This proves (1.22). Using this equality in (4.3) we deduce

$$\chi(M) = \int_M \boldsymbol{e}_{h^\varepsilon}(M),$$

i.e., we have proved the Gauss-Bonnet theorem for the metric h^ε. Now let $\varepsilon \to 0$. As we have mentioned, Theorem 1.8 implies that $h^\varepsilon \to g$ so in the limit, the above equality reduced to the Gauss-Bonnet theorem for the original metric g.

Appendix A. Jets of the distance function

Suppose that (M, g) is a smooth, m-dimensional manifold, $\boldsymbol{p}_0 \in M$, U is an open, geodesically convex neighborhood of $\boldsymbol{p}_0$ and $(x^1, \dots, x^m)$ are normal coordinates on U centered at $\boldsymbol{p}_0$. We have a smooth function

$$\eta : U \times U \to [0, \infty), \ \ \eta(\boldsymbol{p}, \boldsymbol{q}) = \operatorname{dist}_g(\boldsymbol{p}, \boldsymbol{q})^2.$$

We want to investigate the partial derivatives of r at $(\boldsymbol{p}_0, \boldsymbol{p}_0)$. Using the above normal coordinates we regard η as a function $\eta = \eta(x, y)$ defined in an open neighborhood of $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^m$.

If $f = f(t^1, \dots, t^N)$ is a smooth function defined in a neighborhood of $0 \in \mathbb{R}^N$ and k is a nonnegative integer, then we denote by $[f]_k$ the degree k-homogeneous part in the Taylor expansion of f at 0, i.e.,

$$[f]_k = \frac{1}{k!} \sum_{|\alpha|=k} \partial_t^\alpha f\big|_{t=0} t^\alpha \in \mathbb{R}[t^1, \dots, t^N].$$

In the coordinates (x^i) the metric g has the form (using Einstein's summation convention throughout)

$$g = g_{ij} dx^i dx^j,$$

where g_{ij} satisfy the estimates [18, Cor. 9.8]

$$g_{k\ell} = \delta_{k\ell} - \frac{1}{3} R_{ikj\ell}(0) x^i x^j + O(|x|^3). \tag{A.1}$$

We deduce that

$$g^{k\ell} = \delta_{k\ell} + \frac{1}{3} R_{ikj\ell}(0) x^i x^j + O(|x|^3). \tag{A.2}$$

The function η satisfies a Hamilton-Jacobi equation, [31, p. 171],

$$g^{k\ell} \frac{\partial \eta(x, y)}{\partial x^k} \frac{\partial \eta(x, y)}{\partial x^\ell} = 4\eta(x, y), \ \ \forall x, y. \tag{A.3}$$

Moreover, η satisfies the obvious symmetry conditions

$$\eta(x, y) = \eta(y, x), \quad \eta(0, x) = \eta(x, 0) = |x|^2 := \sum_{i=1}^m (x^i)^2. \tag{A.4}$$

As shown in [7, Lemma 2.2] we have

$$[\eta]_2 = |x - y|^2 = \sum_{i=1}^m (x^i - y^i)^2. \tag{A.5}$$

The symmetries (A.4) suggest the introduction of new coordinates (u, v) on $U \times U$,

$$u_i = x^i - y^i, \ \ v_j = x^j + y^j.$$

Then

$$x^i = \frac{1}{2}(u_i + v_i), \ \ y^j = \frac{1}{2}(v_j - u_j), \ \ \partial_{x^i} = \partial_{u_i} + \partial_{v_i}.$$

The equality (A.2) can be rewritten as

$$g^{k\ell}(x) = \delta^{k\ell} + \frac{1}{12} \sum_{i,j} R_{ikj\ell}(u_i + v_i)(u_j + v_j) + O(3). \tag{A.6}$$

The symmetry relations (A.4) become

$$\eta(u,v) = \eta(-u,v),\ \ \eta(u,u) = |u|^2, \tag{A.7}$$

while (A.5) changes to

$$[\eta]_1 = 0,\ \ [\eta]_2 = |u|^2. \tag{A.8}$$

The equality (A.3) can be rewritten

$$\sum_{k,l} g^{kl}(x) \underbrace{\big(\,\eta'_{u_k} + \eta'_{v_k}\,\big)}_{=:A_k} \underbrace{\big(\,\eta'_{u_\ell} + \eta'_{v_\ell}\,\big)}_{=:A_\ell} = 4\eta. \tag{A.9}$$

Note that

$$[A_k]_0 = [A_\ell]_0 = [g^{k\ell}]_1 = 0, \tag{A.10}$$

while (A.8) implies that

$$[A_k]_1 = 2u^k.$$

We deduce

$$4[\eta]_3 = \sum_{k,\ell} [g^{kl}]_0 \big(\,[A_k]_1[A_\ell]_2 + [A_k]_2[A_\ell]_1\,\big) = \sum_k 2[A_k]_2[A_k]_1 = 4\sum_k u_k[A_k]_2.$$

We can rewrite this last equality as a differential equation for $[\eta]_3$ namely

$$[\eta]_3 = \sum_k u_k(\partial_{u_k} + \partial_{v_k})[\eta]_3.$$

We set $P = [\eta]_3$ so that P is a homogeneous polynomial of degree 3 in the variables u, v. Moreover, according to (A.7) the polynomial P is even in u and $P(u,u) = 0$. Thus P has the form

$$P = \underbrace{\sum_i C_i(u)v_i}_{=:P_2} + P_0(v),$$

where $C_i(u)$ is a homogeneous polynomial of degree 2 in the variables u, and $P_0(v)$ is homogeneous of degree 3 in the variables v.

We have

$$\sum_k u_k\partial_{v_k} P_2 = \underbrace{\sum_k C_k(u)u_k}_{=:Q_3},\ \ Q_1 := \sum_k u_k\partial_{v_k}P_0,\ \ \sum_k u_k\partial_{u_k}P_0 = 0,$$

and the classical Euler equations imply

$$\sum_k u_k\partial_{u_k}P_2 = 2P_2.$$

We deduce

$$P = 2P_2 + Q_3 + Q_1,$$

where the polynomials Q_3 and Q_1 are odd in the variable u. Since P is even in the variable u we deduce

$$Q_3 + Q_1 = 0,$$

so that $P_2 + P_0 = P = 2P_2$. Hence $P_2 = P_0 = 0$ and thus

$$[\eta]_3 = 0. \tag{A.11}$$

In particular

$$[A_k]_2 = 0,\ \ \forall k. \tag{A.12}$$

Going back to (A.9) and using (A.10) and (A.12) we deduce

$$
\begin{aligned}
4[\eta]_4 = \sum_{k,\ell}[g^{k\ell}]_2[A_k]_1[A_\ell]_1 + \sum_{k,\ell}[g^{k\ell}]_0\big([A_k]_1[A_\ell]_3 + [A_k]_3[A_\ell]_1\big) \\
= 4\sum_{k,\ell}[g^{k\ell}]_2 u_k u_\ell + 2\sum_k u_k[A_k]_3.
\end{aligned}
\tag{A.13}
$$

We set $P = [\eta]_4$. The polynomial P is homogeneous of degree 4 in the variables u, v, and it is even in the variable u. We can write $P = P_0 + P_2 + P_4$, where

$$
P_4 = \sum_k c_{ijkl}u_iu_ju_ku_\ell,\ \ P_2 = \sum_{i,j} Q_{ij}(u)v_iv_j,
$$

and P_0 is homogeneous of degree 4 in the variables v, $Q_{ij}(u)$ is a homogeneous quadratic polynomial in the variables u. We have

$$
\sum_k u_k[A_k]_3 = \sum_k u_k(\partial_{u_k} + \partial_{v_k})P.
$$

We have

$$
\sum_k u_k\partial_{u_k}P_{2\nu} = 2\nu P_{2\nu},\ \ \nu = 0,1,2,\ \ \sum_k u_k\partial_{v_k}P_4 = 0,
$$

$$
\sum_k u_k\partial_{v_k}P_2 = \sum_{k,i,j} u_kQ_{ij}(\delta_{ki}v_j + \delta_{kj}v_i) = \sum_{k,j}\big(Q_{kj}u_kv_j + Q_{jk}v_ju_k\,\big)
$$

Using these equalities in (A.13) we deduce

$$
\begin{aligned}
4P_4 + 4P_2 + 4P_0 = 4\sum_{k,\ell}[g^{k\ell}]_2u_ku_\ell + 4P_4 + 2P_2 + \sum_k u_k\partial_{v_k}P_0 \\
+ \sum_{k,j}\Big(Q_{jk} + Q_{kj}\Big)u_kv_j.
\end{aligned}
$$

This implies $P_0 = 0$ so that $P = P_4 + P_2$, and we can then rewrite the above equality as

$$
P_2 = 2\sum_{k,\ell}[g^{k\ell}]_2u_ku_\ell + \sum_{k,j}\Big(Q_{jk} + Q_{kj}\Big)u_kv_j. \tag{A.14}
$$

Note that the equality $r(u,u) = |u|^2$ implies $P(u,u) = 0$ so that

$$
P_4(u) = P_4(u,u) = -P_2(u,u).
$$

Therefore it suffices to determine P_2. This can be achieved using the equality (A.6) in (A.14). We have

$$
\begin{aligned}
2\sum_{k,\ell}[g^{k\ell}]_2u_ku_\ell &= \frac{1}{6}\sum_{i,j,k,\ell}R_{ikj\ell}(u_i+v_i)(u_j+v_j)u_ku_\ell \\
&= \frac{1}{6}\sum_{i,j}\underbrace{\Big(\sum_{k,\ell}R_{ikj\ell}u_ku_\ell\Big)}_{\widehat{Q}_{ij}(u)}v_iv_j + \sum_j S_j(u)v_j,
\end{aligned}
$$

where $S_j(u)$ denotes a homogeneous polynomial of degree 3 in u. The equality (A.14) can now be rewritten as

$$
\sum_{i,j}Q_{ij}(u)v_iv_j = \frac{1}{6}\sum_{i,j}\widehat{Q}_{ij}(u)v_iv_j + \sum_j S_j(u)v_j + \frac{1}{2}\sum_{k,j}\Big(Q_{jk} + Q_{kj}\Big)u_kv_j.
$$

From this we read easily

$$Q_{ij}(u) = \frac{1}{6}\widehat{Q}_{ij}(u) = \frac{1}{6}\sum_{k,\ell} R_{ikj\ell}u_k u_\ell.$$

This determines P_2.

$$P_2(u,v) = \frac{1}{6}\sum_{i,j}\widehat{Q}_{ij}(u)v_i v_j. \tag{A.15}$$

As we have indicated above P_2 determines P_4.

$$P_4(u) = -P_2(u,u) = -\frac{1}{6}\sum_{i,j,k,\ell} R_{ikj\ell}u_i u_j u_k u_\ell. \tag{A.16}$$

The skew symmetries of the Riemann tensor imply that $P_4 = 0$ so that

$$[\eta]_4(u,v) = \frac{1}{6}\sum_{i,j}\widehat{Q}_{ij}(u)v_i v_j,\ \ \widehat{Q}_{ij}(u) = \sum_{k,\ell} R_{ikj\ell}u_k u_\ell. \tag{A.17}$$

Example A.1. Suppose that M is a surface, i.e., $m = 2$. Set

$$K = R_{1212} = R_{2121} = -R_{1221}.$$

Note that K is the Gaussian curvature of the surface. Then

$$\widehat{Q}_{11} = \sum_{k,\ell} R_{1k1\ell}u_k u_\ell = Ku_2^2,\ \ \widehat{Q}_{22} = \sum_{k,\ell} R_{2k2\ell}u_k u_\ell = Ku_1^2.$$

$$\widehat{Q}_{12} = \sum_{k,\ell} R_{1k2\ell}u_k u_\ell = -Ku_1u_2 = \widehat{Q}_{21}.$$

Hence

$$P_2(u,v) = \frac{K}{6}(u_2^2v_1^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2) = \frac{K}{6}(u_1v_2 - u_2v_1)^2.$$

□

Appendix B. Spectral estimates

As we have already mentioned, the correlation function

$$\mathscr{E}^\varepsilon(\boldsymbol{p},\boldsymbol{q}) = \sum_{k\geq 0} w_\varepsilon(\sqrt{\lambda_k})\Psi_k(\boldsymbol{p})\Psi_k(\boldsymbol{q})$$

is the Schwartz kernel of the smoothing operator $w_\varepsilon(\sqrt{\Delta})$. In this appendix we present in some detail information about the behavior along the diagonal of this kernel as $\varepsilon \to 0$. We will achieve this by relying on the wave kernel technique pioneered by L. Hörmander, [20].

The fact that such asymptotics exist and can be obtained in this fashion is well known to experts; see e.g [13] or [33, Chap.XII]. However, we could not find any reference describing these asymptotics with the level of specificity needed for the considerations in this paper.

Theorem B.1. *Suppose that $w \in \mathcal{S}(\mathbb{R})$ is an even, nonnegative Schwartz function, and (M,g) is a smooth, compact, connected m-dimensional Riemann manifold. We define*

$$\mathscr{E}^\varepsilon : M\times M \to \mathbb{R},\ \ \mathscr{E}^\varepsilon(\boldsymbol{p},\boldsymbol{q}) = \sum_{k\geq 0} w(\varepsilon\sqrt{\lambda_k})\Psi_k(\boldsymbol{p})\Psi_k(\boldsymbol{q}),$$

where $(\Psi_k)_{k\geq 1}$ is an orthonormal basis of $L^2(M,g)$ consisting of eigenfunctions of Δ_g.

Fix a point $\boldsymbol{p}_0 \in M$ and normal coordinates at $\boldsymbol{p}_0$ defined in an open neighborhood $\mathcal{O}_0$ of $\boldsymbol{p}_0$. The restriction of $\mathscr{E}^\varepsilon$ to $\mathscr{E}^\varepsilon$ to $\mathcal{O}_0 \times \mathcal{O}_0$ can be viewed as a function $\mathscr{E}^\varepsilon(x,y)$ defined in an open neighborhood of $(0,0)$ in $\mathbb{R}^m \times \mathbb{R}^m$. Fix multi-indices $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^m$. Then

$$\partial_x^\alpha \partial_y^\beta \mathscr{E}^\varepsilon(x,y)|_{x=y=0} = \varepsilon^{-m-2d(\alpha,\beta)} \frac{\boldsymbol{i}^{|\alpha|-|\beta|}}{(2\pi)^m} \left(\int_{\mathbb{R}^m} w(|x|) x^{\alpha+\beta} dx + O(\varepsilon^2) \right), \quad \varepsilon \to 0, \tag{B.1}$$

where

$$d(\alpha,\beta) := \left\lfloor \frac{|\alpha+\beta|}{2} \right\rfloor .$$

Moreover, the constant implied by the symbol $O(\varepsilon)$ in (B.1) uniformly bounded with respect to $\boldsymbol{p}_0$.

Proof. For the reader's convenience and for later use, we go in some detail through the process of obtaining these asymptotics. We skip many analytical steps that are well covered in [22, Chap. 17] or [28].

Observe that for any smooth $f : M \to \mathbb{R}$ we have

$$w_\varepsilon(\sqrt{\Delta}) f = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{w_\varepsilon}(t) e^{\boldsymbol{i}t\sqrt{\Delta}} f dt = \frac{1}{2\pi\varepsilon} \int_{\mathbb{R}} \widehat{w}\left(\frac{t}{\varepsilon}\right) e^{\boldsymbol{i}t\sqrt{\Delta}} f dt. \tag{B.2}$$

The Fourier transform $\widehat{w}(t)$ is a Schwartz function so $\widehat{w}(t/\varepsilon)$ is really small for t outside a small interval around 0 and ε sufficiently small. Thus a good understanding of the kernel of $e^{\boldsymbol{i}t\sqrt{\Delta}}$ for t sufficiently small could potentially lead to a good understanding of the Schwartz kernel of $w_\varepsilon(\sqrt{\Delta})$.

Fortunately, good short time asymptotics for the wave kernel are available. We will describe one such method going back to Hadamard, [19, 31]. Our presentation follows closely [22, §17.4] but we also refer to [28] where we have substantially expanded the often dense presentation in [22].

To describe these asymptotics we need to introduce some important families homogeneous generalized functions (or distributions) on $\mathbb{R}$. We will denote by $C^{-\infty}(\Omega)$ the space of generalized functions on the smooth manifold Ω, defined as the dual of the space compactly supported 1-densities, [17, Chap. VI].

For any $a \in \mathbb{C}$, $\mathbf{Re}\, a > 1$ we define $\chi_+^a : \mathbb{R} \to \mathbb{R}$ by

$$\chi_+^a(x) = \frac{1}{\Gamma(a+1)} x_+^a, \quad x_+ = \max(x,0).$$

Observe that we have the following equality in the sense of distributions

$$\frac{d}{dx}\chi_+^{a+1} = \chi_+^a(x), \quad \mathbf{Re}\, a > 1.$$

We can use this to define for any $a \in \mathbb{C}$

$$\chi_+^a := \frac{d^k}{dx^k}\chi_+^{a+k} \in C^{-\infty}(\mathbb{R}), \quad k > 1 - \mathbf{Re}\, a.$$

For $\mathbf{Re}\, a > 0$ we denote by $|\chi|^a$ the generalized function defined by the locally integrable function

$$|\chi|^a(x) = \frac{1}{\Gamma(\frac{a+1}{2})} |x|^a .$$

The correspondence $a \mapsto |\chi|^a$ is a holomorphic map $\{\mathbf{Re}\, z > 0\} \to C^{-\infty}(\mathbb{R})$ which admits a holomorphic extension to the whole complex plane, [15, Chap. 1], [28]. This is a temperate generalized function, and its Fourier transform is given by, [15, 28],

$$\widehat{|\chi|^a}\,(\xi) = \sqrt{\pi}2^{a+1}|\chi|^{-(a+1)}(\xi),\ \ \forall a \in \mathbb{C}. \tag{B.3}$$

Denote by $K_t(x,y)$ the Schwartz kernel of $e^{\boldsymbol{i}t\sqrt{\Delta}}$. We then have the following result [22, §17.4] or [28].

Theorem B.2. *Set $n := m+1$, and let*

$$\eta(x,y) = \mathrm{dist}_g(x,y)^2,\ \ x,y \in M.$$

There exists a positive constant $c > 0$, smaller than the injectivity radius of (M,g), such that for $\mathrm{dist}_g(x,y) < c$ we have the following asymptotic expansion as $t \to 0$

$$K_t(\boldsymbol{p},\boldsymbol{q}) \sim \sum_{k=1}^{\infty} U_k(\boldsymbol{p},\boldsymbol{q}) d_m(2k) \mathcal{H}_k(t,\boldsymbol{p},\boldsymbol{q}),\ \ |t| < c, \tag{B.4}$$

where for $\mathbf{Re}\, a > 0$ we have

$$\mathcal{H}_a(t,\boldsymbol{p},\boldsymbol{q}) = \partial_t \Big(\chi_+^{a-\frac{n}{2}}\big(t_+^2 - \eta(\boldsymbol{p},\boldsymbol{q})\big) - \chi_+^{a-\frac{n}{2}}\big(t_-^2 - \eta(\boldsymbol{p},\boldsymbol{q})\big) \Big),$$

$$d_m(2a) = \frac{\Gamma(\frac{2a+1}{2})}{\pi^{\frac{m}{2}}\Gamma(2a)}.$$

Let us explain in more detail the meaning of the above result. The functions U_k are smooth functions defined in the neighborhood $\mathrm{dist}_g(\boldsymbol{p},\boldsymbol{q}) < c$ of the diagonal in $M \times M$. For fixed $\boldsymbol{q}$, the functions $\boldsymbol{p} \mapsto V_k(\boldsymbol{p}) := U_k(\boldsymbol{p},\boldsymbol{q})$ are determined as follows.

Fix normal coordinates x at $\boldsymbol{q}$, set $|g| := \det(g_{ij})$, and

$$h(x) := -\frac{1}{2} g\big(\nabla \log|g|,\ x\big) = -\frac{1}{2}\sum_{j,k} g^{jk} x^j \partial_{x^k} \log|g|.$$

Then $V_k(x)$ are the unique solutions of the differential recurrences

$$V_1(0) = 1,\ \ 2x\cdot\nabla V_1 = hV_1,\ \ |x| < c, \tag{B.5}$$

$$\frac{1}{k} x \cdot \nabla V_{k+1} + \left(1 - \frac{1}{2k}h\right) V_{k+1} = -\Delta_g V_k,\ \ V_{k+1}(0) = 0,\ \ |x| < c,\ \ k \geq 1. \tag{B.6}$$

We have the following important equality

$$\lim_{\mathrm{dist}_g(\boldsymbol{p},\boldsymbol{q}) \to 0} \mathcal{H}_a(t,\boldsymbol{p},\boldsymbol{q}) = |\chi|^{2a-2-m}(t),\ \ \forall a \in \mathbb{C}. \tag{B.7}$$

The asymptotic estimate (B.4) signifies that for any positive integer μ there exists a positive integer $N(\mu)$ so that for any $N \geq N(\mu)$ the tail

$$\widetilde{\mathcal{T}}_N(t,\boldsymbol{p},\boldsymbol{q}) := K_t(\boldsymbol{p},\boldsymbol{q}) - \sum_{k=1}^{N} U_k(\boldsymbol{p},\boldsymbol{q}) d_m(2k) \mathcal{H}_k(t,\boldsymbol{p},\boldsymbol{q})$$

belongs to $C^\mu\big((-c,c)\times M \times M\big)$ and satisfies the estimates

$$\big\|\partial_t^j \widetilde{\mathcal{T}}_N(t,-,-)\big\|_{C^{\mu-j}(M\times M)} \leq C_j |t|^{2N-n-1-\mu},\ \ |t| \leq c,\ \ ,j \leq \mu,\ \ N \geq N(\mu). \tag{B.8}$$

Fix a point $\boldsymbol{p}_0 \in M$ and normal coordinates at $\boldsymbol{p}_0$ defined in a neighborhood $\mathcal{O}_0$ of $\boldsymbol{p}_0$. Then we can identify a point $(\boldsymbol{p}, \boldsymbol{q}) \in \mathcal{O}_0 \times \mathcal{O}_0$ with a point (x, y) in a neighborhood of $(0, 0)$ in $\mathbb{R}^m \times \mathbb{R}^m$.

Using (B.2) we deduce

$$\partial_x^\alpha \partial_y^\beta \mathscr{E}^\varepsilon(x,y)|_{x=y} = \frac{1}{\varepsilon}\Big\langle \underbrace{\partial_x^\alpha \partial_y^\beta K_t(x,y)|_{x=y}}_{=:K_t^{\alpha,\beta}}, \widehat{w}\left(\frac{t}{\varepsilon}\right)\Big\rangle. \tag{B.9}$$

Choose an even, nonnegative cutoff function $\rho \in C_0^\infty(\mathbb{R})$ such that

$$\rho(t) = \begin{cases} 1, & |t| \leq \frac{c}{4}, \\ 0, & |t| \geq \frac{c}{2}, \end{cases}$$

where $c > 0$ is the constant in Theorem B.2. Then

$$\partial_x^\alpha \partial_y^\beta \mathscr{E}^\varepsilon(x,y)|_{x=y} = \frac{1}{\varepsilon}\Big\langle K_t^{\alpha,\beta}, \rho(t)\widehat{w}\left(\frac{t}{\varepsilon}\right)\Big\rangle + \frac{1}{\varepsilon}\Big\langle K_t^{\alpha,\beta}, \big(1-\rho(t)\big)\widehat{w}\left(\frac{t}{\varepsilon}\right)\Big\rangle.$$

Let us observe that that for any $N > 0$

$$\frac{1}{\varepsilon}\Big\langle K_t^{\alpha,\beta}, \big(1-\rho(t)\big)\widehat{w}\left(\frac{t}{\varepsilon}\right)\Big\rangle = O(\varepsilon^N) \ \text{ as } \varepsilon \to 0$$

Thus

$$\forall N > 0 \ \ \partial_x^\alpha \partial_y^\beta \mathscr{E}^\varepsilon(x,y)|_{x=y} \sim \frac{1}{\varepsilon}\Big\langle K_t^{\alpha,\beta}, \rho(t)\widehat{w}\left(\frac{t}{\varepsilon}\right)\Big\rangle + O(\varepsilon^N), \ \ \varepsilon \to 0. \tag{B.10}$$

On the other hand

$$\partial_x^\alpha \partial_y^\beta K_t(x,y) \sim \sum_{k=1}^\infty d_m(2k) \partial_x^\alpha \partial_y^\beta \left\{ U_k(x,y) \mathcal{H}_k(t,x,y) \right\}. \tag{B.11}$$

Recall that

$$d(\alpha,\beta) = \left\lfloor \frac{1}{2}|\alpha+\beta| \right\rfloor.$$

One can show (see [7, 28])

$$\partial_x^\alpha \partial_y^\beta K_t(x,y)|_{x=y=0} \sim \sum_{k=0}^\infty A_{m,\alpha,\beta,k} |\chi|^{-m-2d(\alpha,\beta)+2k}(t), \tag{B.12}$$

where $A_{m,\alpha,\beta,0}$ is a universal constant depending *only* on m, α, β, which is equal to 0 if $|\alpha+\beta|$ is odd.

Lemma B.3. *(a) For any $r \in \mathbb{Z}$ and any $N > 0$ we have*

$$\frac{1}{\varepsilon}\langle |\chi|^r, \rho\widehat{w_\varepsilon}\rangle = \varepsilon^r\Big(\langle |\chi|^r, \widehat{w}\rangle + O(\varepsilon^N)\Big) \ \ \textit{as } \varepsilon \to 0.$$

(b) For every positive integer r we have

$$\langle |\chi|^{-r}, \widehat{w}\rangle = \frac{\sqrt{\pi}2^{1-r}}{\Gamma(\frac{r}{2})} \int_{\mathbb{R}} |\tau|^{r-1} w(\tau) d\tau.$$

Proof. (a) For transparency we will use the integral notation for the pairing between a generalized function and a test function. We have

$$\big\langle |\chi|^r, \eta\widehat{w_\varepsilon}\,\big\rangle = \frac{1}{\varepsilon}\int_{\mathbb{R}} |\chi|^r(t)\rho(t)\widehat{w}(t/\varepsilon)dt = \int_{\mathbb{R}} |\chi|^r(\varepsilon t)\rho(\varepsilon t)\widehat{w}(t)dt$$

$$= \varepsilon^r \int_{\mathbb{R}} |\chi|^r(t)\rho(\varepsilon t)\widehat{w}(t)dt = \varepsilon^r \langle |\chi|^r, \rho_\varepsilon \widehat{w}\rangle, \quad \rho_\varepsilon(t) = \rho(\varepsilon t).$$

Now observe that $\rho_\varepsilon\widehat{w} - \widehat{w} = \widehat{w}(\rho_\varepsilon - 1) \to 0$ in $\mathcal{S}(\mathbb{R})$. More precisely for $k \geq 0$ we have

$$\frac{\partial^k}{\partial k}(\rho_\varepsilon - 1) = O(\varepsilon^N t^N) \ \text{ as } \varepsilon \to 0.$$

This implies that

$$\big\langle |\chi|^r, \widehat{w}(\rho_\varepsilon - 1)\,\big\rangle = O(\varepsilon^N) \ \text{ as } \varepsilon \to 0,$$

so that

$$\langle |\chi|^r, \rho_\varepsilon\widehat{w}\rangle = \langle |\chi|^r, \widehat{w}\rangle + \big\langle |\chi|^r, \widehat{w}(\rho_\varepsilon - 1)\,\big\rangle = \langle |\chi|^r, \widehat{w}\rangle + O(\varepsilon^N) \ \text{ as } \varepsilon \to 0.$$

(b) We have

$$\langle |\chi|^{-r}, \widehat{w}\rangle = \big\langle\, \widehat{|\chi|^{-r}}, w\,\big\rangle \overset{(B.3)}{=} \sqrt{\pi}2^{1-r}\big\langle |\chi|^{r-1}(\tau), w(\tau)\big\rangle$$

$$= \frac{\sqrt{\pi}2^{1-r}}{\Gamma(\frac{r}{2})}\int_{\mathbb{R}} |\tau|^{r-1}w(\tau)d\tau.$$

□

Using (B.10) and the above lemma we deduce

$$\partial_x^\alpha\partial_y^\beta \mathscr{E}^\varepsilon(x,y)|_{x=y} = D_{m,\alpha,\beta}\varepsilon^{-m-2d(\alpha,\beta)} + O\left(\varepsilon^{-m-2d(\alpha,\beta)+2}\right) \ \text{ as } \varepsilon \to 0, \tag{B.13}$$

where $D_{m,\alpha,\beta}$ is a universal constant that depends only on m, α, β which is $= 0$ if $|\alpha+\beta|$ is odd,

$$D_{m,\alpha,\beta} = A_{m,\alpha,\beta,0}\frac{\sqrt{\pi}2^{1-r}}{\Gamma(\frac{r}{2})}\int_{\mathbb{R}} |\tau|^{r-1}w(\tau)d\tau, \ \ r = m + 2d(\alpha,\beta). \tag{B.14}$$

To determine the constant $D_{m,\alpha,\beta}$ it suffices to compute it for one particular m-dimensional Riemann manifold. Assume that (M,g) is the torus T^m equipped with the flat metric

$$g = \sum_{i=1}^m (d\theta^i)^2, \ \ 0 \leq \theta^i \leq 2\pi.$$

The eigenvalues of the corresponding Laplacian Δ_m are

$$|\vec{k}|^2, \ \ \vec{k} = (k_1,\dots,k_m) \in \mathbb{Z}^m.$$

Denote by $\prec$ the lexicographic order on $\mathbb{Z}^m$. For $\vec{\theta} = (\theta^1,\dots,\theta^m) \in \mathbb{R}$ and $\vec{k} \in \mathbb{Z}^m$ we set

$$\langle \vec{k}, \vec{\theta}\rangle := \sum_{j=1}^m k_j\theta^j.$$

A real orthornormal basis of $L^2(\mathbb{T}^m)$ is given by the functions

$$\Psi_{\vec{k}}(\vec{\theta}) = \frac{1}{(2\pi)^{\frac{m}{2}}}\begin{cases} 1, & \vec{k} = \vec{0} \\ 2^{\frac{1}{2}}\sin\langle \vec{k}, \vec{\theta}\rangle, & \vec{k} \succ \vec{0}, \\ 2^{\frac{1}{2}}\cos\langle \vec{k}, \vec{\theta}\rangle, & \vec{k} \prec \vec{0}. \end{cases}$$

Then

$$\mathscr{E}^\varepsilon(\vec{\theta},\vec{\varphi}) = \frac{1}{(2\pi)^m}\sum_{\vec{k}\in\mathbb{Z}^m} w(\varepsilon|\vec{k}|)e^{\boldsymbol{i}\langle\vec{k},\vec{\theta}-\vec{\varphi}\rangle},$$

so that

$$\partial^\alpha_{\vec{\theta}}\partial^\beta_{\vec{\varphi}}\mathscr{E}^\varepsilon(\vec{\theta},0) = \frac{\boldsymbol{i}^{|\alpha|-|\beta|}}{(2\pi)^m}\sum_{\vec{k}\in\mathbb{Z}^m} w_\varepsilon(|\vec{k}|)\vec{k}^{\alpha+\beta}e^{\boldsymbol{i}\langle\vec{k},\vec{\theta}\rangle}.$$

Define

$$W_m, u_\varepsilon : \mathbb{R}^m\to\mathbb{R},\ \ W_m(x) = w(|x|),\ \ u_\varepsilon(x) = W_m(\varepsilon x)x^{\alpha+\beta}.$$

Using the Poisson summation formula [21, §7.2] we deduce

$$\partial^\alpha_{\vec{\theta}}\partial^\beta_{\vec{\varphi}}\mathscr{E}^\varepsilon(0,0) = \frac{\boldsymbol{i}^{|\alpha|-|\beta|}}{(2\pi)^m}\sum_{\vec{\nu}\in\mathbb{Z}^m}\widehat{u}_\varepsilon(2\pi\vec{\nu}).$$

Observe that

$$\begin{aligned}\widehat{u}_\varepsilon(\xi) &= \int_{\mathbb{R}^m} e^{-\boldsymbol{i}\langle\xi,x\rangle}w(\varepsilon|x|)x^{\alpha+\beta}dx = (\boldsymbol{i}\partial_\xi)^{\alpha+\beta}\left(\int_{\mathbb{R}^m}e^{-\boldsymbol{i}\langle\xi,x\rangle}W_m(\varepsilon x)dx\right)\\ &= \varepsilon^{-m}(\boldsymbol{i}\partial_\xi)^{\alpha+\beta}\left(\int_{\mathbb{R}^m}e^{-\boldsymbol{i}\langle\frac{1}{\varepsilon}\xi,y\rangle}W_m(y)dy\right) = \varepsilon^{-m}(\boldsymbol{i}\partial_\xi)^{\alpha+\beta}\widehat{W}_m\left(\frac{1}{\varepsilon}\xi\right).\end{aligned}$$

Hence

$$\partial^\alpha_{\vec{\theta}}\partial^\beta_{\vec{\varphi}}\mathscr{E}^\varepsilon(\vec{\theta},0) = \frac{\boldsymbol{i}^{|\alpha|-|\beta|}}{(2\pi\varepsilon)^m}\sum_{\vec{\nu}\in\mathbb{Z}^m}\left\{(\boldsymbol{i}\partial_\xi)^{\alpha+\beta}\widehat{W}_m\left(\frac{1}{\varepsilon}\xi\right)\right\}_{\xi=2\pi\vec{\nu}}.$$

As $\varepsilon\to 0$ we have

$$\partial^\alpha_{\vec{\theta}}\partial^\beta_{\vec{\varphi}}\mathscr{E}^\varepsilon(0,0) = \varepsilon^{-m-|\alpha+\beta|}\frac{\boldsymbol{i}^{|\alpha|-|\beta|}}{(2\pi)^m}\Big((\boldsymbol{i}\partial_\xi)^{\alpha+\beta}\widehat{W}_m(0) + O(\varepsilon^N)\Big),\ \ \forall N.$$

Now observe that

$$(\boldsymbol{i}\partial_\xi)^{\alpha+\beta}\widehat{W}_m(0) = \int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta}dx.$$

so that

$$\partial^\alpha_{\vec{\theta}}\partial^\beta_{\vec{\varphi}}\mathscr{E}^\varepsilon(0,0) = \varepsilon^{-m-|\alpha+\beta|}\frac{\boldsymbol{i}^{|\alpha|-|\beta|}}{(2\pi)^m}\left(\int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta}dx + O(\varepsilon^N)\right),\ \ \forall N. \tag{B.15}$$

This shows that

$$D_{m,\alpha,\beta} = \frac{\boldsymbol{i}^{|\alpha|-|\beta|}}{(2\pi)^m}\int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta}dx. \tag{B.16}$$

This completes the proof of Theorem B.1. □

Remark B.4. Note that

$$\int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta}dx = \left(\int_{|x|=1}x^{\alpha+\beta}dA(x)\right)\underbrace{\left(\int_0^\infty w(r)r^{m+|\alpha+\beta|-1}dr\right)}_{=:I_{m,\alpha,\beta}(w)}.$$

On the other hand, according to [25, Lemma 9.3.10] we have

$$\int_{|x|=1}x^{\alpha+\beta}dA(x) = Z_{m,\alpha,\beta} := \begin{cases}\frac{2\prod_{i=1}^k\Gamma(\frac{\alpha_i+\beta_i+1}{2})}{\Gamma(\frac{m+|\alpha+\beta|}{2})}, & \alpha+\beta\in(2\mathbb{Z}_{\geq 0})^m,\\ 0, & \text{otherwise.}\end{cases} \tag{B.17}$$

We can now rewrite (B.16) as

$$D_{m,\alpha,\beta} = \varepsilon^{-m-|\alpha+\beta|}\frac{\boldsymbol{i}^{|\alpha|-|\beta|}Z_{m,\alpha,\beta}}{(2\pi)^m}I_{m,\alpha,\beta}(w). \tag{B.18}$$

□

Theorem B.5. *Fix a point $\boldsymbol{p}\in M$ and normal coordinates (x^i) near $\boldsymbol{p}$. For $i\neq j$ we denote by $K_{ij}(\boldsymbol{p})$ the sectional curvature of g at $\boldsymbol{p}$ along the plane spanned by $\partial_{x^i},\partial_{x^j}$. For any multi-induces $\alpha,\beta\in(\mathbb{Z}_{\geq 0})^m$ we set*

$$\mathscr{E}^\varepsilon_{\alpha;\beta} := \partial_x^\alpha\partial_y^\beta\mathscr{E}^\varepsilon(x,y)|_{x=y=0}.$$

Then there exists a universal constant $\mathcal{Z}_m$ that depends only on the dimension of M and the weight w such that

$$\mathscr{E}^\varepsilon_{ii;jj} - \mathscr{E}^\varepsilon_{ij;ij} = \mathcal{Z}_m K_{ij}(\boldsymbol{p})\varepsilon^{-m-2}\big(1+O(\varepsilon^2)\big) \ \text{ as } \varepsilon\to 0. \tag{B.19}$$

Proof. Using (B.12) we deduce

$$\mathscr{E}^\varepsilon_{ii;jj} - \mathscr{E}^\varepsilon_{ij;ij} \sim \frac{1}{\varepsilon}\Big\langle K_t^{ii,jj} - K_t^{ij,ij}, \eta(t)\widehat{w}\left(\frac{t}{\varepsilon}\right)\Big\rangle + O(\varepsilon^N), \ \ \varepsilon\to 0 \tag{B.20}$$

On the other hand from (B.9) we conclue

$$K_t^{ii,jj} - K_t^{ij,ij} \sim \sum_{k=1}^\infty d_m(2k)\left(\partial^2_{x^i}\partial^2_{y^j} - \partial^2_{x^ix^j}\partial^2_{y^iy^j}\right)\{\,U_k(x,y)\mathcal{H}_k(t,x,y)\,\}\,|_{x=y=0} \tag{B.21}$$

To investigate the above asymptotics we use the technology in [28].

Let us introduce some notations. For a positive integer k we denote by ∂^k a generic mixed-partial derivative of order k in the variables x^i, y^j. We denote by $\partial^k\eta$ the collection of k-th order derivatives of $\eta(x,y)$. $\mathcal{P}_i(X)$ will denote a homogeneous polynomial of degree i in the variables X, while $\mathcal{P}_k(X)\mathcal{P}_\ell(Y)$ will denote a polynomial which is homogeneous of degree k in the variables X and of degree ℓ in the variables Y. We then have the equalities

$$\mathcal{H}_a = \mathcal{P}_1(\partial\eta)\mathcal{H}_{a-1}, \tag{B.22}$$

$$\partial^2\mathcal{H}_a = \mathcal{P}_2(\partial\eta)\mathcal{H}_{a-2} + \mathcal{P}_1(\partial^2\eta)\mathcal{H}_{a-1}, \tag{B.23}$$

$$\partial^3\mathcal{H}_a = \mathcal{P}_3(\partial\eta)\mathcal{H}_{a-3} + \mathcal{P}_1(\partial\eta)\mathcal{P}_1(\partial^2\eta)\mathcal{H}_{a-2} + \mathcal{P}_1(\partial^3\eta)\mathcal{H}_{a-1}, \tag{B.24}$$

$$\begin{aligned}\partial^4\mathcal{H}_a &= \mathcal{P}_4(\partial\eta)\mathcal{H}_{a-4} + \big(\,\mathcal{P}_2(\partial\eta)\mathcal{P}_1(\partial^2\eta)\,\big)\mathcal{H}_{a-3}\\ &+\big(\,\mathcal{P}_2(\partial^2\eta) + \mathcal{P}_1(\partial\eta)\mathcal{P}_1(\partial^3\eta)\,\big)\mathcal{H}_{a-2} + \mathcal{P}_1(\partial^4\eta)\mathcal{H}_{a-1}.\end{aligned} \tag{B.25}$$

To simplify the presentation we will assume that in (B.19) we have $i=1$, $j=2$. Also, we will denote by $O(1)$ a function $f(x,y)$ such that $f(x,y)|_{x=y=0}=0$. The computations in Section A show that

$$\mathcal{P}_j(\partial\eta) = \mathcal{P}_k(\partial^3\eta) = O(1). \tag{B.26}$$

In particular, the above equalities show that the 1st and 3rd order derivatives of $\mathcal{H}^a$ are $O(1)$. We have

$$\begin{aligned}\partial^2_{x^1}\partial^2_{y^2}(U_k\mathcal{H}_k) &= \partial^2_{x^1}\left(\,(\partial^2_{y^2}U_k)\mathcal{H}_k + 2\partial_{y^2}U_k\partial_{y^2}\mathcal{H}_k + U_k\partial^2_{y^2}\mathcal{H}_k\right)\\ &= \big(\partial^2_{x^1}\partial^2_{y^2}U_k\big)\mathcal{H}_k + \big(\partial^2_{y^2}U_k\big)\big(\partial^2_{x^1}\mathcal{H}_k\big) + \big(\partial^2_{x^1}U_k\big)\big(\partial^2_{y^2}\mathcal{H}_k\big)\\ &\quad +4(\partial^2_{x^1y^2}U_k)(\partial^2_{x^1y^2}\mathcal{H}_k) + U_k\partial^2_{x^1}\partial^2_{y^2}\mathcal{H}_k + O(1),\end{aligned} \tag{B.27}$$

$$\begin{aligned}\partial^2_{x^1x^2}\partial^2_{y^1y^2}(U_k\mathcal{H}_k) &= \partial^2_{x^1x^2}\Big(\big(\partial^2_{y^1y^2}U_k\big)\mathcal{H}_k + \partial_{y^1}U_k\partial_{y^2}\mathcal{H}_k + \partial_{y^2}U_k\partial_{y^1}\mathcal{H}_k + U_k\partial^2_{y^1y^2}\mathcal{H}_k \Big)\\ &= \big(\partial^2_{x^1x^2}\partial^2_{y^1y^2}U_k\big)\mathcal{H}_k + \big(\partial^2_{y^1y^2}U_k\big)\big(\partial^2_{x^1x^2}\mathcal{H}_k\big)\\ &+ \partial^2_{x^2y^1}U_k\partial^2_{x^1y^2}\mathcal{H}_k + \partial^2_{x^1y^1}U_k\partial^2_{x^2y^2}\mathcal{H}_k + \partial^2_{x^2y^2}U_k\partial_{x^1y^1}\mathcal{H}_k + \partial^2_{x^1y^2}U_k\partial^2_{x^2y^1}\mathcal{H}_k\\ &+ \partial^2_{x^1x^2}U_k\partial^2_{y^1y^2}\mathcal{H}_k + U_k\partial^2_{x^1x^2}\partial^2_{y^1y^2}\mathcal{H}_k + O(1). \end{aligned}\tag{B.28}$$

Using (B.22)-(B.25) we deduce that

$$\Big(\partial^2_{x^1}\partial^2_{y^2} - \partial^2_{x^1x^2}\partial^2_{y^1y^2}\Big)(U_k\mathcal{H}_k)_{x=y=0} = \sum_{j=0}^{4} T_k^j\mathcal{H}_{k-j}|_{x=y=0},$$

where the coefficients T_k^j are polynomials in the derivatives of U_k and η at $(x,y) = (0,0)$. Using (B.22)-(B.25) we deduce

$$T_k^4 = T_k^3 = 0.$$

Moreover, in view to (B.26), the terms in T_k^2 are due only to the 4-th order derivatives of $\mathcal{H}_k$. Upon inspecting (B.27) and (B.28) we see that the 4-th order derivatives of $\mathcal{H}_k$ are multiplied by U_k. According to (B.6) the function U_k is $O(1)$ if $k > 1$. Hence $T_k^2 = 0$ for $k > 1$. We deduce

$$\begin{aligned}K_t^{ii,jj} - K_t^{ij,ij} &\sim \sum_{k=1}^{\infty} d_m(2k)\left(T_k^0\mathcal{H}_k + T_k^1\mathcal{H}_{k-1} + T_k^2\mathcal{H}_{k-2}\right)|_{x=y=0}\\ &= B_{-1}\mathcal{H}_{-1}|_{x=y=0} + B_0\mathcal{H}_0|_{x=y=0} + B_1\mathcal{H}_1|_{x=y=0} + \cdots,\end{aligned}$$

where

$$B_{-1} = d_m(2)T_1^2,\;\; B_0 = d_m(2)T_1^1,\;\; B_1 = d_m(2)T_1^0 + d_m(4)T_2^1, \dots .$$

The term B_{-1} can be alternatively described as

$$B_{-1} = A_{m,ii;jj,0} - A_{m,ij;ij,0},$$

where the coefficients $A_{m,\alpha,\beta,0}$ are defined as in (B.12). Using (B.14) and (B.16) we deduce

$$B_{-1} = 0.$$

To compute T_1^1 we observe first that

$$\eta(x-y) = \sum_i (x^i - y^i)^2 + \text{higer order terms.}\tag{B.29}$$

Using (B.23) we can simplify (B.27) and (B.28) in the case $k = 1$ as follows.

$$\partial^2_{x^1}\partial^2_{y^2}(U_1\mathcal{H}_1) = \big(\partial^2_{x^1}\partial^2_{y^2}U_1\big)\mathcal{H}_1 + U_1\partial^2_{x^1}\partial^2_{y^2}\mathcal{H}_1 + O(1),\tag{B.30}$$

$$\begin{aligned}\partial^2_{x^1x^2}\partial^2_{y^1y^2}(U_1\mathcal{H}_1) &= \big(\partial^2_{x^1x^2}\partial^2_{y^1y^2}U_1\big)\mathcal{H}_1 + \partial^2_{x^1y^1}U_1\partial^2_{x^2y^2}\mathcal{H}_1\\ &+\partial^2_{x^2y^2}U_1\partial_{x^1y^1}\mathcal{H}_1 + U_1\partial^2_{x^1x^2}\partial^2_{y^1y^2}\mathcal{H}_1 + O(1).\end{aligned}\tag{B.31}$$

Using (B.23), (B.25) and (B.29) we deduce that

$$\begin{aligned}T_1^1 &= \Big(\partial^2_{x^1}\partial^2_{y^2} - \partial^2_{x^1x^2}\partial^2_{y^1y^2}\Big)\,\eta|_{(0,0)}\\ &+2\Big(\partial^2_{x^1}U_1 + \partial^2_{y^2}U_1\Big)|_{(0,0)} + 2\Big(\partial^2_{x^1y^1}U_1 + \partial^2_{x^2y^2}U_1\Big)|_{(0,0)}.\end{aligned}$$

Using the transport equation (B.5) we obtain as in [10, VI.3] that U_1 coincides with the function $\varphi(x,y)$ in [10, VI.3 Eq.(33)] or the function $u_0(x,y)$ in [6, p. 380]. For our purposes an explicit description of U_1 is not needed. All we care is that

$$U_1(x,y) = U_1(y,x), \;\; U_1(x,x) \equiv 1.$$

These conditions imply that the Hessian of $U_1(x,y)$ at $(0,0)$ is a quadratic form in the variables $u_i = (x^i - y^i)$ so that

$$\partial^2_{x^1} U_1(0,0) + \partial^2_{x^1 y^1} U_1(0,0) = \partial^2_{y^2} U_1(0,0) + \partial^2_{x^2 y^2} U_1(0,0) = 0. \tag{B.32}$$

Hence

$$T_1^1 = \left(\partial^2_{x^1}\partial^2_{y^2} - \partial^2_{x^1x^2}\partial^2_{y^1y^2} \right) \eta|_{(0,0)}.$$

Using (A.17) we conclude that

$$T_1^1 = ZR_{1212} = ZK_{12}(\boldsymbol{p}),$$

where Z is a universal constant, independent of (M,g). Hence

$$K_t^{ii,jj} - K_t^{ij,ij} \sim d_m(2) Z K_{12}(\boldsymbol{p}) \mathcal{H}_0|_{x=y=0} + \sum_{k\geq 1} B_k \mathcal{H}_k|_{x=y=0}.$$

The equality (B.19) now follows from the above equality by using (B.20), (B.7) and Lemma B.3. □

Appendix C. Gaussian measures and Gaussian vectors

For the reader's convenience we survey here a few basic facts about Gaussian measures. For more details we refer to [8]. A *Gaussian measure* on $\mathbb{R}$ is a Borel measure $\gamma_{\mu,v}$, $v \geq 0$, $m \in \mathbb{R}$, of the form

$$\gamma_{\mu,v}(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{(x-\mu)^2}{2v}} dx.$$

The scalar μ is called the *mean*, while v is called the *variance*. We allow v to be zero in which case

$$\gamma_{\mu,0} = \delta_\mu = \text{the Dirac measure on } \mathbb{R} \text{ concentrated at } \mu.$$

For a real valued random variable X we write

$$X \in \boldsymbol{N}(\mu, v) \tag{C.1}$$

if the probability measure of X is $\gamma_{\mu,v}$.

Suppose that $\boldsymbol{V}$ is a finite dimensional vector space. A *Gaussian measure* on $\boldsymbol{V}$ is a Borel measure γ on $\boldsymbol{V}$ such that, for any $\xi \in \boldsymbol{V}^\vee$, the pushforward $\xi_*(\gamma)$ is a Gaussian measure on $\mathbb{R}$,

$$\xi_*(\gamma) = \gamma_{\mu(\xi), v(\xi)}.$$

One can show that the map $\boldsymbol{V}^\vee \ni \xi \mapsto \mu(\xi) \in \mathbb{R}$ is linear, and thus can be identified with a vector $\boldsymbol{\mu}_\gamma \in \boldsymbol{V}$ called the *barycenter* or *expectation* of γ that can be alternatively defined by the equality

$$\boldsymbol{\mu}_\gamma = \int_{\boldsymbol{V}} \boldsymbol{v} d\gamma(\boldsymbol{v}).$$

Moreover, there exists a nonnegative definite, symmetric bilinear map

$$\boldsymbol{\Sigma} : \boldsymbol{V}^\vee \times \boldsymbol{V}^\vee \to \mathbb{R} \;\text{ such that }\; v(\xi) = \boldsymbol{\Sigma}(\xi,\xi), \;\; \forall \xi \in \boldsymbol{V}^\vee.$$

The form $\boldsymbol{\Sigma}$ is called the *covariance form* and can be identified with a linear operator $\boldsymbol{S}: \boldsymbol{V}^{\vee} \to \boldsymbol{V}$ such that

$$\boldsymbol{\Sigma}(\xi, \eta) = \langle \xi, \boldsymbol{S}\eta \rangle, \ \ \forall \xi, \eta \in \boldsymbol{V}^{\vee},$$

where $\langle -, - \rangle : \boldsymbol{V}^{\vee} \times \boldsymbol{V} \to \mathbb{R}$ denotes the natural bilinear pairing between a vector space and its dual. The operator $\boldsymbol{S}$ is called the *covariance operator* and it is explicitly described by the integral formula

$$\langle \xi, \boldsymbol{S}\eta \rangle = \boldsymbol{\Sigma}(\xi, \eta) = \int_{\boldsymbol{V}} \langle \xi, \boldsymbol{v} - \boldsymbol{\mu}_\gamma \rangle \langle \eta, \boldsymbol{v} - \boldsymbol{\mu}_\gamma \rangle d\gamma(\boldsymbol{v}).$$

The Gaussian measure is said to be *nondegenerate* if $\boldsymbol{\Sigma}$ is nondegenerate, and it is called *centered* if $\boldsymbol{\mu} = 0$. A Gaussian measure on $\boldsymbol{V}$ is uniquely determined by its covariance form and its expectation.

Example C.1. Suppose that $\boldsymbol{U}$ is an n-dimensional Euclidean space with inner product $(-,-)$. We use the inner product to identify $\boldsymbol{U}$ with its dual $\boldsymbol{U}^{\vee}$. If $A: \boldsymbol{U} \to \boldsymbol{U}$ is a symmetric, positive definite operator, then

$$\boldsymbol{\gamma}_A(d\boldsymbol{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det A}} e^{-\frac{1}{2}(A^{-1}\boldsymbol{u}, \boldsymbol{u})}\,|d\boldsymbol{u}| \tag{C.2}$$

is a centered Gaussian measure on $\boldsymbol{U}$ with covariance form described by the operator A. □

If $\boldsymbol{V}$ is a finite dimensional vector space equipped with a Gaussian measure γ and $\boldsymbol{L}: \boldsymbol{V} \to \boldsymbol{U}$ is a linear map, then the pushforward $\boldsymbol{L}_*\gamma$ is a Gaussian measure on $\boldsymbol{U}$ with expectation $\boldsymbol{\mu}_{\boldsymbol{L}_*\gamma} = \boldsymbol{L}(\boldsymbol{\mu}_\gamma)$ and covariance form

$$\boldsymbol{\Sigma}_{\boldsymbol{L}_*\gamma} : \boldsymbol{U}^{\vee} \times \boldsymbol{U}^{\vee} \to \mathbb{R}, \ \ \boldsymbol{\Sigma}_{\boldsymbol{L}_*\gamma}(\eta, \eta) = \boldsymbol{\Sigma}_\gamma(\boldsymbol{L}^{\vee}\eta, \boldsymbol{L}^{\vee}\eta), \ \ \forall \eta \in \boldsymbol{U}^{\vee},$$

where $\boldsymbol{L}^{\vee}: \boldsymbol{U}^{\vee} \to \boldsymbol{V}^{\vee}$ is the dual (transpose) of the linear map $\boldsymbol{L}$. Observe that if γ is nondegenerate and $\boldsymbol{L}$ is surjective, then $\boldsymbol{L}_*\gamma$ is also nondegenerate.

Suppose $(\mathcal{S}, \mu)$ is a probability space. A *Gaussian* random vector on $(\mathcal{S}, \mu)$ is a (Borel) measurable map

$$X : \mathcal{S} \to \boldsymbol{V}, \ \ \boldsymbol{V} \text{ finite dimensional vector space}$$

such that $X_*\mu$ is a Gaussian measure on $\boldsymbol{V}$. We will refer to this measure as the *associated Gaussian measure*, we denote it by γ_X and we denote by $\boldsymbol{\Sigma}_X$ (respectively $\boldsymbol{S}(X)$) its covariance form (respectively operator),

$$\boldsymbol{\Sigma}_X(\xi_1, \xi_2) = \boldsymbol{E}\big(\, \langle \xi_1, X - \boldsymbol{E}(X)\,\rangle\, \langle \xi_2, X - \boldsymbol{E}(X)\,\rangle\,\big).$$

Note that the expectation of γ_X is precisely the expectation of X. The random vector is called *nondegenerate*, respectively *centered*, if the Gaussian measure γ_X is such.

Let us point out that if $X : \mathcal{S} \to \boldsymbol{U}$ is a Gaussian random vector and $\boldsymbol{L}: \boldsymbol{U} \to \boldsymbol{V}$ is a linear map, then the random vector $\boldsymbol{L}X : \mathcal{S} \to \boldsymbol{V}$ is also Gaussian. Moreover

$$\boldsymbol{E}(\boldsymbol{L}X) = \boldsymbol{L}\boldsymbol{E}(X), \ \ \boldsymbol{\Sigma}_{\boldsymbol{L}X}(\xi, \xi) = \boldsymbol{\Sigma}_X(\boldsymbol{L}^{\vee}\xi, \boldsymbol{L}^{\vee}\xi), \ \ \forall \xi \in \boldsymbol{V}^{\vee},$$

where $\boldsymbol{L}^{\vee}: \boldsymbol{V}^{\vee} \to \boldsymbol{U}^{\vee}$ is the linear map dual to $\boldsymbol{L}$. Equivalently, $\boldsymbol{S}(\boldsymbol{L}X) = \boldsymbol{L}\boldsymbol{S}(X)\boldsymbol{L}^{\vee}$.

Suppose that $X_j : \mathcal{S} \to \boldsymbol{V}_1$, $j = 1, 2$, are two *centered* Gaussian random vectors such that the direct sum $X_1 \oplus X_2 : \mathcal{S} \to \boldsymbol{V}_1 \oplus \boldsymbol{V}_2$ is also a centered Gaussian random vector with associated Gaussian measure

$$\gamma_{X_1 \oplus X_2} = p_{X_1 \oplus X_2}(\boldsymbol{x}_1, \boldsymbol{x}_2)|d\boldsymbol{x}_1 d\boldsymbol{x}_2|.$$

We obtain a bilinear form

$$\boldsymbol{cov}(X_1, X_2) : \boldsymbol{V}_1^{\vee} \times \boldsymbol{V}_2^{\vee} \to \mathbb{R}, \ \ \boldsymbol{cov}(X_1, X_2)(\xi_1, \xi_2) = \boldsymbol{\Sigma}(\xi_1, \xi_2),$$

called the *covariance form*. The random vectors X_1 and X_2 are independent if and only if they are uncorrelated, i.e.,

$$\boldsymbol{cov}(X_1, X_2) = 0.$$

We can then identify $\boldsymbol{cov}(X_1, X_2)$ with a linear operator $\boldsymbol{Cov}(X_1, X_2) : \boldsymbol{V}_2 \to \boldsymbol{V}_1$, via the equality

$$\begin{aligned}\boldsymbol{E}\big(\, \langle \xi_1, X_1\rangle\langle \xi_2, X_2\rangle \,\big) &= \boldsymbol{cov}(X_1, X_2)(\xi_1, \xi_2)\\ &= \big\langle\, \xi_1, \boldsymbol{Cov}(X_1, X_2)\xi_2^\dagger \,\big\rangle, \;\; \forall \xi_1 \in \boldsymbol{V}_1^\vee, \;\; \xi_2 \in \boldsymbol{V}_2^\vee,\end{aligned}$$

where $\xi_2^\dagger \in \boldsymbol{V}_2$ denotes the vector metric dual to ξ_2. The operator $\boldsymbol{Cov}(X_1, X_2)$ is called the *covariance operator* of X_1, X_2.

The conditional random variable $(X_1|X_2 = x_2)$ has probability density

$$p_{(X_1|X_2=\boldsymbol{x}_2)}(\boldsymbol{x}_1) = \frac{p_{X_1\oplus X_2}(\boldsymbol{x}_1, \boldsymbol{x}_2)}{\int_{\boldsymbol{V}_1} p_{X_1\oplus X_2}(\boldsymbol{x}_1, \boldsymbol{x}_2)|d\boldsymbol{x}_1|}.$$

For a measurable function $f : \boldsymbol{V}_1 \to \mathbb{R}$ the conditional expectation $\boldsymbol{E}(f(X_1)|X_2 = \boldsymbol{x}_2)$ is the (deterministic) scalar

$$\boldsymbol{E}(f(X_1)|X_2 = \boldsymbol{x}_2) = \int_{\boldsymbol{V}_1} f(\boldsymbol{x}_1)p_{(X_1|X_2=\boldsymbol{x}_2)}(\boldsymbol{x}_1)|d\boldsymbol{x}_1|.$$

If X_2 is nondegenerate, the *regression formula*, [5], implies that the random vector $(X_1|X_2 = x_2)$ is a Gausian vector with covariance operator

$$\boldsymbol{S}(Y) = \boldsymbol{S}(X_1) - \boldsymbol{Cov}(X_1, X_2)\boldsymbol{S}(X_2)^{-1}\, \boldsymbol{Cov}(X_2, X_1), \tag{C.3}$$

and mean

$$\boldsymbol{E}(X_1|X_2 = x_2) = Cx_2, \tag{C.4}$$

where C is given by

$$C = \boldsymbol{Cov}(X_1, X_2)\boldsymbol{S}(X_2)^{-1}. \tag{C.5}$$

Appendix D. A class of random symmetric matrices

We denote by Sym_m the space of real symmetric $m \times m$ matrices. This is an Euclidean space with respect to the inner product

$$(A, B) := \mathrm{tr}(AB).$$

This inner product is invariant with respect to the action of $\mathrm{SO}(m)$ on Sym_m. We set

$$\widehat{\boldsymbol{E}}_{ij} := \begin{cases} \boldsymbol{E}_{ij}, & i = j\\ \frac{1}{\sqrt{2}}E_{ij}, & i < j. \end{cases}.$$

The collection $(\widehat{\boldsymbol{E}}_{ij})_{i\leq j}$ is a basis of Sym_m orthonormal with respect to the above inner product. We set

$$\hat{a}_{ij} := \begin{cases} a_{ij}, & i = j\\ \sqrt{2}a_{ij}, & i < j. \end{cases}$$

The collection $(\hat{a}_{ij})_{i\leq j}$ the orthonormal basis of $\mathrm{Sym}_m^\vee$ dual to $(\widehat{\boldsymbol{E}}_{ij})$. The volume density induced by this metric is

$$|dA| := \prod_{i\leq j} d\widehat{a}_{ij} = 2^{\frac{1}{2}\binom{m}{2}} \prod_{i\leq j} da_{ij}.$$

Throughout the paper we encountered a 2-parameter family of Gaussian probability measures on Sym_m. More precisely for any real numbers u, v such that

$$v > 0, mu + 2v > 0,$$

we denote by $\mathrm{Sym}_m^{u,v}$ the space Sym_m equipped with the centered Gaussian measure $d\boldsymbol{\Gamma}_{u,v}(A)$ uniquely determined by the covariance equalities

$$\boldsymbol{E}(a_{ij}a_{k\ell}) = u\delta_{ij}\delta_{k\ell} + v(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \;\; \forall 1 \leq i, j, .k, \ell \leq m.$$

In particular we have

$$\boldsymbol{E}(a_{ii}^2) = u + 2v, \;\; \boldsymbol{E}(a_{ii}a_{jj}) = u, \quad \boldsymbol{E}(a_{ij}^2) = v, \;\; \forall 1 \leq i \neq j \leq m,$$

while all other covariances are trivial. The ensemble $\mathrm{Sym}_m^{0,v}$ is a rescaled version of of the Gaussian Orthogonal Ensemble (GOE) and we will refer to it as GOE_m^v.

For $u > 0$ the ensemble $\mathrm{Sym}_m^{u,v}$ can be given an alternate description. More precisely a random $A \in \mathrm{Sym}_m^{u,v}$ can be described as a sum

$$A = B + \; X\mathbb{1}_m, \;\; B \in \mathrm{GOE}_m^v, \;\; X \in \boldsymbol{N}(0, u), \quad B \text{ and } X \text{ independent.}$$

We write this

$$\mathrm{Sym}_m^{u,v} = \mathrm{GOE}_m^v \,\hat{+} \boldsymbol{N}(0, u)\mathbb{1}_m, \tag{D.1}$$

where $\hat{+}$ indicates a sum of *independent* variables.

The Gaussian measure $d\boldsymbol{\Gamma}_{u,v}$ coincides with the Gaussian measure $d\boldsymbol{\Gamma}_{u+2v,u,v}$ defined in [27, App. B]. We recall a few facts from [27, App. B].

The probability density $d\boldsymbol{\Gamma}_{u,v}$ has the explicit description

$$d\boldsymbol{\Gamma}_{u,v}(A) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}\sqrt{D(u,v)}} e^{-\frac{1}{4v}\operatorname{tr} A^2 - \frac{u'}{2}(\operatorname{tr} A)^2}|dA|,$$

where

$$D(u, v) = (2v)^{(m-1)+\binom{m}{2}}\big(mu + 2v\,\big),$$

and

$$u' = \frac{1}{m}\left(\frac{1}{mu + 2v} - \frac{1}{2v}\right) = -\frac{u}{2v(mu + 2v)}.$$

In the special case GOE_m^v we have $u = u' = 0$ and

$$d\boldsymbol{\Gamma}_{0,v}(A) = \frac{1}{(2\pi v)^{\frac{m(m+1)}{4}}} e^{-\frac{1}{4v}\operatorname{tr} A^2}|dA|. \tag{D.2}$$

We have a *Weyl integration formula* [3] which states that if $f : \mathrm{Sym}_m \to \mathbb{R}$ is a measurable function which is invariant under conjugation, then the the value $f(A)$ at $A \in \mathrm{Sym}_m$ depends only on the eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ of A and we have

$$\boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\, f(X)\,\big) = \frac{1}{\boldsymbol{Z}_m(v)} \int_{\mathbb{R}^m} f(\lambda_1, \ldots, \lambda_m) \underbrace{\left(\prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j|\right) \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}}}_{=:Q_{m,v}(\lambda)} |d\lambda_1 \cdots d\lambda_m|, \tag{D.3}$$

where the normalization constant $\boldsymbol{Z}_m(v)$ is defined by

$$\boldsymbol{Z}_m(v) = \int_{\mathbb{R}^m} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}} |d\lambda_1 \cdots d\lambda_m|$$

$$= (2v)^{\frac{m(m+1)}{4}} \underbrace{\int_{\mathbb{R}^m} \prod_{1\leq i<j\leq m} |\lambda_i - \lambda_j| \prod_{i=1}^m e^{-\frac{\lambda_i^2}{2}} |d\lambda_1 \cdots d\lambda_m|}_{=:\boldsymbol{Z}_m}.$$

The precise value of $\boldsymbol{Z}_m$ can be computed via Selberg integrals, [3, Eq. (2.5.11)], and we have

$$\boldsymbol{Z}_m = (2\pi)^{\frac{m}{2}} m! \prod_{j=1}^m \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1}{2})} = 2^{\frac{m}{2}} m! \prod_{j=1}^m \Gamma\left(\frac{j}{2}\right). \tag{D.4}$$

For any positive integer n we define the *normalized* 1-point corelation function $\rho_{n,v}(x)$ of GOE_n^v to be

$$\rho_{n,v}(x) = \frac{1}{\boldsymbol{Z}_n(v)} \int_{\mathbb{R}^{n-1}} Q_{n,v}(x, \lambda_2, \dots, \lambda_n) d\lambda_1 \cdots d\lambda_n.$$

For any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ we have [11, §4.4]

$$\frac{1}{n} \boldsymbol{E}_{\mathrm{GOE}_n^v}\big(\operatorname{tr} f(X)\big) = \int_{\mathbb{R}} f(\lambda) \rho_{n,v}(\lambda) d\lambda. \tag{D.5}$$

The equality (D.5) characterizes $\rho_{n,v}$. Let us observe that for any constant $c > 0$, if

$$A \in \mathrm{GOE}_n^v \Longleftrightarrow cA \in \mathrm{GOE}_n^{c^2 v}.$$

Hence, for any Borel set $B \subset \mathbb{R}$ we have

$$\int_{cB} \rho_{n,c^2v}(x) dx = \int_B \rho_{n,v}(y) dy.$$

We conclude that

$$c\rho_{n,c^2v}(cy) = \rho_{n,v}(y), \quad \forall n, c, y. \tag{D.6}$$

The behavior of the 1-point correlation function $\rho_{n,v}(x)$ for n large is described by *Wigner's semicircle theorem* [3, Thm.2.1.1]. It states that, for any $v > 0$, the sequence of probability measures on $\mathbb{R}$

$$\rho_{n,vn^{-1}}(x) dx = n^{\frac{1}{2}} \rho_{n,v}(n^{\frac{1}{2}} x) dx$$

converges weakly as $n \to \infty$ to the semicircle distribution

$$\rho_{\infty,v}(x)|dx| = \boldsymbol{I}_{\{|x|\leq 2\sqrt{v}\}} \frac{1}{2\pi v} \sqrt{4v - x^2} |dx|.$$

The expected value of the absolute value of the determinant of of a random $A \in \mathrm{GOE}_m^v$ can be expressed neatly in terms of the correlation function $\rho_{m+1,v}$. More precisely, we have the following result first observed by Y.V. Fyodorov [14] in a context related to ours. Set

$$\boldsymbol{C}_m(v) := 2^{\frac{3}{2}} (2v)^{\frac{m+1}{2}} \Gamma\left(\frac{m+3}{2}\right).$$

Lemma D.1. *Suppose $v > 0$. Then for any $c \in \mathbb{R}$ we have*

$$\boldsymbol{E}_{\mathrm{GOE}_m^v}\big(|\det(A - c\mathbb{1}_m)|\big) = \boldsymbol{C}_m(v) e^{\frac{c^2}{4v}} \rho_{m+1,v}(c).$$

Proof. Using the Weyl integration formula we deduce

$$\boldsymbol{E}_{\mathrm{GOE}_m^v}\big(|\det(A - c\mathbb{1}_m)|\big) = \frac{1}{\boldsymbol{Z}_m(v)} \int_{\mathbb{R}^m} \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}} |c - \lambda_i| \prod_{i\leq j} |\lambda_i - \lambda_j| d\lambda_1 \cdots d\lambda_m$$

$$= \frac{e^{\frac{c^2}{4v}}}{\boldsymbol{Z}_m(v)} \int_{\mathbb{R}^m} e^{-\frac{c^2}{4v}} \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}} |c-\lambda_i| \prod_{i\leq j} |\lambda_i - \lambda_j| d\lambda_1 \cdots d\lambda_m$$

$$= \frac{e^{\frac{c^2}{4v}} \boldsymbol{Z}_{m+1}(v)}{\boldsymbol{Z}_m(v)} \frac{1}{\boldsymbol{Z}_{m+1}(v)} \int_{\mathbb{R}^m} Q_{m+1,v}(c,\lambda_1,\dots,\lambda_m) d\lambda_1 \cdots d\lambda_m$$

$$= \frac{e^{\frac{c^2}{4v}} \boldsymbol{Z}_{m+1}(v)}{\boldsymbol{Z}_m(v)} \rho_{m+1,v}(c) = v^{\frac{m+1}{2}} \frac{e^{\frac{c^2}{4v}} \boldsymbol{Z}_{m+1}}{\boldsymbol{Z}_m} \rho_{m+1,v}(c)$$

$$= (m+1)\sqrt{2}(2v)^{\frac{m+1}{2}} e^{\frac{c^2}{4v}} \Gamma\left(\frac{m+1}{2}\right) \rho_{m+1,v}(c) = 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}} \Gamma\left(\frac{m+3}{2}\right) e^{\frac{c^2}{4v}} \rho_{m+1,v}(c).$$

□

The above result admits the following generalization, [4, Lemma 3.2.3].

Lemma D.2. *Let $u, v > 0$. Set $\theta^+_{m,v}(x) := \rho_{m+1,v}(x) e^{\frac{x^2}{4v}}$. Then*

$$\boldsymbol{E}_{\mathrm{Sym}_m^{u,v}}\big(\,|\det(A - c\mathbb{1}_m)|\,\big) = \boldsymbol{C}_m(v) \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{\frac{(c-x)^2}{4v} - \frac{x^2}{2u}} dx \tag{D.7a}$$

$$= \boldsymbol{C}_m(v)(\gamma_u * \theta^+_{m+1,v})(c). \tag{D.7b}$$

In particular, if $u = 2kv$, $k < 1$ we have

$$\boldsymbol{E}_{\mathrm{Sym}_m^{2kv,v}}\big(\,|\det(A - c\mathbb{1}_m)|\,\big) = \boldsymbol{C}_m \frac{1}{\sqrt{2\pi k}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{-\frac{(x+t_k^2 c)^2}{4vt_k^2} + \frac{(t_k^2+1)c^2}{4v}} dx,$$

($\lambda := c - x$)

$$= \boldsymbol{C}_m(v) \frac{1}{\sqrt{2\pi k}} \int_{\mathbb{R}} \rho_{m+1,v}(\lambda) e^{-\frac{1}{4vt_k^2}(\lambda - (t_k^2+1)c)^2 + \frac{(t_k^2-1)c^2}{4v}} d\lambda$$

where $t_k^2 := \frac{k}{1-k}$.

Proof. Recall the equality (D.1) $\mathrm{Sym}_m^{u,v} = \mathrm{GOE}_m^v \hat{+} \boldsymbol{N}(0,u)\mathbb{1}_m$. We deduce that

$$\boldsymbol{E}_{\mathrm{Sym}_m^{u,v}}\big(\,|\det(A - c\mathbb{1}_m)|\,\big) = \boldsymbol{E}\big(\det(B + (X-c)\mathbb{1})|\,\big)$$

$$= \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\,|\det(B - (c-X)\mathbb{1}_m)|\,\big|\, X = x\big) e^{-\frac{x^2}{2u}} dx$$

$$= \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\,|\det(B - (c-x)\mathbb{1}_m)|\,\big) e^{-\frac{x^2}{2u}} dx$$

$$= \boldsymbol{C}_m(v) \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{\frac{(c-x)^2}{4v} - \frac{x^2}{2u}} dx.$$

Now observe that if $u = 2kv$ then

$$\frac{(c-x)^2}{4v} - \frac{x^2}{2u} = -\frac{x^2}{4kv} + \frac{1}{4v}(x^2 - 2cx + c^2)$$

$$= \frac{1}{4v}\left(-\frac{1}{t_k^2}x^2 - 2cx - c^2 t_k^2\right) + \frac{c^2(1+t_k^2)}{4v} = -\frac{1}{4vt_k^2}(x + t_k^2 c)^2 + \frac{c^2(1+t_k^2)}{4v}.$$

□

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