J. London Math. Soc. (2) 69 (2004) 593–607
DOI: 10.1112/S0024610704005228

SEIBERG–WITTEN INVARIANTS AND SURFACE SINGULARITIES. II: SINGULARITIES WITH GOOD $\mathbb{C}^*$-ACTION

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Abstract

A previous conjecture is verified for any normal surface singularity which admits a good $\mathbb{C}^*$-action. This result connects the Seiberg–Witten invariant of the link (associated with a certain 'canonical' spin^c structure) with the geometric genus of the singularity, provided that the link is a rational homology sphere.

As an application, a topological interpretation is found of the generalized Batyrev stringy invariant (in the sense of Veys) associated with such a singularity.

The result is partly based on the computation of the Reidemeister–Turaev sign-refined torsion and the Seiberg–Witten invariant (associated with any spin^c structure) of a Seifert 3-manifold with negative orbifold Euler number and genus zero.

1. *Introduction*

The main motivation for writing this paper was [**28**], where the authors formulated a very general conjecture which relates the topological and the analytical invariants of a complex normal surface singularity whose link is a rational homology sphere.

Let $(X, 0)$ be a normal two-dimensional analytic singularity. From a topological point of view, it is completely characterized by its link M, which is an oriented 3-manifold. Moreover, by a result of Neumann [**30**], any decorated resolution graph of $(X, 0)$ carries the same information as M. A property of $(X, 0)$ will be called *topological* if it can be determined from M, or equivalently, from any resolution graph of $(X, 0)$. For example, for a given resolution, if we take the canonical divisor K, and the number $\#\mathcal{V}$ of irreducible components of the exceptional divisor, then $K^2 + \#\mathcal{V}$ is independent of the choice of the resolution, hence it is an invariant of the link M (cf. Subsection 2.4).

Let us recall some definitions regarding the *analytical* structure of $(X, 0)$. Consider the line bundle $\Omega^2_{X\setminus\{0\}}$ of holomorphic 2-forms on $X\setminus\{0\}$. If this line bundle is holomorphically trivial then we say that $(X, 0)$ is Gorenstein. If one of its powers is holomorphically trivial then we say that $(X, 0)$ is $\mathbb{Q}$-Gorenstein. Let $\pi : \tilde{X} \longrightarrow X$ be a resolution over a sufficiently small Stein representative X of the germ $(X, 0)$. Then $p_g := \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is finite and independent of the choice of π. It is called the *geometric genus* of $(X, 0)$.

Our goal is to investigate in which instances one can express the geometric genus in terms of topological invariants of the link. By the conjecture [**28**], if the link of a $\mathbb{Q}$-Gorenstein singularity is a rational homology sphere, then it determines p_g (by the precise formula (1) below. The conjecture has grown partly from the work of

Received 9 June 2003; revised 7 November 2003.

2000 *Mathematics Subject Classification* 14B05, 14J17, 32S25, 57M27, 57R57 (primary), 14E15, 32S45, 57M25 (secondary).

The first author is partially supported by NSF grant DMS-0088950; the second author is partially supported by NSF grant DMS-0071820.

Artin on rational singularities [**2**, **3**], and of Laufer, Wagreich, Yau and Némethi on elliptic singularities [**18**, **27**, **43**], and partly from the work of Fintushel and Stern [**10**], and of Neumann and Wahl [**32**] connecting the signature of the Milnor fiber with the Casson invariant of the link, provided that the link is an integral homology sphere. The conjecture formulated in [**32**] about this connection had a crucial influence on our project. For more details and historical data the reader is invited to consult [**28**]. (For a related set of conjectures, see also [**33**].)

In the conjecture [**28**], the main new ingredient is the Seiberg–Witten invariant of the link M associated with a 'canonical' spin^c structure. We recall (see [**28**] and the references therein, or Subsections 2.2 and 2.3 below) that the Seiberg–Witten invariant associated with any spin^c structure σ of M is a rational number $\mathbf{sw}^0_M(\sigma)$. In [**28**] we introduced a 'canonical' spin^c structure σ_{can} of M as follows. The (almost) complex structure on $X\setminus\{0\}$ induces a natural spin^c structure on $X\setminus\{0\}$. Then σ_{can}, by definition, is its restriction to M. The point is that σ_{can} depends only on the topology of M.

In general, it is very difficult to compute $\mathbf{sw}^0_M(\sigma)$ from its original analytic definition. In this paper we will replace it by the invariant $\mathbf{sw}^{\text{TCW}}_M(\sigma)$ which is defined topologically, and conjecturally equals $\mathbf{sw}^0_M(\sigma)$ (cf. also [**35**]); $\mathbf{sw}^{\text{TCW}}_M(\sigma)$ is the sign-refined Reidemeister–Turaev torsion of M associated with σ normalized by the Casson–Walker invariant (see Subsection 2.3). If M is an integral homology sphere then σ_{can} is the unique spin^c structure of M, and $-\mathbf{sw}^{\text{TCW}}_M(\sigma_{\text{can}})$ is the Casson invariant of M.

In this paper we prove the conjecture [**28**] for singularities with good $\mathbb{C}^*$-action. A complex affine algebraic variety X admits a $\mathbb{C}^*$-action if and only if the affine coordinate ring A admits a grading $A=\bigoplus_k A_k$. Following Orlik and Wagreich, we say that the action is *good* if $A_k=0$ for $k<0$ and $A_0=\mathbb{C}$. This means that the point 0 corresponding to the maximal ideal $\bigoplus_{k>0} A_k$ is the only fixed point of the action. Additionally, we assume that $(X,0)$ is normal. (Notice also that by [**29**], if a normal surface singularity admits a good $\mathbb{C}^*$-action and its link is a rational homology sphere then it is $\mathbb{Q}$-Gorenstein.)

For these singularities we prove the following.

THEOREM 1.1. *Let $(X,0)$ be a normal surface singularity with a good $\mathbb{C}^*$-action whose link is a rational homology sphere. Then*

$$\mathbf{sw}^{\text{TCW}}_M(\sigma_{\text{can}})-\frac{K^2+\#\mathcal{V}}{8}=p_g. \tag{1}$$

In particular, if $(X,0)$ is Gorenstein and has a smoothing with Milnor fiber F, then its signature $\sigma(F)$ satisfies $-\mathbf{sw}^{\text{TCW}}_M(\sigma_{\text{can}})=\sigma(F)/8$.

The last statement follows from (1) and from the well-known formula $8p_g+\sigma(F)+K^2+\#\mathcal{V}=0$ (valid for smoothings of Gorenstein singularities), see for example [**22**]. The proof of (1) is based, in part, on Pinkham's formula [**36**] for p_g expressed in terms of the Seifert invariants of the link (cf. also Dolgachev's work about weighted homogeneous singularities; see for example [**6**]). On the other hand, in the proof we use the formulae for $K^2+\#\mathcal{V}$ and the Reidemeister–Turaev torsion determined in [**28**], and a formula for the Casson–Walker invariant proved in [**20**]. The Fourier transform of Reidemeister–Turaev torsion is very closely related to the (equivariant) Poincaré series associated with the graded coordinate ring of the universal abelian cover of $(X, 0)$. In the proof we also borrow some techniques used by Neumann in his investigation of this Poincaré series [**29**]. Nevertheless, our proof and results provide new interpretations of several of the coefficients of the Poincaré series (see Section 4).

The theorem has the following corollary, which can also be applied for singularities without $\mathbb{C}^*$-action (but with some other type of additional rigidity properties).

COROLLARY 1.2. *Assume that the link of a normal surface singularity* $(X, 0)$ *is a rational homology sphere Seifert* 3*-manifold. If* $(X, 0)$ *is rational, or minimally elliptic, or Gorenstein elliptic, then the identity* (1) *holds.*

Indeed, in the case of these singularities, all the numerical invariants involved in the conjecture are characterized by the link. Moreover, each family (with fixed topological type) contains a special representative which admits a good $\mathbb{C}^*$-action. In fact, the above corollary can automatically be extended to any family of singularities with these two properties.

The paper is organized as follows. In Section 2 we review the needed definitions and results. For a more complete picture and list of references the reader is invited to consult [**28**]. The first theorem of Section 3 determines the Reidemeister–Turaev torsion of M (associated with an arbitrary spin^c structure) in terms of the Seifert invariants of M. This result, via equations (2) and (6), provides the complete Seiberg–Witten invariant of a Seifert manifold with $e < 0$. This is really remarkable (even independent of Theorem 1.1), since, in general, Seiberg–Witten computations are difficult (see for example [**26**], [**35**] or [**28**] for the Seifert case) and only sporadic cases were completely clarified. The next result, Theorem 3.2, connects four topological invariants of the link: the Reidemeister–Turaev torsion, the Casson–Walker invariant, the Dolgachev–Pinkham invariant DP_M (which is the topological candidate for p_g), and finally $K^2 + \#\mathcal{V}$ (which can be identified with the Gompf invariant, cf. Subsection 2.4). This result implies Theorem 1.1 via Pinkham's result [**36**] (cf. equation 2).

Finally, in Section 4, we analyze more closely the relationship with some of the coefficients of the Laurent expansions appearing in this paper (for example, of the Poincaré series), provided that $(X, 0)$ is a complete intersection or hypersurface singularity. In the second case, using some results of Saito [**38**] and Ebeling [**9**], we also give a topological/geometrical interpretation of the (generalized) Batyrev *stringy Euler number* [**4**] (as generalized by Veys in [**41**]). (This suggests a possible connection with Arnold's strange duality and with the mirror symmetry of $K3$ surfaces, see the comments in [**9**] and [**7**].)

2. *Preliminaries*

2.1. *The canonical* spin^c *structure of* M

Let $(X, 0)$ be a normal surface singularity. Its link M is a compact oriented 3-manifold. In this paper we assume that M is a rational homology sphere, and we write $H := H_1(M, \mathbb{Z})$.

The almost complex structure on $X\setminus\{0\}$ determines a spin^c structure on $X\setminus\{0\}$, whose restriction $\sigma_{\rm can}\in \mathrm{Spin}^c(M)$ to M depends only on the oriented C^∞ type of M [**28**].

2.2. *The Seiberg–Witten invariants of M*

To describe the Seiberg–Witten invariants one has to consider additional geometric data belonging to the space of parameters

$$\mathcal{P}=\{u=(g,\eta);\ g=\text{Riemann metric},\ \eta=\text{closed two-form}\}.$$

Then for each spin^c structure σ on M one defines the (σ,g,η)-*Seiberg–Witten monopoles*. For a generic parameter u, the Seiberg–Witten invariant $\mathbf{sw}_M(\sigma,u)$ is the signed monopole count. This integer depends on the choice of the parameter u and thus it is not a topological invariant. To obtain an invariant of M, one needs to alter this monopole count by the *Kreck–Stolz* invariant $\mathrm{KS}_M(\sigma,u)$, cf. [**21**] (or see [**17**] for the original 'spin version'). Then, by [**5**, **21**, **23**], the rational number

$$\tfrac{1}{8}\mathrm{KS}_M(\sigma,u)+\mathbf{sw}_M(\sigma,u)$$

is independent of u and thus it is a topological invariant of the pair (M,σ). We denote this *modified Seiberg–Witten invariant* by $\mathbf{sw}^0_M(\sigma)$.

At present there is intense activity to replace the analytic definition of $\mathbf{sw}^0_M$ by a topological one. In this paper we will consider the candidate $\mathbf{sw}^{\rm TCW}_M$ which involves the sign-refined Reidemeister–Turaev torsion of M.

2.3. *The Reidemeister–Turaev torsion and the Casson–Walker invariant*

For any spin^c structure σ on M, we denote by

$$\mathcal{T}_{M,\sigma}=\sum_{h\in H}\mathcal{T}_{M,\sigma}(h)\,h\in\mathbb{Q}[H]$$

the sign-refined *Reidemeister–Turaev torsion* associated with σ (see [**40**]). We think of $\mathcal{T}_{M,\sigma}$ as a function $H\longrightarrow\mathbb{Q}$ given by $h\longmapsto\mathcal{T}_{M,\sigma}(h)$. Let $\lambda(M)$ be the Casson–Walker invariant of M normalized as in [**20**, § 4.7]. Then one defines (see for example [**35**])

$$\mathbf{sw}^{\rm TCW}_M(\sigma)=-\frac{1}{|H|}\lambda(M)+\mathcal{T}_{M,\sigma}(1). \tag{2}$$

Below we will present a formula for $\mathcal{T}_{M,\sigma}$ in terms of the Fourier transform. For this, consider the Pontryagin dual $\hat{H}:=\mathrm{Hom}(H,U(1))$ of H. Then a function $f:H\longrightarrow\mathbb{C}$ and its Fourier transform $\hat{f}:\hat{H}\longrightarrow\mathbb{C}$ satisfy

$$\hat{f}(\chi)=\sum_{h\in H}f(h)\bar{\chi}(h),\quad f(h)=\frac{1}{|H|}\sum_{\chi\in\hat{H}}\hat{f}(\chi)\chi(h).$$

It is known that $\hat{\mathcal{T}}_{M,\sigma}(1)=0$.

2.4. *M as a plumbed manifold*

Let $\pi:\tilde{X}\longrightarrow X$ be a resolution of the singular point $0\in X$ such that the exceptional divisor $E:=\pi^{-1}(0)$ is a normal crossing divisor with irreducible components $\{E_v\}_{v\in\mathcal{V}}$. Let $\Gamma(\pi)$ be the dual resolution graph associated with π

decorated with the self-intersection numbers $\{E_v \cdot E_v\}_v$. $\Gamma(\pi)$ can be identified with a plumbing graph, and M with a plumbed 3-manifold constructed from $\Gamma(\pi)$. Since M is a rational homology sphere, $\Gamma(\pi)$ is a tree and any E_v is rational. Denote by δ_v the degree of any vertex v (that is $\#\{w : E_w \cdot E_v = 1\}$).

Let D_v be a small transversal disc to E_v in $\tilde{X}$. In fact, ∂D_v can be considered as the generic fiber of the S^1-bundle over E_v used in the plumbing construction of M. Consider the elements $g_v := [\partial D_v]$ $(v \in \mathcal{V})$ in H. It is not difficult to verify that they generate H.

Next, we define the *canonical cycle* Z_K of $(X, 0)$ associated with the resolution π. This is a *rational* cycle $Z_K = \sum_{v\in\mathcal{V}} r_v E_v$, $r_v \in \mathbb{Q}$, supported by the exceptional divisor E, and defined by (the adjunction formula)

$$Z_K \cdot E_v = E_v \cdot E_v + 2 \qquad \text{for any } v \in \mathcal{V}.$$

Since the matrix $\{E_v \cdot E_w\}_{v,w}$ is non-degenerate, this system has a unique (rational) solution. We write K^2 for $Z_K \cdot Z_K$, and $\#\mathcal{V}$ for the number of irreducible components of E. Then $K^2 + \#\mathcal{V}$ does not depend on the choice of the resolution π; it is an invariant of M.

[One can define on M a canonical *contact structure* ξ_{can} induced by the natural almost complex structure on $TM \oplus \mathbb{R}_M$ with $c_1(\xi_{\mathrm{can}})$ torsion (see for example [**13**, p. 420]). On the other hand, in [**12**], Gompf associates with such a contact structure ξ an invariant $\theta_M(\xi)$. It turns out that $\theta_M(\xi_{\mathrm{can}}) = K^2 + \#\mathcal{V} - 2$ (see [**28**, 4.8]).]

If $(X, 0)$ is a normal surface singularity with a good $\mathbb{C}^*$-action, then M is a Seifert 3-manifold, and the minimal resolution graph is star-shaped. Hence, it is convenient to express the topological invariants of M in terms of the Seifert invariants. In the following subsections we recall briefly some definitions, notations and needed properties.

2.5. *The Seifert invariants* [**16**, **29**, **31**]

Consider a Seifert fibration $\pi : M \longrightarrow \Sigma$. In our situation, since M is a rational homology sphere, the base space Σ has genus zero ($\Sigma \approx S^2$).

Consider a set of points $\{x_i\}_{i=1}^{\nu}$ in such a way that the set of fibers $\{\pi^{-1}(x_i)\}_i$ contains the set of singular fibers. Set $O_i := \pi^{-1}(x_i)$. Let D_i be a small disc in X containing x_i, $\Sigma' := \Sigma \setminus \bigcup_i D_i$ and $M' := \pi^{-1}(\Sigma')$. Now, $\pi : M' \longrightarrow \Sigma'$ admits sections; let $s : \Sigma' \longrightarrow M'$ be one of them. Let $Q_i := s(\partial D_i)$ and let H_i be a circle fiber in $\pi^{-1}(\partial D_i)$. Then in $H_1(\pi^{-1}(D_i), \mathbb{Z})$ one has $H_i \sim \alpha_i O_i$ and $Q_i \sim -\beta_i O_i$ for some integers $\alpha_i > 0$ and β_i with $(\alpha_i, \beta_i) = 1$. The set $\{(\alpha_i, \beta_i)\}_{i=1}^{\nu}$ constitutes the set of (*un-normalized*) *Seifert invariants*. The number

$$e := -\sum \beta_i/\alpha_i$$

is called the *orbifold Euler number* of M. M is a link of singularity if and only if $e < 0$. Replacing the section by another one, a different choice changes each β_i within its residue class modulo α_i in such a way that the sum $e = -\sum_i(\beta_i/\alpha_i)$ is constant.

The set of *normalized* Seifert invariants $\{(\alpha_i, \omega_i)\}_{i=1}^{\nu}$ are defined as follows. Write

$$e = b + \sum \omega_i/\alpha_i \tag{3}$$

for some integer b, and $0\leqslant\omega_i<\alpha_i$ with $\omega_i\equiv-\beta_i(\operatorname{mod}\alpha_i)$. Clearly, these properties define $\{\omega_i\}_i$ uniquely. Notice that $b\leqslant e<0$. In the sequel we assume that $\nu\geqslant 3$. (Recall that for cyclic quotient singularities (1) was verified in [**28**].)

For each i, consider the continued fraction $\alpha_i/\omega_i=b_{i1}-1/(b_{i2}-1/(\ldots-1/b_{i\nu_i})\ldots)$. Then (a possible) plumbing graph of M is a star-shaped graph with ν arms. The central vertex has decoration b and the arm corresponding to the index i has ν_i vertices decorated by $-b_{i1},\ldots,-b_{i\nu_i}$ (the vertex decorated by $-b_{i1}$ is connected by the central vertex).

We will distinguish those vertices $v\in\mathcal{V}$ of the graph which have $\delta_v\neq 2$. We will denote by $\bar v_0$ the central vertex, and by $\bar v_i$ the end-vertex of the ith arm for all $1\leqslant i\leqslant\nu$. Then $g_{\bar v_0}$ is exactly the class of the generic fiber. The group H has the presentation

$$H=\operatorname{ab}\left\langle g_{\bar v_0},g_{\bar v_1},\ldots,g_{\bar v_\nu}\;\middle|\; g_{\bar v_0}^{-b}=\prod_{i=1}^{\nu}g_{\bar v_i}^{\omega_i},\ g_{\bar v_0}=g_{\bar v_i}^{\alpha_i}\ \text{for all } i\right\rangle. \tag{4}$$

Let $\alpha:=\operatorname{lcm}(\alpha_1,\ldots,\alpha_\nu)$. The order of the group H and the order o of the subgroup $\langle g_{\bar v_0}\rangle$ can be determined by (cf. [**29**])

$$|H|=\alpha_1\ldots\alpha_\nu|e|,\quad o:=|\langle g_{\bar v_0}\rangle|=\alpha|e|. \tag{5}$$

2.6. *Invariants computed from the plumbing graph*

In the sequel we will also use Dedekind sums. They are defined as follows [**37**]. Let $\lfloor x\rfloor$ be the integer part, and $\{x\}:=x-\lfloor x\rfloor$ the fractional part of x. Then

$$\boldsymbol{s}(h,k)=\sum_{\mu=0}^{k-1}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

where $((x))$ denotes the Dedekind symbol

$$((x))=\begin{cases}\{x\}-1/2 & \text{if } x\in\mathbb{R}\setminus\mathbb{Z}\\ 0 & \text{if } x\in\mathbb{Z}.\end{cases}$$

Assume that M is a Seifert manifold with $e<0$ and genus zero. Then one has the following formulae for some of its invariants.

(1) The Casson–Walker invariant [**20**, (6.1.1)]:

$$-\frac{24}{|H|}\lambda(M)=\frac{1}{e}\left(2-\nu+\sum_{i=1}^{\nu}\frac{1}{\alpha_i^2}\right)+e+3+12\sum_{i=1}^{\nu}\boldsymbol{s}(\beta_i,\alpha_i). \tag{6}$$

(2) $K^2+\#\mathcal{V}$ [**28**, (5.4)]:

$$K^2+\#\mathcal{V}=\frac{1}{e}\left(2-\nu+\sum_{i=1}^{\nu}\frac{1}{\alpha_i}\right)^2+e+5+12\sum_{i=1}^{\nu}\boldsymbol{s}(\beta_i,\alpha_i). \tag{7}$$

(3) The coefficient r_0 of $E_{\bar v_0}$ in Z_K (see for example [**28**, 5.2 and 5.5]):

$$r_0=1+\frac{1}{e}\left(2-\nu+\sum_{i=1}^{\nu}\frac{1}{\alpha_i}\right). \tag{8}$$

$R:=r_0-1$ is called the 'exponent of $(X,0)$', and $-R$ the 'log discrepancy of $E_{\bar v_0}$'. Some authors (see for example [**29**]) prefer to use the notation $\chi_M:=2-\nu+\sum_{i=1}^{\nu}(1/\alpha_i)=eR$ as well.

(4) The Reidemeister–Turaev sign-refined torsion: For any $\chi \in \hat{H}$ (and $t \in \mathbb{C}$) set

$$\hat{P}_\chi(t) := \frac{\left(t^{\alpha}\chi(g_{\bar{v}_0}) - 1\right)^{\nu-2}}{\prod_{i=1}^{\nu}\left(t^{\alpha/\alpha_i}\,\chi(g_{\bar{v}_i}) - 1\right)}. \tag{9}$$

Recall that $\mathrm{Spin}^c(M)$ is an H-torsor. For an arbitrary $\sigma \in \mathrm{Spin}^c(M)$ take the unique $h_\sigma \in H$ so that $h_\sigma \cdot \sigma_{\mathrm{can}} = \sigma$. Then, by [**28**, 5.7, 5.8], one has

$$\hat{\mathcal{T}}_{M,\sigma}(\bar{\chi}) = \bar{\chi}(h_\sigma)\cdot \lim_{t\to 1}\hat{P}_\chi(t) \qquad \text{for any } \chi \in \hat{H}\setminus\{1\}. \tag{10}$$

(5) The geometric genus of $(X,0)$: Let M be a Seifert manifold with $e<0$ and Seifert invariants as above. Define the Dolgachev–Pinkham (topological) invariant of M by

$$\mathrm{DP}_M := \sum_{l\geqslant 0}\max\left(0,\ -1+lb-\sum_{i=1}^{\nu}\left\lfloor \frac{-l\omega_i}{\alpha_i}\right\rfloor\right). \tag{11}$$

Assume that $(X,0)$ is a normal surface singularity with a good $\mathbb{C}^*$-action (see for example [**36**]) such that its link M is a rational homology sphere. Then, by [**36**, (5.7)]

$$p_g(X,0) = \mathrm{DP}_M. \tag{12}$$

3. *The main results*

In [**28**] (cf. (9), (10)) we determined the Fourier transforms $\hat{\mathcal{T}}_{M,\sigma}(\chi)$ ($\chi \neq 1$) of the Reidemeister–Turaev torsion of M in terms of a regularization limit. The first theorem of this section provides $\mathcal{T}_{M,\sigma}$ (for any spin^c structure σ) in terms of the Seifert invariants of M.

THEOREM 3.1. *Assume that M is a Seifert 3-manifold with $e<0$ and genus zero. Fix an arbitrary* spin^c *structure $\sigma \in \mathrm{Spin}^c(M)$ characterized by $h_\sigma \cdot \sigma_{\mathrm{can}} = \sigma$. Write h_σ as $g_{\bar{v}_0}^{a_0} g_{\bar{v}_1}^{a_1}\dots g_{\bar{v}_\nu}^{a_\nu}$ for some integers $a_0, a_1, \dots, a_\nu$. Finally, define $\tilde{a} := \alpha\cdot(a_0 + \sum_i a_i/\alpha_i)$. Then*

$$\frac{1}{|H|}\sum_{\chi\in\hat{H}}\bar{\chi}(h_\sigma)\hat{P}_\chi(t) = \sum_{l\geqslant -\tilde{a}/o}\max\left(0,\ 1+a_0-lb+\sum_{i=1}^{\nu}\left\lfloor\frac{-l\omega_i+a_i}{\alpha_i}\right\rfloor\right)t^{ol+\tilde{a}}.$$

Therefore,

$$\mathcal{T}_{M,\sigma}(1) = \lim_{t\to 1}\left(\sum_{l\geqslant -\tilde{a}/o}\max\left(0, 1+a_0-lb+\sum_{i=1}^{\nu}\left\lfloor\frac{-l\omega_i+a_i}{\alpha_i}\right\rfloor\right)t^{ol+\tilde{a}} - \frac{1}{|H|}\cdot\hat{P}_1(t)\right).$$

In particular, for $\sigma = \sigma_{\mathrm{can}}$ or for $h_\sigma = 1$, one can take $a_0 = a_1 = \dots = a_\nu = \tilde{a} = 0$; hence

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1) = \lim_{t\to 1}\left(\sum_{l\geqslant 0}\max\left(0,\ 1-lb+\sum_{i=1}^{\nu}\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right)t^{ol} - \frac{1}{|H|}\cdot\hat{P}_1(t)\right).$$

Proof. Notice that there is a ‘mysterious’ similarity between our formula (9), (10) for the Fourier transform of the Reidemeister–Turaev torsion, and the formula [**29**, 4.2] of Neumann of the Poincaré series of the graded affine ring associated with the universal abelian cover of $(X,0)$. The idea of the next proof is based on Neumann's computation of this graded ring in [**29**, p. 241].

Using the identity $g_{\bar v_i}^{\alpha_i}=g_{\bar v_0}$ in H (cf. (4)), first write $\bar\chi(h_\sigma)\hat P_\chi(t)$ as

$$\begin{aligned}
&\bar\chi(h_\sigma)\cdot\big(1-t^{\alpha}\chi(g_{\bar v_0})\big)^{-2}\prod_i\frac{1-\big(t^{\alpha/\alpha_i}\chi(g_{\bar v_i})\big)^{\alpha_i}}{1-t^{\alpha/\alpha_i}\chi(g_{\bar v_i})}\\
&=\bar\chi(h_\sigma)\left(\sum_{s_0=0}^{\infty}(1+s_0)\chi(g_{\bar v_0})^{s_0}\,t^{\alpha s_0}\right)\cdot\prod_i\sum_{s_i=0}^{\alpha_i-1}t^{s_i\alpha/\alpha_i}\big(\chi(g_{\bar v_i})\big)^{s_i}\\
&=\sum(1+s_0)\,t^{\alpha s_0+\sum_i\alpha s_i/\alpha_i}\;\chi\big(g_{\bar v_0}^{s_0-a_0}g_{\bar v_1}^{s_1-a_1}\dots g_{\bar v_\nu}^{s_\nu-a_\nu}\big),
\end{aligned}$$

where the (unmarked) sum is over $s_0\geqslant 0$ and $0\leqslant s_i<\alpha_i$ for each i. However $\sum_{\chi\in\hat H}\chi(h)$ is non-zero only if $h=1$, and in that case it is $|H|$. Using the group structure (4) one finds that all the relations in H have the form

$$g_{\bar v_0}^{l_1+\ldots+l_\nu-lb}\prod_i g_{\bar v_i}^{-\omega_i l-\alpha_i l_i}=1,$$

where $l_1,\dots,l_\nu$ and l are integers. Therefore, $g_{\bar v_0}^{s_0-a_0}g_{\bar v_1}^{s_1-a_1}\dots g_{\bar v_\nu}^{s_\nu-a_\nu}=1$ if and only if $s_0=a_0+l_1+\ldots+l_\nu-lb$ and $s_i=a_i-\omega_i l-\alpha_i l_i$ $(1\leqslant i\leqslant\nu)$ for some integers $l_1,\dots,l_\nu,l$. Since $0\leqslant s_i<\alpha_i$ one obtains

$$l_i=\left\lfloor\frac{-l\omega_i+a_i}{\alpha_i}\right\rfloor.$$

In particular,

$$1+s_0=1+a_0-lb+\sum_i\left\lfloor\frac{-l\omega_i+a_i}{\alpha_i}\right\rfloor,$$

and only those integers l are allowed for which this number $1+s_0$ is $\geqslant 1$. It is easy to see that this cannot happen for $l<-\tilde a/o$. Indeed, for such an l (cf.(5)),

$$a_0-lb+\sum_i\left\lfloor\frac{-l\omega_i+a_i}{\alpha_i}\right\rfloor\leqslant a_0-lb+\sum_i\frac{-l\omega_i+a_i}{\alpha_i}=\frac{\tilde a+lo}{\alpha}<0.$$

The exponent $\alpha(s_0+\sum_i s_i/\alpha_i)$ of t is $-l\alpha e+\tilde a=lo+\tilde a$ again by (5). Finally, recall that $\hat{\mathcal T}_{M,\sigma}(1)=0$ (cf. Subsection (2.3)); hence $\mathcal T_{M,\sigma}(1)$ follows from the Fourier inversion (Subsection 2.3 and (10)). □

Theorem 1.1 is a consequence of the following key identity.

THEOREM 3.2. *Let M be a Seifert 3-manifold with $e<0$ of genus zero (that is, M is a rational homology 3-sphere). Then the invariants $\mathcal T_{M,\sigma_{\rm can}}(1)$, $\lambda(M)$, $K^2+\#\mathcal V$ and DP_M are connected by the identity*

$$\mathcal T_{M,\sigma_{\rm can}}(1)-\frac{\lambda(M)}{|H|}=\frac{K^2+\#\mathcal V}{8}+\mathrm{DP}_M.$$

The proof is carried out in several steps.

COROLLARY 3.3. *Theorem* 3.1 *implies that*

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)-\mathrm{DP}_M=\lim_{t\to 1}\left(\sum_{l\geqslant 0}\left(1-lb+\sum_{i=1}^{\nu}\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right)t^{ol}-\frac{1}{|H|}\cdot\hat{P}_1(t)\right).$$

Proof. Use (11), Theorem 3.1 and the identity $\max(0,x)-\max(0,-x)=x$. □

On the right-hand side we have a difference of two series, both having poles of order two at $t=1$. The following results provide their Laurent series at $t=1$. In fact, we prefer to expand the series in terms of the powers of t^o-1 (instead of $t-1$).

PROPOSITION 3.4. *Recall the definition of* χ_M *after* (8). *Then*

$$\sum_{l\geqslant 0}\left(1-lb+\sum_{i=1}^{\nu}\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right)t^{ol}=\frac{-e}{(t^o-1)^2}+\frac{-e-\chi_M/2}{t^o-1}+\frac{2-\chi_M}{4}$$
$$+\sum_{i=1}^{\nu}\boldsymbol{s}(\beta_i,\alpha_i)+R(t),$$

with $\lim_{t\to 1}R(t)=0$.

Proof. The first step is as in [**29**, p. 241]. The left-hand side of the equation, via (3) becomes

$$\sum_{l\geqslant 0}\left(-le+\frac{\chi_M}{2}\right)t^{ol}+\sum_{i=1}^{\nu}\sum_{l\geqslant 0}\left(-\left\{\frac{-l\omega_i}{\alpha_i}\right\}+\frac{\alpha_i-1}{2\alpha_i}\right)t^{ol}.$$

Evidently

$$\sum_{l\geqslant 0}\left(-le+\frac{\chi_M}{2}\right)t^{ol}=\frac{-et^o}{(1-t^o)^2}+\frac{\chi_M/2}{1-t^o},$$

which gives the non-holomorphic part. The second contribution is a sum over $1\leqslant i\leqslant\nu$.

For each fixed i, write $l=\alpha_i m+q$ with $m\geqslant 0$ and $0\leqslant q<\alpha_i$. Using the notation $\sum_q:=\sum_{q=0}^{\alpha_i-1}$ and $\sum_q':=\sum_{q=1}^{\alpha_i-1}$, the ith summand is

$$\sum_q\left(-\left\{\frac{-q\omega_i}{\alpha_i}\right\}+\frac{\alpha_i-1}{2\alpha_i}\right)\sum_{m\geqslant 0}t^{o\alpha_i m+oq}=\frac{\sum_q\left(-\left\{\frac{-q\omega_i}{\alpha_i}\right\}+\frac{\alpha_i-1}{2\alpha_i}\right)t^{oq}}{1-t^{o\alpha_i}}.$$

Separating the two cases $q=0$ and $q>0$, and using the definition of the Dedekind symbol and the identity $\{-x\}=1-\{x\}$ for $x\notin\mathbb{Z}$, this is transformed into

$$A(t):=\frac{\frac{\alpha_i-1}{2\alpha_i}+\sum_q'\left(\left(\frac{q\omega_i}{\alpha_i}\right)\right)t^{oq}-\frac{1}{2\alpha_i}\left(t^o+t^{2o}+\ldots+t^{(\alpha_i-1)o}\right)}{1-t^{o\alpha_i}}.$$

By L'Hospital's theorem (and some simplifications),

$$\lim_{t\to 1}A(t)=-\sum_q'\left(\left(\frac{q\omega_i}{\alpha_i}\right)\right)\frac{q}{\alpha_i}+\frac{\alpha_i-1}{4\alpha_i}.$$

Since $\sum_q'((q\omega_i/\alpha_i))=0$ and $\omega_i\equiv-\beta_i \pmod{\alpha_i}$, the result follows from the definition of the Dedekind symbol and the Dedekind sums. $\square$

REMARK 3.5 (cf. also [**29**, p. 242]). In fact,

$$\sum_{l\geqslant 0}\left(1-lb+\sum_{i=1}^{\nu}\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right)t^{ol}=\frac{-e}{(t^o-1)^2}+\frac{-e-\chi_M/2}{t^o-1}$$
$$+\sum_{i=1}^{\nu}\frac{1}{\alpha_i}\sum_{\xi\in\mathbb{Z}_{\alpha_i}}{}'\frac{1}{(1-\xi)(1-\xi^{\omega_i}t^o)},$$

where the last sum is over $\xi^{\alpha_i}=1$, $\xi\neq 1$. Since we do not need this statement now, we will skip its proof. The interested reader can prove it easily using the expression $A(t)$ above and property (16c) of the Dedekind symbol from [**37**, p. 14].

PROPOSITION 3.6. *$\hat{P}_1(t)/|H|$ has the Laurent expansion*

$$\frac{\hat{P}_1(t)}{|H|}=\frac{-e}{(t^o-1)^2}+\frac{-e-\chi_M/2}{t^o-1}+E+Q(t),$$

where $\lim_{t\to 1}Q(t)=0$ *and*

$$E:=-\frac{(e+1)(e+5)}{12e}+\frac{1}{4}\sum_i\left(1-\frac{1}{\alpha_i}\right)+\frac{1}{12e}\sum_i\left(1-\frac{1}{\alpha_i}\right)\left(4+\frac{1}{\alpha_i}\right)$$
$$-\frac{1}{4e}\sum_{i<j}\left(1-\frac{1}{\alpha_i}\right)\left(1-\frac{1}{\alpha_j}\right).$$

Proof. First notice that one has the Taylor expansion

$$\frac{t^\gamma-1}{t^\tau-1}=\frac{\gamma}{\tau}+\frac{\gamma}{2o\tau}(\gamma-\tau)\cdot(t^o-1)+\frac{\gamma}{o\tau}(\gamma-\tau)\left(\frac{2\gamma-\tau}{12o}-\frac{1}{4}\right)\cdot(t^o-1)^2+\dots.$$

Now, use this formula $\nu+2$ times in the expression

$$\hat{P}_1(t)=\frac{1}{(t^o-1)^2}\cdot\left(\frac{t^o-1}{t^\alpha-1}\right)^2\cdot\prod_i\frac{t^\alpha-1}{t^{\alpha/\alpha_i}-1}.$$

A long (but elementary) computation, together with (5), gives the result. $\square$

Proof of Theorem 3.2. Apply (6) and (7), respectively Corollary 3.3, Proposition 3.4 and Proposition 3.6. $\square$

COROLLARY 3.7. *Assume that $(X,0)$ is Gorenstein and admits a smoothing with Milnor fiber F. Then the topological Euler characteristic $\chi(F)$ of F satisfies*

$$2\cdot\chi(F)=24\cdot\mathfrak{T}_{M,\sigma_{\mathrm{can}}}(1)+\frac{1}{e}\left(2-\nu+\sum_i\frac{1}{\alpha_i^2}\right)-\frac{1}{e}\left(2-\nu+\sum_i\frac{1}{\alpha_i}\right)^2.$$

(*Notice that the first Betti number of F is zero because of* [**15**]. *Hence* $\chi(F)=1+\mu$, *where μ is the Milnor number of the smoothing.*)

Proof. By the generalization of Steenbrink [**39**] of Laufer's formula [**19**] one has $\mu=12p_g+K^2+\#\mathcal{V}$. Then use Theorem 1.1 (or Theorem 3.2) and (6), (7). $\square$

Assume that in Corollary 3.7 one has $|H|=1$. Then in the identity $\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=0$, and $-1/e=\alpha=\prod_i \alpha_i$; cf. (5). Moreover, in this case, by [**29**], $(X,0)$ is a Brieskorn–Hamm complete intersection $X(\alpha_1,\dots,\alpha_\nu)$. For such singularities the Milnor number is computed in [**14**, (3.10.b)]. This formula agrees with our identity (simplified for $|H|=1$).

4. *Examples*

4.1. *The Poincaré series*

Let $(X,0)$ be a normal singularity with a good $\mathbb{C}^*$-action and affine graded coordinate ring $A=\bigoplus_k A_k$. Then its Poincaré series is defined by $p_{(X,0)}(t)=\sum_k \dim(A_k)t^k$. The point is that the last expression from Theorem 3.1 involves exactly $p_{(X,0)}(t^o)$, provided that the genus of $(X,0)$ is zero. More precisely (cf. [**29**, p. 241]),

$$p_{(X,0)}(t)=\sum_{l\geqslant 0}\max\left(0,1-lb+\sum_{i=1}^{\nu}\left\lfloor\frac{-l\omega_i}{\alpha_i}\right\rfloor\right)\,t^l.$$

Moreover, if $(X_{ab},0)$ denotes the universal abelian cover of $(X,0)$, then $p_{(X_{ab},0)}(t)=\hat{P}_1(t)$ (cf. [**29**, p. 240]). Therefore, Theorem 3.1 reads as

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=\lim_{t\to 1}\big(p_{(X,0)}(t^o)-p_{(X_{\mathrm{ab}},0)}(t)/|H|\big). \tag{13}$$

Notice that for many special families, the Poincaré series $p_{(X,0)}(t)$ is computed very explicitly, see for example [**42**]. For $p_{(X_{ab},0)}(t)=\hat{P}_1(t)$ one can use the expression (9) which provides it in terms of the Seifert invariants of M.

Notice that both Poincaré series are rational functions with pole of order 2 at $t=1$ (the Laurent expansion of $\hat{P}_1(t)$ is given in Proposition 3.6). In the following examples we will emphasize the first three terms of the corresponding Laurent expansions.

4.2. *Complete intersections*

The above formula provides $p_{(X,0)}(t)$ in terms of the Seifert invariants. Nevertheless, if $(X,0)\subset(\mathbb{C}^n,0)$ is a complete intersection with weights $q_1,\dots,q_n$ (where $\gcd(q_1,\dots,q_n)=1$) and degrees $d_1,\dots,d_{n-2}$, then one also has (cf. for example with [**42**])

$$p_{(X,0)}(t)=\frac{\prod_{i=1}^{n-2}(1-t^{d_i})}{\prod_{j=1}^{n}(1-t^{q_j})}.$$

Similarly as in Proposition 3.6, one can determine its Laurent expansion at $t=1$,

$$p_{(X,0)}(t)=\frac{\prod_i d_i}{\prod_j q_j}\left[\frac{1}{(t-1)^2}+\frac{\sum_i d_i-\sum_j q_j+2}{2(t-1)}+F+U(t)\right],$$

where $\lim_{t\to 1}U(t)=0$ and

$$\begin{aligned}F:=&\tfrac{1}{6}\sum_i d_i^2+\tfrac14\sum_{i<j}d_id_j+\tfrac14\sum_i d_i+\tfrac{1}{12}\sum_j q_j^2+\tfrac14\sum_{i<j}q_iq_j-\tfrac14\sum_j q_j\\ &-\tfrac14\sum_{i,j}d_iq_j+\tfrac{1}{12}.\end{aligned}$$

By (13), the coefficients of $(t^o-1)^k$ $(k=-2,-1)$ of $p_{(X,0)}(t^o)$ and $\hat{P}_1(t)/|H|$ should agree. By this (and Proposition 3.6), we recover two identities well known by specialists (see [**8**], or [**42**] for the hypersurface case):

$$-e=\prod_i d_i/\prod_j q_j \quad \text{and} \quad R=\sum_i d_i-\sum_j q_j,$$

where R is the 'exponent of $(X,0)$', see (8). On the other hand, the interpretation of the coefficients of $(t^o-1)^0$ as the Reidemeister–Turaev torsion is the novelty of the present paper:

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=-e\cdot F-E,$$

where F is given above and E in Proposition 3.6.

4.3. *Hypersurface singularities*

Assume that $(X,0)\subset(\mathbb{C}^3,0)$ is a hypersurface weighted homogeneous singularity whose link is a rational homology sphere. Then (13) can be rewritten in a rather surprising form. Denote by $\Delta(t)$ the characteristic polynomial of the algebraic monodromy operator. By a recent result of Ebeling [**9**], $\Delta(t)$ is in a very subtle relationship with $p_{(X,0)}(t)$. In order to explain this, we need to recall Saito's duality [**38**].

Fix an integer $h>0$ and assume that $\phi(t)$ is a rational function of the form

$$\phi(t)=\prod_{m|h}(1-t^m)^{\chi_m}. \tag{14}$$

Then Saito in [**38**] defined a dual rational function (with respect to the integer h) by

$$\phi^*(t)=\prod_{k|h}(1-t^k)^{-\chi_{h/k}}.$$

In this paper we wish to eliminate the dependency of the duality on the integer h, and we will use the following principle. If $\phi(t)$ is a rational function of the form $\prod_m(1-t^m)^{\chi_m}$, then we take $h=h(\phi):=\mathrm{lcm}\{m:\chi_m\neq 0\}$, and we define $\phi^*(t)$ using this $h=h(\phi)$. It is not difficult to prove that if $\gcd\{m:\chi_m\neq 0\}=1$ then $h(\phi^*)=h(\phi)$ and $\phi^{**}(t)=\phi(t)$.

Let M be the link of $(X,0)$ with Seifert invariants as above. Following Ebeling, we write

$$\psi(t):=(1-t)^{2-\nu}\prod_{i=1}^{\nu}(1-t^{\alpha_i}),$$

and $\phi(t):=p_{(X,0)}(t)\cdot\psi(t)$. Notice that $\Delta(t)$ can always be written in the form (14) with $\gcd\{m:\chi_m\neq 0\}=1$ (in fact, $\chi_1\neq 0$, cf. [**1**]). Then by [**9**], ϕ also has the form (14), and

$$\Delta(t)=\phi^*(t) \quad \text{or} \quad \Delta^*(t)=\phi(t).$$

[In [**9**] it is not explicitly stated that $h=h(\phi)$, but it can be verified using [**25**].]

This has the following connection with our result. Notice that $\psi^*(t)=\hat{P}_1(t)$ and by (13)

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=\lim_{t\to 1}\left[\phi(t^o)\psi^{-1}(t^o)-\frac{1}{|H|}\psi^*(t)\right]=\lim_{t\to 1}\left[\phi(t^o)-\frac{1}{|H|}\psi^*(t)\psi(t^o)\right]\cdot\psi^{-1}(t^o).$$

Since

$$\lim_{t\to 1}\psi^{-1}(t^o)(t^o-1)^2=\lim_{t\to 1}\frac{(t^o-1)^\nu}{\prod_i(t^{\alpha_i o}-1)}=\prod_i\frac{1}{\alpha_i}=\frac{|e|}{|H|}$$

via Ebeling's result and $\psi^*(t)=\hat{P}_1(t)$, one obtains

$$\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)=\frac{|e|}{|H|}\cdot\lim_{t\to 1}\frac{\Delta^*(t^o)-\hat{P}_1(t)\hat{P}_1^*(t^o)/|H|}{(t^o-1)^2}. \tag{15}$$

In the following expressions it is convenient to introduce the notation

$$R_p:=(-e)^{\min\{p,0\}}\cdot\Bigl(2-\nu+\sum_i\alpha_i^p\Bigr),\qquad p\in\mathbb{Z}.$$

For example, $R_{-1}=-R$ (cf. (8)). Similarly as in Proposition 3.6, one can write the Taylor expansion at $t=1$ (with respect to t^o-1):

$$\begin{aligned}\frac{1}{|H|}\hat{P}_1(t)\hat{P}_1^*(t^o)&=\frac{1}{|H|}\Bigl(\frac{1-t^\alpha}{1-t^o}\Bigr)^{\nu-2}\cdot\prod_i\frac{1-t^{o\alpha_i}}{1-t^{\alpha/\alpha_i}}\\&=|H|\Bigl(1+X_1(t^o-1)+\frac{2X_2+X_1^2}{2}(t^o-1)^2+\dots\Bigr)\end{aligned}$$

where

$$X_1:=\tfrac{1}{2}(R_1-R_{-1}),\quad X_2:=\tfrac{1}{24}(R_2-R_{-2})-\tfrac{1}{4}(R_1-R_{-1}).$$

Therefore, by (15),

$$\frac{\Delta^*(t)}{|H|}=1+X_1(t-1)+\Bigl(\frac{2X_2+X_1^2}{2}-\frac{1}{e}\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)\Bigr)(t-1)^2+\dots$$

In this expansion, the first term, namely $\Delta^*(1)=|H|$, is not very deep; it can also be deduced from the definition of Saito's duality and from $\Delta(1)=|H|$. However, the second term has an interesting interpretation. If we examine our list in Subsection 2.6, we realize that the numerical expression R_1 is not involved in it, but it is involved in the stringy Euler number computation of Veys [**41**, (6.4)]. More precisely, let e_{st} denote Batyrev's stringy Euler characteristic of $(X,0)$ (cf. [**4**]) as generalized by Veys in [**41**]. It can be defined by the resolution of $(X,0)$, and a possible geometrical/topological interpretation was sought. For our germ $(X,0)$ (provided that it is not strictly log canonical), [**41**, (6.4.ii)] reads as $e_{\mathrm{st}}=-R_1/R$. Hence, our result provides

$$\frac{d\Delta^*}{dt}(1)=|H|\cdot R\cdot(1-e_{\mathrm{st}})/2.$$

Finally, the next Taylor coefficient of $\Delta^*(t)$ involves $\mathcal{T}_{M,\sigma_{\mathrm{can}}}(1)$ (and also R_2, for which we know no other geometrical interpretation).

We end with the following remark. It is interesting to compare the Taylor expansions

$$\frac{\Delta^*(t)}{|H|}=1+\frac{R}{2}(1-e_{\mathrm{st}})(t-1)+\dots\quad\text{and}\quad\frac{\Delta(t)}{|H|}=1+\frac{\mu}{2}(t-1)+\dots.$$

Here μ is the Milnor number of $(X,0)$ (and the second expansion follows easily from [**1**]). This shows that the Milnor number, respectively $R(1-e_{\mathrm{st}})$, correspond to each other via some duality (which extends Arnold's strange duality).

For another application of [**9**] (in the spirit of the present paper), see [**24**].

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