AN INVITATION TO TAME GEOMETRY

LIVIU I. NICOLAESCU

CONTENTS

1. GOALS

- Describe large categories of nice spaces and maps.
- Describe some of the nice properties of these nice spaces and maps.
- Describe nice applications of these nice spaces and maps.

2. "REASONABLE" CATEGORIES OF SPACES

The spaces belonging to reasonable category $\mathcal{S}$ should be subsets of some Euclidean space so that

$$\mathcal{S} = \bigcup_{n\geq 1} \mathcal{S}^n, \;\; \mathcal{S}^n = \text{the collection of spaces in } \mathcal{S} \text{ which are subsets of } \mathbb{R}^n.$$

Via the inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ we can regard $\mathcal{S}^n$ as a subcollection of $\mathcal{S}^{n+1}$. A reasonable category should satisfy the following requirements.

$\boldsymbol{E}_1$. The collection $\mathcal{S}^n$ contains all the real algebraic subsets of $\mathbb{R}^n$, i.e., the subsets described by finitely many polynomial equations.
$\boldsymbol{E}_2$. The collection $\mathcal{S}^n$ contains all the closed affine halfspaces.
$\boldsymbol{P}_1$. The collection $\mathcal{S}^n$ is closed under all the boolean operatioons $\cup, \cap, \setminus$, i.e.,

$$A, B \in \mathcal{S}^n \Longrightarrow A\cup B, A\cap B, A\setminus B \in \mathcal{S}^n.$$

$\boldsymbol{P}_2$. If $A \in \mathcal{S}^m$ and $B \in \mathcal{S}^n$ then $A\times B \in \mathcal{S}^{m+n}$.
$\boldsymbol{P}_3$. If $A \in \mathcal{S}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map, then $T(A) \in \mathcal{S}^n$.
$\boldsymbol{M}$. If $A, B \in \mathcal{S}$ then an $\mathcal{S}$-morphism $A \to B$ is a map $f : A \to B$ such that its graph $\Gamma_f \subset A\times B$ belongs to $\mathcal{S}$.

Date: April 1, 2008. Notes for a colloquium talk at the University of New Orleans.

We will refer to the sets in $\mathcal{S}$ as $\mathcal{S}$-*definable* or *constructible sets*, and to the morphisms as *definable* or *constructible maps*.

3. Connection with human language

The boolean operations $\cup, \cap, \setminus$ correspond to the logical operators AND, OR, NOT = $\wedge, \vee, \neg$. The projection $\pi : A\times B \to B$ corresponds to the existential quantifier $\exists$. For example, if $Z\subset A\times B$, then $\pi(Z)$ can be described as

$$Z = \bigl\{ a\in A;\;\; \exists b\in B;\;\; (a,b)\in Z\,\bigr\}.$$

Note that the universal quantifier can pe expressed as a composition $\neg\exists\neg$. We obtain the following *metaprinciple*

If $\mathcal{S}$ is a reasonable category and a set A is defined by a statement involving only the basic logic operators and $\mathcal{S}$-definable sets, then A is also $\mathcal{S}$ definable.

Example 3.1. Suppose that $\mathcal{S}$ is a reasonable category, $f : A\to B$ is $\mathcal{S}$-definable, and $S\subset B$ is also definable. Then

$$f^{-1}(S) := \bigl\{ a\in A;\;\; \exists s\in S;\;\; (a,s)\in\Gamma_f\,\bigr\}$$

is definable. Note that if $a\in\mathbb{R}^n$, and ε_0 then the map

$$F:\mathbb{R}^n\to\mathbb{R},\;\; F(x) = |x-a|^2-\varepsilon^2$$

is definable since it is a polynomial. The set $S=(0,\infty)$ is definable because it is the complement of a closed halfspace and thus $F^{-1}(S)$ is definable. Note that this set is precisely the open Euclidean ball $B_a(\varepsilon)$ of center a and radius ε. □

Example 3.2. Suppose $A\in\mathcal{S}^m$, $B\in\mathcal{S}^n$. Consider the set

$$A_0 := \bigl\{ a\in A;\;\; (a,b)\in A\times B,\;\; \forall b\in B\,\bigr\}.$$

Then

$$A\setminus A_0 = \bigl\{\, a\in A;\;\; \exists b\in B;\;\; (a,b)\not\in A\times B\,\bigr\} = \pi\bigl(\,\mathbb{R}^{m+n}\setminus(A\times B)\,\bigr),$$

where $\pi:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$ denotes the natural projection. This shows that A_0 is $\mathcal{S}$-definable. □

Example 3.3. Suppose $\mathcal{S}$ is a reasonable category, and $A\in\mathcal{S}^n$. We denote by $\boldsymbol{cl}(A)$ its closure in $\mathbb{R}^n$. Observe that

$$\boldsymbol{cl}(A) = \bigl\{ x\in\mathbb{R}^n;\;\; \forall\varepsilon\in(0,\infty), \exists a\in A\cap B_a(\varepsilon)\,\bigr\}.$$

We can rewrite the above description by saying that $\boldsymbol{cl}(A$ consists of all $x\in\mathbb{R}^n$ such that the following statement is true

$$\forall\varepsilon\Bigl(\,\varepsilon>0\Rightarrow\exists a(a\in A)\wedge\bigl(\,|x-a|<\varepsilon\,\bigr)\,\Bigr).$$

This shows that $\boldsymbol{cl}(A)$ is $\mathcal{S}$-definable because the logical operator $\Rightarrow$ can also be rewritten as $\vee\neg$. □

4. Examples of reasonable categories

Example 4.1 (Semialgebraic sets)**.** Consider the collection $\mathcal{S}_{\mathrm{alg}}$ consisting of *semialgebraic* subsets. More precisely

$$A\in\mathcal{S}^n_{\mathrm{alg}}\Longleftrightarrow A=\bigcup_{k=1}^N A_k,$$

where for every $k=1,\dots,N$ the set A_k described by finitely many polynomial inequalities.

A theorem of Tarski-Seidenberg (1950's) states that $\mathcal{S}_{alg}$ is a reasonable category, and in fact, it is the *smallest* reasonable category. Observe that every set $A \in \mathcal{S}_{\rm alg}$ is a finite union of intervals (possible of infinite or zero length). □

Example 4.2. (a) Suppose that $\mathcal{S}$ is a reasonable category, and $\mathcal{A}^k$ is a collection of subsets of $\mathbb{R}^k$. We set $\mathcal{A} := \cup_k \mathcal{A}$, and we denote by $\mathcal{S}[\mathcal{A}]$ the smallest reasonable category containing $\mathcal{S}$ and all the collections $\mathcal{A}^k$. We say that $\mathcal{S}[\mathcal{A}]$ is the category obtained from $\mathcal{S}$ by adjoining the collection $\mathcal{A}$.

(b) We denote by $\mathcal{S}_{\rm an}$ the category obtained from $\mathcal{S}_{\rm alg}$ by adjoining the graphs of real analytic functions

$$f : [0,1]^n \to \mathbb{R}.$$

The sets obtained in this fashion are called *subanalytic sets* and first appeared in the works of A. Gabrielov, R. Hardt and H. Hironaka in late 60s and early 70s.

(c) We denote by $\widehat{\mathcal{S}}_{\rm an}$ the smallest reasonable category $\mathcal{S}$ containing $\mathcal{S}_{\rm an}$, and satisfying the property:

If $f : (0,1) \to \mathbb{R}$ is C^1 and $\mathcal{S}$-definable then so are its antiderivatives.

Note that $\mathcal{S}_{\rm exp} \subset \widehat{\mathcal{S}}_{\rm an}$ because $\log t$ is an antiderivative of $1/t$ so $\log t$ is $\widehat{\mathcal{S}}_{\rm an}$-definable, and e^t is the inverse of $\log t$ and thus it is also $\widehat{\mathcal{S}}_{\rm an}$-definable. □

5. Tame categories

A reasonable category $\mathcal{S}$ is called *tame* or *o-minimal* (order minimal) if it satisfies the condition

$\boldsymbol{T}$. Any set $A \subset \mathcal{S}^1$ is a finite union of intervals.

Example 5.1. (a) The Tarski-Seidenberg theorem in the 50s implies that the category $\mathcal{S}_{\rm alg}$ is tame.
(b) Work of Garbrielov, Hardt, Hironaka in the 70s implies that $\mathcal{S}_{\rm an}$ is tame.
(c) Work of Khovanski and Wilkie in the 90s implies that $\widehat{\mathcal{S}}_{\rm an}$ is tame. □

In the sequel we will refer interchangeably to the spaces in $\widehat{\mathcal{S}}_{\rm an}$ as tame, or definable or, constructible. Let us point out that any compact real analytic manifold is a tame set.

6. Properties of tame sets and maps

We list some nice properties of tame sets and maps. For proofs we refer to [1, 2].

(1) If $f : (0,1) \to \mathbb{R}$ is a tame map (not necessarily continuous, then for any positive integer p there exists a partition

$$0 = a_0 < a_1 < \cdots < a_N = 1, \quad N = N(p)$$

such that the restriction of f to every subinterval (a_k, a_{k+1}) is of class C^p and weakly monotone.

(2) Suppose $A, B \subset \mathbb{R}^N$ are compact tame sets and $f : A \to B$ is a tame map. Then f is continuous if and only if its graph is a closed subset of $A \times B$.

(3) Any tame set A is a disjoint union of finitely many real analytic subsets

$$A = \bigsqcup_{k=1}^{N} S_k. \tag{6.1}$$

If we define $\dim A = \max \dim S_k$ then the dimension of A is independent of the choice of stratification (6.1). Moreover

$$\dim A > \dim\big(\boldsymbol{cl}(A) \setminus A\big).$$

(4) If $f : A \to B$ is a continuous map then there exists a finite partition of B into finitely many real analytic manifolds $B = \bigsqcup_{k=1}^{N} B_k$ such that each of the maps $f : f^{-1}(B_k) \to B_k$ is a tamely trivial fiber bundle. In particular, there can be only finitely many topological types amongst the fibers of f. (See Fig 1.)

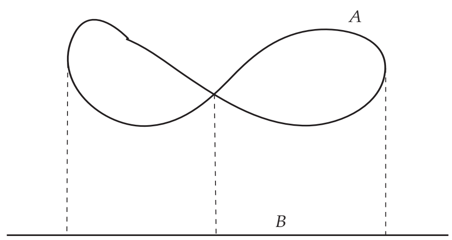


FIGURE 1. *A piecewise fibration.*

(5) Any tame set A can be triangulated which means that there exists pair $(F, \{\Delta_i\}_{i\in I})$ where F is a tame homeomorphism from A to a tame subset M of some Euclidean space $\mathbb{R}^N$ and $\{\Delta_i\}_{i\in I}$ is a *finite* family of mutually disjoint affine open simplices in $\mathbb{R}^N$, of various dimensions, such that

$$M = \bigcup_{i\in I} \Delta_i$$

and for every $i, j \in I$ the intersection $\boldsymbol{cl}(\Delta_i) \cap \boldsymbol{cl}(\Delta_j)$ is either empty, or it is a common face of $\boldsymbol{cl}(\Delta_i)$ and $\boldsymbol{cl}(\Delta_j)$.

7. EULER CHARACTERISTIC

Suppose A is a tame set. For any triangulation $\mathcal{T} = (F, \{\Delta_i\}_{i\in I})$. we set

$$\boldsymbol{\chi}_{\mathrm{t}}(\mathcal{T}) := \sum_{i\in I} (-1)^{\dim \Delta_i}$$

The integer $\boldsymbol{\chi}_{\mathrm{t}}(\mathcal{T})$ is independent of the triangulation $\mathcal{T}$, and it is called the *tame Euler characteristic* of A. If A is a tame locally compact subset of $\mathbb{R}^n$ then

$$\boldsymbol{\chi}_{\mathrm{t}}(A) = \sum_{k\geq 0} H_c^k(A, \mathbb{R}),$$

where $H_c^\bullet$ denotes the cohomology with compact supports. Equivalently, $\boldsymbol{\chi}_{\mathrm{t}}(A)$ is the Euler characteristic of the Borel-Moore homology of A. The o-minimal Euler charateristic is *not* a homotopy invariant. For example, if I is the open inteval $(0, 1)$ then $\boldsymbol{\chi}_{\mathrm{t}}(I) = -1$.

We say that two tame sets A and B are *scissor equivalent* if there exists a tame, but not necessarily continuous, bijection $f : A \to B$. We have the following fundamental result of Lou van der Dries.

Scissor Principle. *Two tame sets A and B are scissor equivalent if and only if they have the same dimension and the same tame Euler characteristic.*

Note that the scissor principle implies that if two tame sets are tamely homeomorphic then they have the same Euler characteristic.

8. "MOTIVIC" INTEGRATION

For every tame set X we denote by $\mathcal{T}_X$ the collection of tame subsets of X. Suppose G is an Abelian group. A *G-valuation* on X is a map $\mu : \mathcal{T}_X \to G$ such that if A, B are two tame subsets of X then

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B), \;\; \forall A, B \in \mathcal{T}_X.$$

Example 8.1. The tame Euler characteristic is a $\mathbb{Z}$-valuation. □

Example 8.2. For every $A \in \mathcal{T}_X$ we denote by $\mathbb{I}_A : X \to \mathbb{Z}$ the characteristic function of A

$$\mathbb{I}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

We denote by $\mathcal{C}_X$ the Abelian subgroup of the additive group $\operatorname{Map}(X, \mathbb{Z})$ generated by the characteristic functions of tame subsets. The functions in $\mathcal{C}_X$ are called *constructible*. Note that $f : X \to \mathbb{Z}$ is constructible if and only if its range is finite, and for every $n \in \mathbb{Z}$ the level set $f^{-1}(n)$ is tame. We have

$$f = \sum_{n \in \mathbb{Z}} n \mathbb{I}_{f^{-1}(n)}.$$

From the equality

$$\mathbb{I}_{A \cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_{A \cap B}$$

we deduce that the map

$$\mathbb{I} : \mathcal{T}_X \to \mathcal{C}_X, \;\; A \mapsto \mathbb{I}_A$$

is a $\mathcal{C}_X$-valuation on X called the *universal valuation* on $\mathcal{T}_X$. Note that any morphism of groups $\Phi : \mathcal{C}_X \to G$ defines a G-valuation φ on X given by

$$\varphi(A) = \Phi(I_A), \;\; \forall A \in \mathcal{T}_X.$$ □

We have the following fundamental theorem, [4].

Groemer Extension Theorem. For every G-valuation φ on X there exists a unique morphism of Abelian groups $\Phi : \mathcal{C}_X \to G$ such that

$$\varphi(A) = \Phi(I_A).$$

The morphism Φ extending the valuation φ is called the *integral with respect to the valuation* φ, and for every $f \in \mathcal{C}_X$ we set

$$\int f d\varphi = \int_X f d\varphi := \Phi(f).$$

We deduce that the Euler characteristic defines a linear map $\mathcal{C}_X \to \mathbb{Z}$ called the *integral with respect to the Euler characteristic*. Let us point out that the construction $X \longmapsto \mathcal{C}_X$ is bi-functorial.

Any tame map $\pi : X \to Y$ induces a pullback morphism

$$\pi^* : \mathcal{C}_Y \to \mathcal{C}_X, \;\; \mathcal{C}_Y \ni f \mapsto \pi^* f := f \circ \pi \in \mathcal{C}_X.$$

Suppose now that $\pi : X \to Y$ is a tame continuous map. For every $y \in Y$ we set $A_y := A \cap \pi^{-1}(y)$ We define a map

$$\pi_* : \mathcal{T}_X \to \mathcal{C}_Y, \;\; \mathcal{T}_X \ni A \mapsto \pi_*(A),$$

$$\pi_*(A) \in \mathcal{C}_y, \;\; \pi_*(A)(y) = \boldsymbol{\chi}_{\mathrm{t}}(A_y) = \int_X \mathbb{I}_{\pi^{-1}(y)} \mathbb{I}_A d\boldsymbol{\chi}_{\mathrm{t}} = \int_{\pi^{-1}(y)} \mathbb{I}_A.$$

Since $\pi|_A$ is a piecewise fibration we deduce that the function $\pi_*(A)$ is indeed constructible. Note also that if A B are two disjoint tame subsets of X then the sets A_y and B_y are disjoint for every y so that

$$\boldsymbol{\chi}_{\rm t}(A_y \cup B_y) = \boldsymbol{\chi}_{\rm t}(A_y) + \boldsymbol{\chi}_{\rm t}(B_y)$$

which shows that the map $\mathcal{T}_X \to \mathcal{C}_Y$ is a $\mathcal{C}_Y$-valuation. We obtain in this fashion a morphism of Abelian groups $\pi_* : \mathcal{C}_Y \to \mathcal{C}_X$ called the *integration along fibers*. For $f \in \mathcal{C}_X$ we have

$$\pi_* f(y) = \int_X \mathbb{I}_{\pi^{-1}(y)} f d\boldsymbol{\chi}_{\rm t} = \int_{\pi^{-1}(y)} f d\boldsymbol{\chi}_{\rm t}\,.$$

The operations π^* and π_* satisfy several desirable properties first formulated by Grothendieck while working with coherent sheaves.

Functoriality. If $X \stackrel{\alpha}{\to} Y \stackrel{\beta}{\to} Z$ are tame continuous maps then

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*, \;\; (\beta \circ \alpha)^* = \alpha^* \circ \beta^*.$$

Projection formula. If $\alpha : X \to Y$ is a tame continuous map $f \in \mathcal{C}_X$ and $g \in \mathcal{C}_Y$ then

$$\alpha_*\bigl(\, f \cdot \alpha^*(g)\,\bigr) = \alpha_*(f) \cdot g.$$

Base change formula. If $X \stackrel{\rho}{\to} S$ and $T \stackrel{\beta}{\to} S$ are tame continuous map and we define

$$T \times_S X := \bigl\{\, (t,x) \in T \times X;\;\; \beta(t) = \rho(x)\,\bigr\},$$

then we have a commutative (cartesian) diagram

$$\begin{array}{ccc} T \times_S X & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \rho} \\ T & \xrightarrow[\beta]{} & S \end{array}$$

and

$$\beta^* \circ \rho_* = (\pi_T)_* \circ \pi_X^*.$$

9. Integral kernels and transforms

Suppose X and Y are tame sets. An *integral kernel* from X to Y is a function $\boldsymbol{K} \in \mathcal{C}_{Y\times X}$. Given such a kernel we define a linear map

$$\mathcal{I}_K : \mathcal{C}_X \to \mathcal{C}_Y, \;\; \mathcal{C}_X \ni f \mapsto (\pi_Y)_*\bigl(\pi_X^*(f) \cdot \boldsymbol{K}\,\bigr) \in \mathcal{C}_Y$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the natural projections. The linear map $\mathcal{I}_K$ is called the *integral transform defined by the kernel* K. We will represent it as a "roof"

$$\begin{array}{ccccc} & & (Y \times X, \boldsymbol{K}) & & \\ & {\scriptstyle \pi_Y}\swarrow & & \searrow{\scriptstyle \pi_X} & \\ Y & & & & X \end{array}$$

More intuitively, for any $f \in \mathcal{C}_X$, the integral transform $\mathcal{I}_K f$ is a constructible function on Y such that

$$\mathcal{I}_K f(y) = \int_X \boldsymbol{K}(y,x) f(x) d\boldsymbol{\chi}_{\rm t}(x).$$

The following result follows rather easily from the properties of the pushforward and the pullback.

Composition formula. If S_0, S_1, S_2 are tame sets, $\boldsymbol{K}_{10} \in \mathcal{C}_{S_1\times S_0}$, $\boldsymbol{K}_{21} \in \mathcal{C}_{S_2\times S_1}$, then

$$\mathcal{I}_{\boldsymbol{K}_{21}} \circ \mathcal{I}_{\boldsymbol{K}_{10}} = \mathcal{I}_{\boldsymbol{K}_{21} * \boldsymbol{K}_{10}}$$

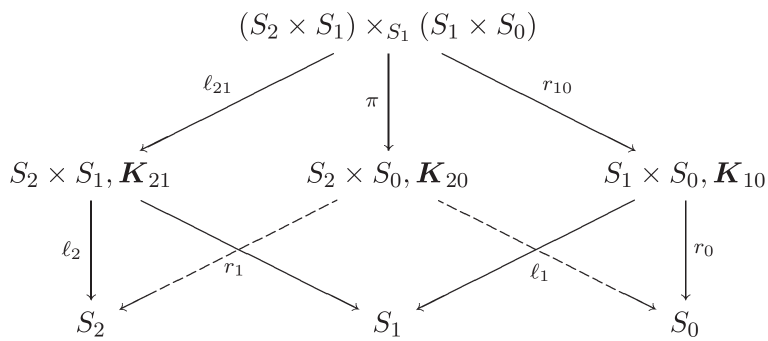
where

$$\boldsymbol{K}_{21} * \boldsymbol{K}_{10}(s_2, s_0) = \int_{S_1} \boldsymbol{K}_{21}(s_2, s_1)\boldsymbol{K}_{10}(s_1, s_0) d\boldsymbol{\chi}_{\mathrm{t}}(s_1).$$

More rigorously $\boldsymbol{K}_{20} = \boldsymbol{K}_{21} * \boldsymbol{K}_{10}$ is given by the equality

$$\boldsymbol{K}_{20} = \pi_*(\ell_{21}^* \boldsymbol{K}_{21} \cdot r_{10}^* \boldsymbol{K}_{10}),$$

where



10. Topological tomography

Denote by $\mathbf{Graff}^1(\mathbb{R}^n)$ the Grassmannian of affine hyperplanes in $\mathbb{R}^n$. This is a constructible set. Note that the lines in $\mathbb{R}^2$ are hyperplanes in $\mathbb{R}^2$.

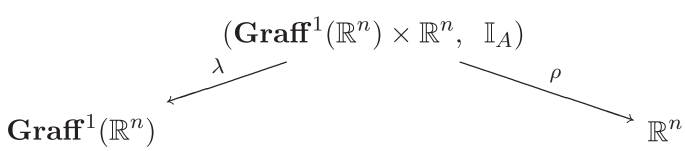
The classical Radon transform associates to a function $f : \mathbb{R}^2 \to \mathbb{R}$ a function $\mathcal{R}f$ on $\mathbf{Graff}^1(\mathbb{R}^1)$ such that the value of $\mathcal{R}f$ on the line L is equal to the integral of f along the line L. The classical Radon inversion formula allows the reconstruction of f from its Radon transform. In particular, if f is the characteristic function of a bounded open set $\Omega \subset \mathbb{R}^2$, then we can completely reconstruct Ω if we know the length of the intersection of any line L with the region Ω.

We want to show that if Ω is a compact tame set, then we can ***completely*** *reconstruct Ω if we know* ***only*** *the number of connected components of the intersection of Ω with any line.*

We have a natural constructible set

$$A = \big\{ (H, x) \in \mathbf{Graff}^1(\mathbb{R}^n) \times \mathbb{R}^n;\ \ H \ni x \big\}$$

We regard the characteristic function of A as a kernel from $\mathbb{R}^n$ to $\mathbf{Graff}^1(\mathbb{R}^n)$, and we denote by $\mathcal{R}_n$ the associated integral transform. Let us compute $\mathcal{R}_n(\mathbb{I}_S)$, where $S \subset \mathbb{R}^n$ is a tame set. We look at the integral transform given by the roof



Note that $\rho^*(\mathbb{I}_S)$ is the characteristic function of the set

$$A_S = \big\{ (H, x) \in \mathbf{Graff}^1(\mathbb{R}^n) \times \mathbb{R}^n;\ \ x \in S \cap H \big\}$$

Then, for any $H_0 \in \mathbf{Graff}^1(\mathbb{R}^n)$ we have a homeomorphism

$$\lambda^{-1}(H_0) \cap A_S \ni (H_0, x) \mapsto x \in H_0 \cap S.$$

We deduce that

$$\lambda_*(\mathbb{I}_{A_S})(H_0) = \boldsymbol{\chi}_{\mathrm{t}}(H_0 \cap S).$$

Hence

$$\mathcal{R}_n(\mathbb{I}_S) : \mathbf{Graff}^1(\mathbb{R}^n) \to \mathbb{Z}$$

is given by

$$\mathcal{R}_n(\mathbb{I}_A) = \boldsymbol{\chi}_{\mathrm{t}}(H \cap S) = \int_{\mathbb{R}^n} \mathbb{I}_{H\cap S} d\,\boldsymbol{\chi}_{\mathrm{t}},$$

so that $\mathcal{R}_n$ is a topological version of the Radon transform. More generally, for every hyperplane H we have

$$\mathcal{R}_n f(H) = \int_{\mathbb{R}^n} \mathbb{I}_H f d\,\boldsymbol{\chi}_{\mathrm{t}}\,.$$

Consider now the dual set

$$A^\dagger := \big\{ (x,H) \in \mathbb{R}^n \times \mathbf{Graff}^1(\mathbb{R}^n);\;\; x \in H, \big\},$$

and denote by $\mathcal{R}_n^\dagger : \mathcal{C}_{\mathbf{Graff}^1(\mathbb{R}^n)} \to \mathcal{C}_{\mathbb{R}^n}$ the integral transform defined by the kernel $\mathbb{I}_{A^\dagger}$. We have the following result due to A. Khovaskii and P. Schapira.

Inversion Formula. For any $f \in \mathcal{C}_{\mathbb{R}^n}$ We have

$$\mathcal{R}_n^\dagger \circ \mathcal{R}_n(f) = (-1)^{n+1} f + \frac{1+(-1)^n}{2}\left(\int_{\mathbb{R}^n} f d\,\boldsymbol{\chi}_{\mathrm{t}} \right) \mathbb{I}_{\mathbb{R}^n}$$

We can express $\int f d\,\boldsymbol{\chi}_{\mathrm{t}}$ in terms of the Radon transform $\mathcal{R}_n f$ as follows. Consider the linear map

$$\pi : \mathbb{R}^n \to \mathbb{R},\;\; (x^1, \dots, x^n) \mapsto x^1.$$

The fiber of π over $t \in \mathbb{R}$ is the hyperplane H_t given by the equation $x^1 = t$.

Denote by c_k the constant map $\mathbb{R}^k \to \{*\}$. Note that $\mathcal{C}_* = \mathbb{Z}$

$$(c_k)_*(f) = \int_{\mathbb{R}^k} f d\,\boldsymbol{\chi}_{\mathrm{t}},\;\; \forall f \in \mathcal{C}_{\mathbb{R}^k}.$$

On the other hand,

$$\int_{\mathbb{R}^n} f d\,\boldsymbol{\chi}_{\mathrm{t}} = (c_n)_*(f) = (c_1)_* \circ \pi_*(f) = \int_{\mathbb{R}} (\pi_* f)(t) d\,\boldsymbol{\chi}_{\mathrm{t}}(t).$$

Now observe that

$$\pi_* f(t) = \int_{\mathbb{R}^n} \mathbb{I}_{\pi^{-1}(t)} f d\,\boldsymbol{\chi}_{\mathrm{t}} = \int_{\mathbb{R}^n} \mathbb{I}_{H_t} f = \mathcal{R}_n(H_t).$$

Hence

$$\int f d\,\boldsymbol{\chi}_{\mathrm{t}} = \int_{\mathbb{R}} \mathcal{R}_n f(H_t) d\,\boldsymbol{\chi}_{\mathrm{t}}(t).$$

so that

$$f = (-1)^{n+1} \mathcal{R}_n^\dagger \circ \mathcal{R}_n(f) + \frac{1+(-1)^n}{2}\left(\int_{\mathbb{R}} \mathcal{R}_n f(H_t) d\,\boldsymbol{\chi}_{\mathrm{t}}(t) \right).$$

If we define

$$B = \big\{ (x,H) \in \mathbb{R}^n \times \mathbf{Graff}^1(\mathbb{R}^n);\;\; H = \pi^{-1}(\,\pi(x)\,)\,\big\}$$

then we deduce that

$$\int_{\mathbb{R}} \mathcal{R}_n f(H_t) d\,\boldsymbol{\chi}_{\mathrm{t}}(t) = \mathcal{I}_{\mathbb{I}_B}(\mathcal{R}_n f).$$

Finally if we set

$$K = (-1)^{n+1}\mathbb{I}_{A^\dagger} + \frac{1+(-1)^n}{2}\mathbb{I}_B \in \mathcal{C}_{\mathbb{R}^n \times \mathbf{Graff}^1(\mathbb{R}^n)}$$

then we deduce that

$$\mathcal{I}_K \circ \mathcal{R}_n(f) = f, \;\; \forall f \in \mathcal{C}_{\mathbb{R}^n}.$$

This proves that the Radon transform is injective.

REFERENCES

[1] M. Coste: *An Introduction to o-minimal geometry*, Real Algebraic and Analytic Geometry Network, http://www.ihp-raag.org/publications.php

[2] L.van der Dries: *Tame Topology and o-minimal Structures*, London Math. Soc. Series, vol. 248, Cambridge University Press, 1998.

[3] A. Khovanskii: *Fewnomials*, Transl. Math. Monogr., **88**, Amer.Math. Soc., 1991.

[4] D.A. Klain, G.-C. Rota: *Introduction to Geometric Probability*, Cambridge University Press, 1997.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.
E-mail address: nicolaescu.1@nd.edu
URL: http://www.nd.edu/˜lnicolae/