# THE DERIVED CATEGORY OF SHEAVES AND THE POINCARÉ-VERDIER DUALITY

LIVIU I. NICOLAESCU

ABSTRACT. I think these are powerful techniques.

CONTENTS



*Date*: Last revised: April, 2005.

Notes for myself and whoever else is reading this footnote.

## 1. Derived categories and derived functors: a short introduction

For a detailed presentation of this subject we refer to [4, 6, 7, 8]. Suppose $\mathcal{A}$ is an Abelian category. We can form the Abelian category $C(\mathcal{A})$ consisting of complexes of objects in $\mathcal{A}$. We denote the objects in $C(\mathcal{A})$ by $A^\bullet$ or $(A^\bullet, d)$. The homology of such a complex will be denoted by $H^\bullet(A^\bullet)$.

A morphism $s \in \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)$ is called a *quasi-isomorphism* (qis brevity) if it induces an isomorphism in co-homology. We will indicate qis-s by using the notation

$$A^\bullet \overset{s}{\rightsquigarrow} B^\bullet.$$

Define a new *additive* category $K(\mathcal{A})$ whose objects coincide with the objects of $C(\mathcal{A})$, i.e. are complexes, but the morphisms are the homotopy classes of morphisms in $C(\mathcal{A})$, i.e.

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)/\simeq,$$

where $\simeq$ denotes the homotopy relation. Often we will use the notation

$$[A^\bullet, B^\bullet] := \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet).$$

The *derived category* of $\mathcal{A}$ will be a category $D(\mathcal{A})$ with the same objects as $K(\mathcal{A})$ but with a much larger class of morphisms. More precisely, a morphism in $\mathrm{Hom}_{D(\mathcal{A})} X^\bullet, Y^\bullet)$ is a "roof", i.e. a diagram of the form

$$X^\bullet \overset{s}{\leftsquigarrow} Z^\bullet \overset{f}{\to} Y^\bullet.$$

This roof should be interpreted as a "fraction"

$$X^\bullet \xrightarrow{f/s} Y^\bullet.$$

Two such roofs

$$f/s = X^\bullet \overset{s}{\leftsquigarrow} Z_0^\bullet \overset{f}{\to} Y^\bullet \;\text{ and }\; g/t = X^\bullet \overset{t}{\leftsquigarrow} Z_1^\bullet \overset{g}{\to} Y^\bullet$$

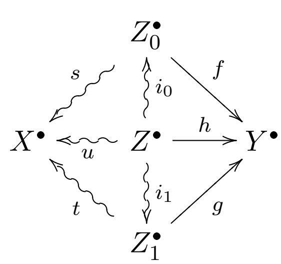
define identical morphisms in $D(\mathcal{A})$, $f/s \sim g/t$, if there exists a roof

$$X^\bullet \overset{u}{\leftsquigarrow} Z^\bullet \overset{h}{\to} Y^\bullet$$

and qis

$$Z^\bullet \overset{i_k}{\rightsquigarrow} Z_k^\bullet, \;\; k = 0, 1$$

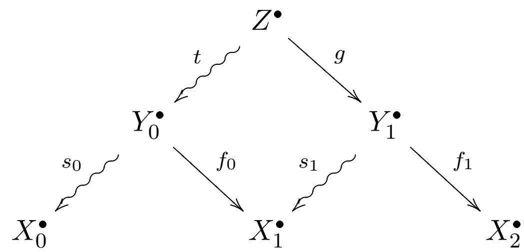
such that the diagram below is homotopy commutative.




The composition of two such morphisms

$$X_0^\bullet \overset{s_0}{\leftsquigarrow} Y_0^\bullet \overset{f_0}{\to} X_1^\bullet, \;\; X_1^\bullet \overset{s_1}{\leftsquigarrow} Y_1^\bullet \overset{f_1}{\to} X_2^\bullet$$

is a roof $X_1 \overset{s_0\circ t}{\leftsquigarrow} Z^\bullet \xrightarrow{f_1\circ g} X_2^\bullet$, where $Y_0^\bullet \overset{t}{\leftsquigarrow} Z^\bullet \xrightarrow{g} Y_1^\bullet$ is a roof such that the diagram below is homotopy commutative.




One can verify that such operations are well defined. $D(\mathcal{A})$ is an additive category. The group operation on $\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet)$ is defined as follows. Any two fractions

$$f/s, g/t \in \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet)$$

have a "common denominator", i.e. there exists a qis $Z^\bullet \overset{u}{\rightsquigarrow} X^\bullet$ and morphisms $Z^\bullet \xrightarrow{f', g'} Y^\bullet$ such that

$$f/s \sim f'/u, \;\; g/t \sim g'u.$$

We then set

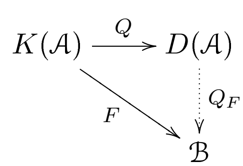
$$f/s + g/t = (f' + g')/u.$$

Note that we have a tautological functor

$$Q : K(\mathcal{A}) \to D(\mathcal{A}), \;\; X^\bullet \mapsto X^\bullet$$

$$f \in \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet) \mapsto f/\mathbb{1}_X = X^\bullet \overset{\mathbb{1}_X}{\leftsquigarrow} X^\bullet \xrightarrow{f} Y^\bullet \in \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet).$$

$Q$ is called the *localization functor* and it has the following universality property. For any additive functor $F : K(\mathcal{A}) \to \mathcal{B}$, $\mathcal{B}$ additive category, such that $F(\varphi)$ is an isomorphism for every qis $X^\bullet \overset{\varphi}{\rightsquigarrow} Y^\bullet$, there exists a unique functor $Q_F : D(\mathcal{A}) \to B$ such that the diagram below is commutative




The category $K(\mathcal{A})$ has several interesting subcategories $K^*(\mathcal{A})$, $* \in \{b, +, -\}$, where $b$ stands for bounded complexes, $+$ for complexes bounded from below and $-$ for complexes bounded from above. Using the same procedure as above we obtain derived categories

$$Q^* : K^*(\mathcal{A}) \to D^*(\mathcal{A})$$

satisfying similar universality properties. We deduce that there exist natural injective functors

$$i^* : D^*(\mathcal{A}) \to D(\mathcal{A})$$

so we can regard $D^*(\mathcal{A})$ as subcategories of $D(\mathcal{A})$. In fact they are *full* subcategories of $D(\mathcal{A})$ (see [6, Prop. 6.15]). This means that for every objects $X^\bullet, Y^\bullet$ in $D^*(\mathcal{A})$ we have

$$\mathrm{Hom}_{D^*(\mathcal{A})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet).$$

This is established using a useful trick called *truncation*. Given a complex $(X^\bullet, d)$ and an integer $n$ we define complexes $\tau_{\leqslant n}X^\bullet$ and $\tau_{\geqslant n}$ as follows.

$$\tau_{\leqslant n}X^p \cong \begin{cases} X^p & \text{if } \; p < n \\ 0 & \text{if } \; p > n \\ \ker d_n & \text{if } \; p = n \end{cases}, \quad \tau_{\geqslant n}X^p \cong \begin{cases} X^p & \text{if } \; p > n \\ 0 & \text{if } \; p < n \\ \operatorname{coker} d_{n-1} & \text{if } \; p = n \end{cases}.$$

Observe that we have natural morphisms of complexes

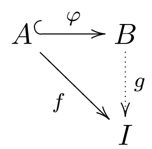
$$\tau_{\leqslant n}X^\bullet \stackrel{\tau_{\leqslant n}}{\longrightarrow} X^\bullet, \quad X^\bullet \stackrel{\tau_{\geqslant n}}{\longrightarrow} \tau_{\geqslant n}X^\bullet.$$

The first morphism is a qis in dimensions $\leqslant n$ and the second one is a qis in dimension $\geqslant n$. In particular, if $H^p(X^\bullet) = 0$ for $p > n$ then $\tau_{\leqslant n}$ is a qis while if $H^p(X^\bullet) = 0$ for $p < n$ then $\tau_{\geqslant n}$ is a qis. If $H^p(X^\bullet) = 0$ for all $p \neq n$ then $X^\bullet$ and $\tau_{\geqslant n}\tau_{\leqslant n}X^\bullet$ are isomorphic in the derived category. The latter is a complex concentrated only in dimension $n$.

Recall that an object $I \in \mathcal{A}$ is called *injective* if the functor

$$\operatorname{Hom}(\bullet, I) : \mathcal{A}^{op} \to \mathbf{Ab}$$

is exact. More precisely, this means that for any monomorphism $A \stackrel{\varphi}{\hookrightarrow} B$ in $\mathcal{A}$ the morphism $\operatorname{Hom}_{\mathcal{A}}(B, I) \to \operatorname{Hom}_{\mathcal{A}}(A, I)$ is surjective. Equivalently, this means that every morphism $f \in \operatorname{Hom}_{\mathcal{A}}(A, I)$ can be extended to a morphism $g \in \operatorname{Hom}_{\mathcal{A}}(B, I)$



The injectives form a full additive category of $\mathcal{A}$ which we denote by $\mathcal{I} = \mathcal{I}_{\mathcal{A}}$. The Abelian category is said to have *enough injectives* if any object of $A$ is a sub-object of an injective.

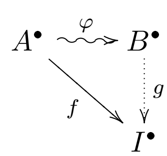
**Theorem 1.1.** *If the Abelian category $\mathcal{A}$ has enough injectives then every object in $C^+(\mathcal{A})$ has an* injective resolution*, i.e. it is quasi-isomorphic to a complex of injectives.* □

The resolution of a complex should be regarded as an abstract incarnation of the geometrical operation of triangulation of spaces. Alternatively, an injective resolution of a complex can be thought of as a sort of an approximation of that complex by a simpler object. The above result should be compared with the more elementary result: every continuous function can be uniformly approximated by step functions.

**Theorem 1.2.** *Suppose $\mathcal{A}$ enough injectives and $I^\bullet \in C^+(\mathcal{I})$. The every qis $A^\bullet \stackrel{f}{\rightsquigarrow} B^\bullet$ between objects in $C^+(\mathcal{A})$ induces an isomorphism*

$$\varphi^* : [B^\bullet, I^\bullet] \to [A^\bullet, I^\bullet].$$

*This means that for every homotopy class of morphisms $f \in [A^\bullet, I^\bullet]$ there* exists *a* unique *homotopy class of morphisms $g \in [B^\bullet, I^\bullet]$ such that the diagram below is homotopy commutative*



For a proof we refer to [7, Thm. I.6.2]. Note that this theorem implies that the quasi-isomorphisms between bounded below complexes of injectives are necessarily homotopy equivalences. This theorem implies the following important result.

**Theorem 1.3.** *If the Abelian category has sufficiently many injectives then the composition*

$$K^+(\mathcal{I}) \to K^+(\mathcal{A}) \xrightarrow{Q} D^+(\mathcal{A})$$

*is an equivalence of categories.*

*Remark* 1.4. The above result can be generalized as follows. First we introduce the notion of *generating subcategory* to be a full additive subcategory $\mathcal{J}$ of $\mathcal{A}$ satisfying the following conditions

(i) Every object of $\mathcal{A}$ is a sub-object of an object in $\mathcal{J}$.
(ii) If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$ such that $A, B \in \mathcal{J}$ then $C \in \mathcal{J}$

We form a category $D^*(\mathcal{J})$ whose objects are complexes $J^\bullet \in K^+(\mathcal{J})$ and whose morphisms are roofs of the form

$$J_1^\bullet \overset{s}{\leftsquigarrow} \text{object in } K^+(\mathcal{J}) \xrightarrow{f} J_2^\bullet.$$

We get a functor

$$Q^{\mathcal{A}}_{\mathcal{J}} : D^+(\mathcal{J}) \to D^+(\mathcal{A}).$$

Then this functor is an equivalence of categories. For a proof we refer to [8, Prop. 1.6.10, 1.7.7]. □

The additive category $K(\mathcal{A})$ has an extra structure which is inherited by $D(\mathcal{A})$. Formally it is equipped with a structure of *triangulated category*. Note first that there exists an automorphism of categories

$$T : K(\mathcal{A}) \to K(\mathcal{A}), \;\; A^\bullet \mapsto A[1]^\bullet := A^{\bullet+1}, \;\; d_{A[1]} = -d_A.$$

In $K(\mathcal{A})$ we can speak of triangles, which are sequences of homotopy classes of morphisms

$$(X, Y, Z; u, v, w) := X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X[1]^\bullet = \begin{array}{ccc} & Z & \\ {\scriptstyle w}\swarrow & & \nwarrow{\scriptstyle v} \\ X & \xrightarrow[u]{} & Y \end{array}$$

The (clockwise) *rotation* of $(X, Y, Z; u, v, w)$ is the triangle

$$\mathcal{R}(X, Y, Z; u, v, w) = (Y, Z, X[1], v, w, -u[1]) = \begin{array}{ccc} & X[1] & \\ {\scriptstyle -u[1]}\swarrow & & \nwarrow{\scriptstyle w} \\ Y & \xrightarrow[v]{} & Z \end{array}$$

The morphisms of triangles are defined in an obvious way, through commutative diagrams.

To a morphism $u \in \mathrm{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$ we can associate its *cone complex* $C(u) \in C(\mathcal{A})$ defined by[1]

$$(C(u)^\bullet, d_u) = (Y^\bullet \oplus X[1]^\bullet, d_u), \;\; d_{C(u)} \begin{bmatrix} y_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} d_Y & -u \\ 0 & d_{X[1]} \end{bmatrix} \cdot \begin{bmatrix} y_n \\ x_{n+1} \end{bmatrix}.$$

[1] **Warning:** There are various sign conventions in the literature. Our convention agrees with the ones in [1, §2.6] and [7]. In [4, 8] the cone of $u$ coincides with our $C(-u)$.

(*Recall that* $d_{X[1]} = -d_X$. ) If we denote by $p(f)$ the projection $Y^\bullet \oplus X[1]^\bullet \twoheadrightarrow X[1]^\bullet$ and by $i(f)$ the inclusion $Y^\bullet \hookrightarrow Y^\bullet \oplus X[1]^\bullet$ we obtain a triangle

$$\Delta_u := \Big(X, Y, C(u), X[1];\ u, i(u), p(u), \Big) := X^\bullet \stackrel{u}{\longrightarrow} Y^\bullet \stackrel{i(f)}{\longrightarrow} C(u)^\bullet \stackrel{p(u)}{\longrightarrow} X[1]^\bullet.$$

We will refer to this as the *triangle associated to a cone*. One can think of the cone complex as a sort of difference between $X$ and $Y$ along $u$.

$$C(u) = \mathrm{dist}_u(Y^\bullet, X^\bullet) = \mathrm{dist}(Y^\bullet \stackrel{u}{\longleftarrow} X^\bullet)$$

Its cohomology is a measure of the difference between the cohomologies of $X$ and $Y$. The short exact sequence of complexes

$$0 \to Y^\bullet \stackrel{i(u)}{\longrightarrow} C(u) \stackrel{p(u)}{\longrightarrow} X[1]^\bullet \to 0$$

implies the following identity between Euler characteristics

$$\chi(H^\bullet(C(u))) = \chi(H^\bullet(Y)) - \chi(H^\bullet(X)).$$

This justifies the interpretation of $C(u)$ as distance between $X$ and $Y$. A *distinguished triangle* will be a triangle in $K(\mathcal{A})$ isomorphic to the triangle of a cone. We denote by $\mathcal{T}_{K(\mathcal{A})}$ the collection of distinguished triangles.

The collection $\mathcal{T} = \mathcal{T}_{K(\mathcal{A})}$ of distinguished triangles in $K(\mathcal{A})$ satisfies a few fundamental properties. We list them below.

**TR1.** (a)(Normalization axiom) Every triangle isomorphic to a triangle in $\mathcal{T}$ is a triangle in $\mathcal{T}$.
(b) For any morphism $X^\bullet \stackrel{u}{\to} Y^\bullet$ there exists a triangle $(X, Y, Z; u, v, w) \in \mathcal{T}$.
(c) $(X, X, 0; \mathbb{1}_X, 0, 0) \in \mathcal{T}$.

**TR2.** (Rotation axiom)

$$(X, Y, Z; u, v, w) \in \mathcal{T} \Longleftrightarrow R(X, Y, Z; u, v, w) = (Y, Z, T(X); v, w, -u[1]) \in \mathcal{T}.$$

To prove this note that for every morphism $u \in \mathrm{Hom}_{C(A)}(X, Y)$ we have a homotopy commutative diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{i(u)} & C(u) & \xrightarrow{p(u)} & X[1] & \xrightarrow{-u[1]} & Y[1] \\ \downarrow{\scriptstyle \mathbb{1}_Y} & & \downarrow{\scriptstyle \mathbb{1}_{C(u)}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathbb{1}_{Y[1]}} \\ Y & \xrightarrow[i(u)]{} & C(u) & \xrightarrow[i(i(u))]{} & C(i(u)) & \xrightarrow[p(i(u))]{} & Y[1] \end{array} \tag{1.1}$$

where $C(i(u)) = C(Y \stackrel{i(u)}{\hookrightarrow} C(u)) = C(u) \oplus Y[1] = Y \oplus X[1] \oplus Y[1]$,

$$d_{C(i(u)} = \left[\begin{array}{ccc} d_Y & -u[1] & -\mathbb{1}_{Y[1]} \\ 0 & d_{X[1]} & 0 \\ 0 & 0 & d_{Y[1]} \end{array}\right].$$

and

$$X[1] \ni x^{n+1} \stackrel{\phi}{\mapsto} \left[\begin{array}{c} 0 \\ x^{n+1} \\ -u(x^{n+1}) \end{array}\right] \in C(i(u))$$

is a homotopy equivalence. More precisely, if we denote by $\psi$ the natural projection $C(i(u)) \to X[1]$ then

$$\psi \circ \phi = \mathbb{1}_{X[1]}.$$

Define

$$s: Y^n \oplus X^{n+1} \oplus Y^{n+1} = C(i(u))^n \longrightarrow C(i(u))[-1]^n = Y^{n-1} \oplus X^n \oplus Y^n, \;\; s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mathbb{1}_{Y^n} & 0 & 0 \end{bmatrix}.$$

Observe that

$$(\mathbb{1}_{C(i(u))^n} - \phi\psi) \begin{bmatrix} y^n \\ x^{n+1} \\ y^{n+1} \end{bmatrix} = \begin{bmatrix} y^n \\ 0 \\ u(x^{n+1}) + y^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbb{1}_Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u[1] & \mathbb{1}_{Y[1]} \end{bmatrix} \begin{bmatrix} y^n \\ x^{n+1} \\ y^{n+1} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} \mathbb{1}_Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u[1] & \mathbb{1}_{Y[1]} \end{bmatrix} = \begin{bmatrix} d_Y & -u[1] & -\mathbb{1}_{Y[1]} \\ 0 & d_{X[1]} & 0 \\ 0 & 0 & d_{Y[1]} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mathbb{1}_{Y^n} & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mathbb{1}_{Y^n} & 0 & 0 \end{bmatrix} \begin{bmatrix} d_Y & -u[1] & -\mathbb{1}_{Y[1]} \\ 0 & d_{X[1]} & 0 \\ 0 & 0 & d_{Y[1]} \end{bmatrix}.$$

Hence

$$\mathbb{1}_{C(i(u))^n} - \phi\psi = s d_{C(i(u))} + d_{C(i(u))} s.$$

This homotopy commutative diagram implies that

$$\mathcal{R}\Delta_u \cong \Delta_{i(u)}.$$

**TR3.** (Completion axiom) For every $(X, Y, Z; u, v, w), (X', Y', Z'; u', v', w') \in \mathcal{T}$ and every $X \xrightarrow{f} X'$ , $Y \xrightarrow{g} Y'$ so that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

we can find $Z \xrightarrow{h} Z'$, *not necessarily unique*, such that the diagram below is commutative

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X') \end{array}$$

**TR4** (The octahedron axiom ) It states among other things that there exists a morphism $w$ such that

$$\operatorname{dist}_{vu}(Z^\bullet, X^\bullet) = \operatorname{dist}_w\Big(\operatorname{dist}_v(Z^\bullet, Y^\bullet), \operatorname{dist}_u(Y^\bullet, X^\bullet)[1]\Big)$$
$$\text{“ = ” } \operatorname{dist}_v(Z^\bullet, Y^\bullet) + \operatorname{dist}_u(Y^\bullet, X^\bullet).$$

For more details see [4, 6].

A triangle in $D(\mathcal{A})$ is called *distinguished* if it is isomorphic (in $D(\mathcal{A})$) to the image via the localization functor $Q$ of a distinguished triangle in $K(\mathcal{A})$. The collection $\mathcal{T}_{D(\mathcal{A})}$ of distinguished triangles in $D(\mathcal{A})$ will continue to satisfy the axioms **TR1**-**TR4**.

**Example 1.5** (Fundamental example)**.** To every short exact sequence of complexes in $C^*(\mathcal{A})$

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0 \tag{1.2}$$

we can associate in a canonical way a distinguished triangle in $D^*(\mathcal{A})$

$$A^\bullet \xrightarrow{f} B\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[1]^\bullet \tag{1.3}$$

First form the *cylinder* of $f$ which is the complex $(Cyl(f), d) = (C(\mathbb{1}_A \oplus -f))$ so that where

$$Cyl(f)^\bullet := A^\bullet \oplus B^\bullet \oplus A^\bullet[1], \;\; d\begin{bmatrix} a^n \\ b^n \\ a^{n+1} \end{bmatrix} = \begin{bmatrix} d_A & 0 & -\mathbb{1}_{A[1]} \\ 0 & d_B & f[1] \\ 0 & 0 & d_{A[1]} \end{bmatrix}\begin{bmatrix} a^n \\ b^n \\ a^{n+1} \end{bmatrix}$$

Denote by $q_B$ the natural projection $Cyl(f)^\bullet \to B^\bullet$, and by $j_B$ the natural inclusion $B^\bullet \hookrightarrow Cyl(f)^\bullet$. One can check that these are chain morphisms and satisfy

$$q_B \circ j_B = \mathbb{1}_B, \;\; j_B \circ q_B \simeq \mathbb{1}_{Cyl(f)}.$$

The morphism $g$ induces a natural map $\gamma : C(f) = B^\bullet \oplus A[1]^\bullet \to C^\bullet$ and we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{i(f)} & C(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathbb{1}_A} & & \downarrow{\scriptstyle j_B} & & \downarrow{\scriptstyle \mathbb{1}_C} & & \\
0 & \longrightarrow & A^\bullet & \xrightarrow{\bar{f}} & Cyl(f) & \xrightarrow{\pi} & C(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathbb{1}_A} & & \downarrow{\scriptstyle q_B} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C^\bullet & \longrightarrow & 0
\end{array}$$

where the last two rows are exact. The morphisms $\mathbb{1}_A$ and $q_B$ are homotopy equivalences and invoking the five lemma we deduce that $\gamma$ is a qis as well. Define now $h \in \mathrm{Hom}_{D^*(A)}(C^\bullet, A[1]^\bullet)$ as the roof

$$C^\bullet \overset{\gamma}{\leftsquigarrow} C(f) \xrightarrow{p(f)} A[1]^\bullet.$$

The top row defines a distinguished triangle quasi-isomorphic to the bottom row

$$(A^\bullet, B^\bullet, C^\bullet; f, g, h) \cong \Delta_f$$

The morphism

$$h \in \mathrm{Hom}_{D(\mathcal{A})}(C^\bullet, A[1]^\bullet)$$

is called the *characteristic class* of the short exact sequence (1.2).

□

We have the following result.

**Proposition 1.6.** *A triangle $(X, Y, Z, u, v, w)$ is distinguished if and only if it is isomorphic to a triangle of the form*

$$(A^\bullet, d_A) \xrightarrow{i} (A^\bullet \oplus B^\bullet, d) \overset{p}{\twoheadrightarrow} (B^\bullet, d_B) \xrightarrow{h} A[1]^\bullet, \tag{1.4}$$

*where $h$ is defined by $(h(b), d_B b) = d(0, b)$, i.e. $h$ is the off-diagonal component of $d$ with respect to the direct sum decomposition*

$$d = \begin{bmatrix} d_A & h \\ 0 & d_B \end{bmatrix}.$$

*The morphism* $h \in \mathrm{Hom}_{K(\mathcal{A})}(B^\bullet, A[1]^\bullet)$ *is an invariant of the homotopy class of* $(A^\bullet \oplus B^\bullet, d)$*. The morphism induced by* $h$ *in cohomology coincides with the connecting morphism of the long exact sequence associated to the short exact sequence*

$$0 \to (A^\bullet, d_A) \xrightarrow{i} (A^\bullet \oplus B^\bullet, d) \overset{p}{\twoheadrightarrow} (B^\bullet, d_B) \to 0.$$

**Proof** The condition $d^2 = 0$ implies that $h \in \mathrm{Hom}_{K(\mathcal{A})}(B^\bullet, A[1]^\bullet)$. Note that

$$(A^\bullet \oplus B^\bullet, d) = C(-h)[-1].$$

and thus we have a distinguished triangle

$$A^\bullet \xrightarrow{i(h)} C(-h)[-1] \xrightarrow{p(h)} B^\bullet \xrightarrow{h} A^\bullet[1].$$

□

A *cohomological functor* on $D^*(\mathcal{A})$ is an additive functor $F : D^*(\mathcal{A}) \to \mathcal{B}$, $\mathcal{B}$ Abelian category, such that for every distinguished triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X[1]^\bullet$$

we get an exact sequence in $\mathcal{B}$

$$F(X^\bullet) \xrightarrow{F(f)} F(Y^\bullet) \xrightarrow{F(g)} F(Z^\bullet).$$

If we set $F^n := F \circ T^n$ so that $F^n(X^\bullet) := F(X[n]^\bullet)$. Using the axiom **TR2** we deduce that for every distinguished triangle $(X, Y, Z, f, g, h)$ we obtain a long exact sequence in $\mathcal{B}$

$$\cdots \to F^n(X^\bullet) \xrightarrow{(-1)^n f[n]} F^n(Y^\bullet) \xrightarrow{(-1)^n g[n]} F^n(Z^\bullet) \xrightarrow{(-1)^n h[n]} F^{n+1}(X^\bullet) \to \cdots \tag{1.5}$$

In particular, a homological functor associates to each short exact sequence in $C^*(\mathcal{A})$

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

the long exact sequence (1.5). A *homologica*l functor is an additive functor $G : D^*(\mathcal{A})^{op} \to \mathcal{B}$ which associates to each distinguished triangle an short exact sequence as above.

*Remark* 1.7 (Food for thought). Suppose the Abelian category is a subcategory of a category of modules. Then the the morphisms of $C(A)$ are actual set-theoretic maps, while the morphisms of $D(\mathcal{A})$ *are not*. Let's call them "virtual maps". A cohomological functor sends "virtual maps" to *genuine maps*! □

**Example 1.8.** We have a tautological homological functor

$$H : D^*(\mathcal{A}) \to \mathcal{A}$$

which associates to each complex $X^\bullet$ its 0-th cohomology $H^0(X^\bullet)$. In this case $H^n(X^\bullet) = H^0(X[n]^\bullet)$ coincides with the $n$-th cohomology of the complex $X^\bullet$. □

**Example 1.9** (Yoneda's Description of Ext)**.** For every object $R^\bullet \in D(\mathcal{A})$ the functors

$$\mathrm{Hom}_{D^*(A)}(R^\bullet, -) \quad \text{and} \quad \mathrm{Hom}_{D^*(\mathcal{A})}(-, R^\bullet)$$

are (co)homological functors. We prove this for the functor $\mathrm{Hom}_{D^*(\mathcal{A})}(-, R^\bullet)$. Suppose we are given a distinguished triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X[1]^\bullet$$

We need to show that the sequence

$$\operatorname{Hom}_{D^*(\mathcal{A})}(Z^\bullet, R^\bullet) \stackrel{g^*}{\to} \operatorname{Hom}_{D^*(\mathcal{A})}(Y^\bullet, R^\bullet) \stackrel{f^*}{\to} \operatorname{Hom}_{D^*(\mathcal{A})}(X^\bullet, R^\bullet)$$

is exact, i.e. $f^*g^* = 0$ and $\ker f^* \subset \operatorname{Im} g^*$. To prove the first part we will show that $gf = 0 \in D^*(\mathcal{A})$. From the normalization axiom we get a distinguished triangle $(X, X, 0;\ \mathbb{1}_X, 0, 0)$. From the completion axiom we can find $0 \stackrel{\phi}{\to} X$ to complete a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\mathbb{1}_X} & X & \xrightarrow{0} & 0 & \longrightarrow & T(X) \\ {\scriptstyle \mathbb{1}_X}\downarrow & & {\scriptstyle f}\downarrow & & \vdots{\scriptstyle \phi} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X) \end{array}$$

so that $g \circ f = \phi \circ 0 = 0$. Suppose $\varphi \in \operatorname{Hom}_{D^*(\mathcal{A})}(Y^\bullet, R^\bullet)$ is in $\ker f^*$, i.e. we have a commutative diagram

$$\begin{array}{ccc} X^\bullet & \xrightarrow{f} & Y^\bullet \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle \varphi} \\ & & R^\bullet \end{array} .$$

We have to show that there exists $\psi : Z^\bullet \to R^\bullet$ such that the diagram below is commutative

$$\begin{array}{ccc} Y^\bullet & \xrightarrow[g]{} & Z^\bullet \\ & {\scriptstyle \varphi}\searrow & \vdots{\scriptstyle \psi} \\ & & R^\bullet \end{array} .$$

The existence of such $\psi$ is postulated by the completion axiom since $\psi$ completes the commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{g} & Z^\bullet & \xrightarrow{h} & X[1]^\bullet \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \varphi} & & \vdots{\scriptstyle \psi} & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow[0]{} & R^\bullet & \xrightarrow[\mathbb{1}_{R^\bullet}]{} & R^\bullet & \longrightarrow & 0 \end{array} .$$

Given a short exact sequence of complexes

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

we obtain a long exact sequence

$$\cdots \to \operatorname{Hom}_{D^*(A)}(Z^\bullet, R^\bullet) \to \operatorname{Hom}_{D^*(A)}(Y^\bullet, R^\bullet) \to$$

$$\to \operatorname{Hom}_{D^*(A)}(X^\bullet, R^\bullet) \to \operatorname{Hom}_{D^*(A)}(Z^\bullet, R[1]^\bullet) \to \cdots .$$

For any two complexes $A^\bullet, B^\bullet \in C^*(\mathcal{A})$ we define the *hyper-Ext*

$$\mathbb{E}\mathrm{xt}^n(A^\bullet, B^\bullet) := \operatorname{Hom}_{D^*(A)}(A^\bullet, B[n]^\bullet) = \operatorname{Hom}_{D^*(A)}(A[k]^\bullet, B[k+n]^\bullet). \tag{1.6}$$

If $A^\bullet, B^\bullet \in K^+(\mathcal{A})$ are complexes of injective objects then

$$\mathbb{E}\mathrm{xt}^n(A^\bullet, B^\bullet) = [A^\bullet, B[n]^\bullet].$$

We have a natural fully faithful functor

$$[-] : \mathcal{A} \to C^*(\mathcal{A})$$

which associates to each object $A$ the complex $[A]$ the complex $[A]^\bullet$ with $[A]^0 = A$ and $[A]^n = 0$ if $n \neq 0$. Then

$$\mathrm{Ext}^n(A, B) = \mathbb{E}\mathrm{xt}^n([A], [B]),$$

where in the left-hand-side we have the classical $\mathrm{Ext}$-functors associated to a pair of objects in an Abelian category.

□

An additive functor $F : \mathcal{A} \to \mathcal{B}$ between Abelian categories is called *left exact* if for any short exact sequence

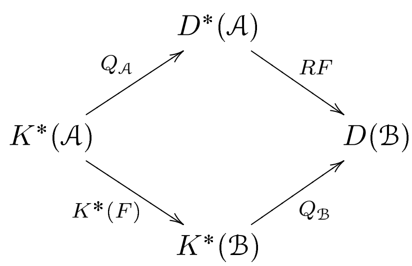
$$0 \to A \to B \to C$$

the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact. $F$ induces a functor $K^*(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$.

**Definition 1.10.** A *derived functor* for $F$ is a pair $(RF, \rho_F)$ with the following properties.
(a) $RF$ is an *exact* functor $D^*(\mathcal{A}) \to D(\mathcal{B})^*$, i.e. an additive functor which maps distinguished triangles to distinguished triangles.
(b) $\rho_F$ is an morphism of functors $Q_B \circ K^*(F) \to RF \circ Q_A$




(c) For any exact functor $G : D^*(\mathcal{A}) \to D(\mathcal{B})^*$ and morphism of functors

$$\rho : \Big( K^*(\mathcal{A}) \stackrel{Q_\mathcal{B} \circ K^*(F)}{\longrightarrow} D(\mathcal{B}) \Big) \to \Big( K^*(\mathcal{A}) \stackrel{G \circ Q_A}{\longrightarrow} D(\mathcal{B}) \Big)$$

there exists a unique morphism

$$c : \Big( D^*(\mathcal{A}) \stackrel{RF}{\longrightarrow} D(\mathcal{B}) \Big) \to \Big( D^*(\mathcal{A}) \stackrel{G}{\longrightarrow} D(\mathcal{B}) \Big)$$

such that

$$\rho = (c \circ Q_A) \circ \rho_F.$$

This definition is a mouthful. We will trade a little bit of rigor in favor of intuition. First of all we will drop the localization functors $Q$ from notation because the category $K^*(\mathcal{A})$ (resp. $K^*(\mathcal{B})$) has the same objects as $D^*(\mathcal{A})$ (resp. $D^*(\mathcal{B})$). These categories differ only through their morphisms. The functor $RF$ assigns to each complex $X^\bullet$ in $C^*(\mathcal{A})$ a complex $RF(X^\bullet)$ in $C(\mathcal{B})$ and to *every roof*

$$\varphi : X^\bullet \stackrel{s}{\leftsquigarrow} \text{something} \stackrel{f}{\to} Y^\bullet \in \mathrm{Hom}_{D^*(\mathcal{A})}(X^\bullet, Y^\bullet)$$

a roof $RF(\varphi) : RF(X^\bullet) \to RF(Y^\bullet)$ in $\mathcal{D}(B)$. These assignments behave nicely with respect to the morphisms in $D^*(\mathcal{A})$ which come from genuine morphisms of complexes. More precisely there exist roofs

$$\rho_F(X^\bullet) : F(X^\bullet) \to RF(X^\bullet)$$

(call them *quasi-resolutions*) such that for every complexes in $X^\bullet$, $Y^\bullet$ in $\mathcal{A}$ and every genuine morphism of complexes we have a commutative diagram in $D(\mathcal{B})$

$$\begin{array}{ccc} RF(X^\bullet) & \xrightarrow{RF(\varphi)} & RF(Y^\bullet) \\ {\scriptstyle \rho_F(X^\bullet)}\uparrow & & \uparrow{\scriptstyle \rho_F(Y^\bullet)} \\ F(X^\bullet) & \xrightarrow{F(\varphi)} & F(Y^\bullet) \end{array} .$$

We can say that $\rho_F$ defines a a *coherent system of quasi-resolutions*.

The universality property states that given a similar exact functor $D^*(\mathcal{A}) \xrightarrow{G} D^*(\mathcal{B})$ equipped as well with a "coherent system of quasi-resolutions" $\rho(X^\bullet) : F(X^\bullet) \to G(X^\bullet)$ there exists a unique systems of "*compatibilities*"

$$D^*(\mathcal{A}) \ni X^\bullet \mapsto c_{X^\bullet} \in \operatorname{Hom}_{D(\mathcal{B})}\big( RF(X^\bullet), G(X^\bullet) \big),$$

such that for any complexes $X^\bullet, Y^\bullet$ and any roof $\varphi$ between them we have the following commutative diagrams.

$$\begin{array}{ccc} G(X^\bullet) & & \\ {\scriptstyle \rho(X^\bullet)}\uparrow & \nwarrow{\scriptstyle c_{X^\bullet}} & \\ F(X^\bullet) & \xrightarrow[\rho_F(X^\bullet)]{} & RF(X^\bullet) \end{array} \quad , \quad \begin{array}{ccc} G(X^\bullet) & \xrightarrow{G(\varphi)} & G(Y^\bullet) \\ {\scriptstyle c_{X^\bullet}}\uparrow & & \uparrow{\scriptstyle c_{Y^\bullet}} \\ RF(X^\bullet) & \xrightarrow[RF(\varphi)]{} & RF(Y^\bullet) \end{array}$$

The derived functor, when it exists, is unique up to isomorphism. We see that to establish the existence of the derived functor we need to have a procedure of constructing resolutions of complexes. We indicate below one such situation.

**Definition 1.11.** The left exact functor $F : \mathcal{A} \to \mathcal{B}$ is said to admit enough $F$-injective objects if there exists a generating subcategory $\mathcal{J}$ of $A$ such that the restriction of $F$ to $\mathcal{J}$ is exact, i.e. if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{J}$ then

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a short exact sequence in $\mathcal{B}$. The subcategory $\mathcal{J}$ is also called a *class of objects adapted to* $F$.

*Remark* 1.12. Observe that given a left exact functor $F : \mathcal{A} \to \mathcal{B}$ and a short exact sequence of injective objects in $\mathcal{A}$

$$0 \to I' \to I \to I'' \to 0$$

the resulting sequence

$$0 \to F(I') \to F(I) \to F(I'') \to 0$$

is also short exact. In particular, if $\mathcal{A}$ has enough injectives it also has enough $F$-injective objects.

□

**Proposition 1.13.** *If $F$ admits enough $F$-injective objects then there exists a derived functor*

$$R^+F : D^+(\mathcal{A}) \to D(\mathcal{B}).$$

It is constructed as follows.

**Step 1:** *Localize along a class of adapted objects*. Fix a class $\mathcal{J}$ of objects adapted to $F$. Form the derived category $D^+(\mathcal{J})$ described in Remark 1.4.

**Step 2:** *The construction of the derived functor.* Fix an equivalence

$$U : D^+(\mathcal{A}) \to D^+(\mathcal{J}).$$

We have a functor

$$K^+(\mathcal{J}) \xrightarrow{K^+F} K^+(\mathcal{B}) \to D^+(\mathcal{B}).$$

By definition $F$ maps acyclic complexes in $\mathcal{J}$ to acyclic complexes in $\mathcal{B}$ and in particular it maps qis's to isomorphisms in $D^+(\mathcal{B})$. We deduce the existence of a functor

$$D^+(\mathcal{J}) \to D^+(\mathcal{B})$$

Composing this functor with the equivalence $U$ we obtain a functor

$$RF^+ : D^+(\mathcal{A}) \to D^+(\mathcal{B}).$$

□

**Definition 1.14.** Suppose the left exact functor $F : \mathcal{A} \to \mathcal{B}$ admits enough $F$-injective objects.

(a) For every complex $X^\bullet$ in $\mathcal{A}$ we define the $F$-hypercohomology of $X^\bullet$ to be the cohomology of $RF^*(X^\bullet)$ (this complex in $\mathcal{B}$ is unique up to a an isomorphism in $D(\mathcal{B})$). We denote this hypercohomology by

$$\mathbb{R}^n F(X^\bullet) :\cong H^n(RF(X^\bullet)).$$

(b) Every object $X \in \mathcal{A}$ can be identified with a complex $[X]$ concentrated in dimension zero. In this case we set

$$R^n F(X) := \mathbb{R}^n F([X])$$

(c) An object $A \in \mathcal{A}$ is called $F$-*acyclic* if $R^n F(A) \cong 0, \forall n > 0$.

□

Observe that if $F$ admits enough injective $F$-injective objects, then the class of $F$-acyclic objects is adapted to $F$. In particular, this implies that the subcategory of $F$-acyclic objects is a generating subcategory and thus to compute $RF(X^\bullet)$ for a complex $X^\bullet \in C^+(\mathcal{A})$ it suffices to find a resolution of $X^\bullet$ by $F$-acyclic objects.

The following result follows immediately from the above discussion.

**Proposition 1.15.** *Suppose $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are Abelian categories and*

$$F : \mathcal{A}_1 \to \mathcal{A}_2, \;\; F_2 : \mathcal{A}_2 \to \mathcal{A}_3$$

*are two left exact functors. Assume $\mathcal{J}_1$ is a class of $F_1$-acyclic and $\mathcal{J}_2$ is a class of $F_2$-acyclic objects such that*

$$F_1(\mathcal{J}_1) \subset \mathcal{J}_2.$$

*Then $\mathcal{J}_1$ is a class of $F_2 \circ F_1$-acyclic objects and we have an isomorphism*

$$R^+(F_2 \circ F_1) \cong R^+F_2 \circ R^+F_1.$$

**Example 1.16** (The hypercohomology spectral sequences)**.** Let us describe a basic procedure for computing the hypercohomology groups.

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, and that $\mathcal{A}$ has enough injective. Denote by $\mathcal{I}$ full additive subcategory of $\mathcal{A}$ consisting of injective objects. Given a complex $X^\bullet \in C^+(\mathcal{A})$ we can find a quasi-isomorphism

$$X^\bullet \overset{\phi}{\rightsquigarrow} I^\bullet$$

and then the hypercohomology groups of $X^\bullet$ are isomorphic to $H^n(F(I^\bullet))$. To compute them we can use the *hypercohomology spectral sequence*. Suppose first $X^n = 0$ for $n < 0$. First we choose a *Cartan-Eilenberg resolution*. This is a double complex $(I^{\bullet,\bullet}, d_I, d_{II})$ together with a morphism

$$\rho : (X^\bullet, d_X) \to (I^{\bullet,0}, d_I)$$

satisfying the following conditions.

(i) The objects $I^{\bullet,\bullet}$ are injective.
(ii) For every $p \geqslant 0$ the $p$-th column complex

$$0 \to X^p \xrightarrow{\rho^n} I^{p,0} \xrightarrow{d_{II}} I^{p,1} \xrightarrow{d_{II}} I^{p,2} \xrightarrow{d_{II}} \cdots$$

is acyclic.
(iii) For every $q \geqslant 0$ we have a $q$-th row complex

$$I^{0,q} \xrightarrow{d_I} I^{1,q} \xrightarrow{d_I} I^{2,q} \xrightarrow{d_I} \cdots$$

whose co-cycles $Z_I^{\bullet,q}$, co-boundaries $B_I^{\bullet,q}$, and cohomologies $H_I^{\bullet,q}$ are injective objects.
(iv) For every $p \geqslant 0$ the complexes $(Z_I^{p,\bullet}, d_{II})$ $(B_I^{\bullet,q}, d_{II})$ and $(H_I^{p,\bullet}, d_{II})$ are resolutions of the objects $Z^p(X^\bullet, d_X)$, $B^p(X^\bullet, d_X)$ and $H^p(X^\bullet, d_X)$.

Form the *total complex*

$$\mathbf{Tot}^m(I^{\bullet,\bullet}) = \oplus_{j+k=m} I^{j,k}, \quad D = d_I + d_{II} : \Sigma(I)^\bullet \to \Sigma(I)^{\bullet+1}.$$

The objects $\mathbf{Tot}^\bullet(I)$ are injective, and the map $\rho : X^\bullet \to \mathbf{Tot}^\bullet(I)$ is a quasi-isomorphism. Thus the cohomology of $F(\mathbf{Tot}^\bullet(I))$ is isomorphic to the $F$-hypercohomology of $X^\bullet$. There are two spectral sequences converging to this cohomology. The first spectral sequence

$${}'E_1^{p,q} = H_{II}^q(F(I^{p,\bullet}))$$

Using condition (ii) of the Cartan-Eilenberg resolution we deduce

$$H_{II}^q(F(I^{p,\bullet})) = R^qF(X^p).$$

We conclude

$${}'E_2^{p,q} = H^p(R^qF(X^\bullet)).$$

As for the second spectral sequence we have

$${}''E_1^{p,q} = H_I^p(F(I^{\bullet,q})).$$

Using the condition (iv) in the definition of a Cartan-Eilenberg complex we deduce

$${}''E_2^{p,q} = R^qF(H^p(X)).$$

This spectral sequence is called the *hypercohomology spectral sequence*.

Suppose we are in the context of Proposition 1.15, where we had two left exact functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}.$$

such that

$$R(G \circ F) \cong RG \circ RF.$$

Arguing in a similar fashion we deduce that for every object $X \in A$ we obtain a spectral sequence $E_r^{\bullet,\bullet}$ which converges to $\mathbb{R}^\bullet(G \circ F)(X)$, whose $E_2$-term is

$$E_2^{p,q} = R^pG(R^qF(X)).$$

□

**Example 1.17** ($R\operatorname{Hom}$)**.** Suppose $\mathcal{A}$ is an Abelian category. We have a functor

$$\operatorname{Hom}^\bullet(-.-) : C(\mathcal{A})^{op} \times C(\mathcal{A}) \to C(\mathbf{Ab})$$
$$(X^\bullet, Y^\bullet) \to \operatorname{Hom}^\bullet(X^\bullet, Y^\bullet),$$

where

$$\operatorname{Hom}^n(X^\bullet, Y^\bullet)^n := \prod_p \operatorname{Hom}_{\mathcal{A}}(X^p, Y[n]^p),$$

with differential

$$D = D_{\operatorname{Hom}} : \operatorname{Hom}^n(X^\bullet, Y^\bullet) \to \operatorname{Hom}^{n+1}(X^\bullet, Y^\bullet)$$
$$\operatorname{Hom}^n(X^\bullet, Y^\bullet) \ni \varphi \mapsto D\varphi = d_Y\varphi - (-1)^n\varphi d_X \in \operatorname{Hom}^{n+1}(X^\bullet, Y^\bullet).$$

One can show that $\operatorname{Hom}^\bullet$ induces a functor

$$\operatorname{Hom}^\bullet : K(\mathcal{A})^{op} \times K(\mathcal{A}) \to K(\mathbf{Ab})$$

and

$$H^n(\operatorname{Hom}^\bullet(X^\bullet, Y^\bullet)) = [X^\bullet, Y[n]^\bullet].$$

Assume that $\mathcal{A}$ has enough injectives. Fix $X^\bullet \in K(\mathcal{A})$. The functor

$$\operatorname{Hom}^\bullet(X^\bullet, -) : K^+(\mathcal{A}) \to K(\mathbf{Ab})$$

enjoys the following properties.

① If $I^\bullet \in K^+(\mathcal{A})$ is a complex of injective objects quasi-isomorphic to zero then the complex $\operatorname{Hom}^\bullet_{C^+(\mathcal{A})}(X^\bullet, I^\bullet)$ is quasi-isomorphic to zero. Indeed we have

$$H^n(\operatorname{Hom}^\bullet_{C^+(\mathcal{A})}(X^\bullet, I^\bullet)) \cong [X^\bullet, I[n]^\bullet].$$

Theorem 1.2 shows that we have isomorphisms

$$[I^\bullet, I^\bullet] \cong [0, I^\bullet], \;\; [I^\bullet, 0] \cong [0, 0] \cong 0$$

so that the quasi-isomorphism $I^\bullet \rightsquigarrow 0$ is a homotopy equivalence whence the desired conclusion.

② If $f : I^\bullet \rightsquigarrow J^\bullet$ is a qis between bounded from below complexes of injective objects then the induced map

$$\bar{f} := \operatorname{Hom}^\bullet_{C^+(\mathcal{A})}(X^\bullet, I^\bullet) \to \operatorname{Hom}_{C^+(\mathcal{A})}(X^\bullet, J^\bullet)$$

is a quasi-isomorphism.

Indeed one can show that the cone of $\bar{f}$ is the complex $\operatorname{Hom}^\bullet_{C^+(\mathcal{A})}(X^\bullet, C(f))$, where $C(f)$ is the cone of $f$ and it is a complex of injective objets. Since $f$ is a qis we deduce $0 \rightsquigarrow C(f)$ and we can now conclude using ①.

If we denote by $\mathcal{I} = \mathcal{I}_{\mathcal{A}}$ the full subcategory of $\mathcal{A}$ consisting of injective objects we deduce that the functor

$$\operatorname{Hom}^\bullet(X^\bullet, -) : K^+(\mathcal{I}_A) \to K^+(\mathbf{Ab})$$

sends quasi-isomorphism to isomorphisms. In particular we obtain a functor

$$R^+_{II}\operatorname{Hom}(X^\bullet, -) : D^+(\mathcal{A}) \to D^+(\mathbf{Ab}).$$

This functor maps distinguished triangles to distinguished triangles. We obtain a bi-functor

$$R_{II}\operatorname{Hom}(-, -) : K(\mathcal{A})^{op} \times D^+(\mathcal{A}) \to D^+(\mathbf{Ab}).$$

Observe that for any complex of injectives $I^\bullet$ and any qis $f : A^\bullet \rightsquigarrow B^\bullet$ we get according to Theorem 1.2 a qis

$$\underline{f} : \operatorname{Hom}^\bullet(B^\bullet, I^\bullet) \to \operatorname{Hom}^\bullet(A^\bullet, I^\bullet).$$

We thus obtain a functor

$$R_I R_{II}^+ \operatorname{Hom} : D(\mathcal{A})^{op} \times D^+(\mathcal{A}) \to D^+(\mathbf{Ab}).$$

Using Theorem 1.2 again we deduce

$$H^n\big(\, R_I R_{II}^+ \operatorname{Hom}(X^\bullet, Y^\bullet)\,\big) = \operatorname{Hom}_{D(\mathcal{A})}(X^\bullet, Y[n]^\bullet) = \mathbb{E}\mathrm{xt}^n(X^\bullet, Y^\bullet).$$

Observe that if $X$ is an object in $A$ identified with a complex concentrated in dimension $0$ then the Hypercohomology spectral sequence shows that there exists a spectral sequence converging to $\mathbb{E}\mathrm{xt}^\bullet(X, Y^\bullet)$ whose $E_2$-term is

$$E_2^{p,q} = \operatorname{Ext}^p(X, H^q(Y^\bullet)).$$

□

**Example 1.18** (Products)**.** Suppose $\mathcal{A}$ is an Abelian category with sufficiently many injectives and $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor to another Abelian category. For any objects $X, Y \in \mathcal{A}$ we have a natural product

$$\operatorname{Ext}^n(X, Y) \times R^p F(X) \to R^{n+p}(Y),\ \ (f, \alpha) \mapsto f \cup \alpha$$

define as follows. Choose injective resolutions $I^\bullet$ for $X^\bullet$ and $J^\bullet$ for $Y$. Then

$$\alpha \in H^n(F(X^\bullet)),\ \ f \in [X^\bullet, Y[p]^\bullet].$$

In particular, $f$ induces morphisms

$$F(f) : F(X^\bullet) \longrightarrow F(Y)[p]^\bullet,\ \ RF(f) : H^n(F(X^\bullet)) \to H^{n+p}(Y^\bullet))$$

We set

$$f \cup \alpha = R^n F(f)(\alpha).$$

□

## 2. Basic operations on sheaves

For every commutative ring $R$ we denote by $\mathbf{pSh}_R(X)$ (resp. $\mathbf{Sh}_R(X)$) the category of presheaves (resp. sheaves) of $R$-modules on $X$. For every open set $U \subset X$ and every presheaf $\mathcal{S}$ over $X$ we will denote he space of sections of $\mathcal{S}$ over $U$ by $\mathcal{S}(U)$ or $\Gamma(U, \mathcal{S})$.

A morphism of (pre)sheaves $f : \mathcal{S}_0 \to \mathcal{S}_1$ is a collection of morphisms

$$f_U : \mathcal{S}_0(U) \to \mathcal{S}_1(U),$$

one for each open set $U \hookrightarrow X$, compatible with the restriction maps. $\mathbf{pSh}_R(X)$ is an additive category, while $\mathbf{Sh}_R(X)$ is an Abelian category.

Note that $\mathbf{Sh}_R(X)$ is naturally a full subcategory of $\mathbf{pSh}_R(X)$. Moreover there exists a functor

$$^+ : \mathbf{pSh}_R(X) \to \mathbf{Sh}_R(X)$$

the *sheafification* which is a left adjoint to the inclusion functor $i : \mathbf{Sh}_R(X) \to \mathbf{pSh}_R$, i.e. there exists an isomorphism

$$\operatorname{Hom}_{\mathbf{Sh}_R}(\mathcal{F}^+, \mathcal{G}) \cong \operatorname{Hom}_{\mathbf{pSh}_R}\big(\mathcal{F}, i(\mathcal{G})\,\big), \tag{2.1}$$

natural in $\mathcal{F} \in \mathbf{pSh}_R$ and $\mathcal{G} \in \mathbf{Sh}_R$.

*Remark* 2.1. In practice, most sheaves are defined as sheafifications of some presheaves. The above isomorphism essentially states that if $\mathcal{G}$ is a sheaf, and $\mathcal{F}^+$ is the sheafification of $\mathcal{F}$ that any morphism $\mathcal{F}^+ \to \mathcal{G}$ is uniquely determined by a collection of morphisms

$$\mathcal{F}(U) \to \mathcal{G}(U)$$

compatible with the restriction maps. In practice, this is how most morphisms of sheaves are described, by first indicating a morphisms of presheaves and then using the tautological isomorphism (2.1). □

We will write $\mathbf{Sh}(X)$ for $\mathbf{Sh}_{\mathbb{Z}}(X)$. For every Abelian group $G$ we denote by $\underline{G} = {}_X\underline{G} \in \mathbf{Sh}(X)$ the associated constant sheaf over $X$. For every sheaf $\mathcal{S} \in \mathbf{Sh}(X)$ and every *open* subset $U \subset X$ we denote by $\mathcal{S}|_U$ the restriction of $\mathcal{S}$ to $U$, i.e. the sheaf

$$U \overset{open}{\hookleftarrow} V \mapsto \mathcal{S}(V).$$

For $\mathcal{F} \in \mathbf{Sh}_R(X)$, $U \subset X$ and $s \in \Gamma(U, \mathcal{F})$ we set

$$\operatorname{supp} s := \{u \in U;\ \ s_u \neq U\}.$$

$\operatorname{supp} s$ is *closed in* $U$, but it may not be closed in $X$.

We denote by $\operatorname{Hom}(\mathcal{S}_0, \mathcal{S}_1) = \operatorname{Hom}_{\mathbf{Sh}(X)}(\mathcal{S}_0, \mathcal{S}_1)$ the Abelian group of morphisms $\mathcal{S}_0 \to \mathcal{S}_1$. We denote by $\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1)$ the *sheaf*

$$U \mapsto \underline{Hom}(\mathcal{S}_0, \mathcal{S}_1)(U) = \operatorname{Hom}_{\mathbb{Z}}(\mathcal{S}_0|_U, \mathcal{S}_1|_U).$$

By definition

$$\operatorname{Hom}(\mathcal{S}_0, \mathcal{S}_1) := \Gamma(X, \underline{Hom}(\mathcal{S}_0, \mathcal{S}_1)).$$

Note that if $\mathcal{S}$ and $\mathcal{T}$ are two sheaves of Abelian groups on $X$ and $x_0 \in X$ then there exists a natural morphism of Abelian groups

$$\underline{Hom}(\mathcal{S}, \mathcal{T})_{x_0} \to \operatorname{Hom}(\mathcal{S}_{x_0}, \mathcal{T}_{x_0}).$$

In general this morphism is neither injective nor surjective.

**Example 2.2.** Suppose $X$ is a smooth manifold and $\mathcal{E}_X$ denotes the sheaf of smooth complex valued functions on $X$. A partial differential operator $P$ defines a morphism $P \in \operatorname{Hom}(\mathcal{E}_X, \mathcal{E}_X)$. Conversely, every endomorphism of the sheaf $\mathcal{E}_X$ is a partial differential operator, [9, 10]. □

Given two sheaves $\mathcal{S}_0, \mathcal{S}_1 \in \mathbf{Sh}_R(X)$ we define their tensor product as the sheaf $\mathcal{S}_0 \otimes_R \mathcal{S}_1$ associated to the presheaf

$$U \mapsto \mathcal{S}_0(U) \otimes \mathcal{S}_1(U).$$

*Remark* 2.3. Let us point out that the natural map

$$\mathcal{S}_0(U) \otimes_R \mathcal{S}_1(U) \to \Gamma(U, \mathcal{S}_0 \otimes_R \mathcal{S}_1)$$

need not be an isomorphism. Consider for example $X = \mathbb{RP}^2$, $\mathcal{S}_0$=$\mathbb{Z}$-orientation sheaf of $\mathbb{RP}^2$ and $\mathcal{S}_1$= the $\mathbb{Z}/2$-orientation sheaf, or equivalently the constant sheaf $\underline{\mathbb{Z}/2}$. Then $\mathcal{S}_0 \otimes \mathcal{S}_1 = \mathcal{S}_1$ however $\mathcal{S}_0(X) = 0$ so that

$$\mathcal{S}_0(X) \otimes \mathcal{S}_1(X) = 0 \not\cong (\mathcal{S}_0 \otimes \mathcal{S}_1)(X) = \mathbb{Z}/2.$$

□

The functors $\underline{Hom}$ and $\otimes$ are related by the *adjunction formula* which states that there exists an isomorphism

$$\operatorname{Hom}(\mathcal{F}\otimes\mathcal{G},\mathcal{H}) \cong \operatorname{Hom}(\mathcal{F},\underline{Hom}(\mathcal{G},\mathcal{H})) \tag{2.2}$$

natural in $\mathcal{F},\mathcal{G},\mathcal{H}\in\mathbf{Sh}_R(X)$.

**Proposition 2.4.** *(a)$\mathbf{Sh}_R(X)$ is an Abelian category which has enough injectives.*
*(b) The functor*

$$\Gamma(X,-):\mathbf{Sh}_R(X)\to {}_R\mathbf{Mod}$$

*is left exact.*

**Proof** (a) Let $\mathcal{F}\in\mathbf{Sh}_R(X)$. For every $x\in X$ we consider an embedding of $\mathcal{F}_x$ in an injective $R$-module $M_x$. Now form

$$\mathcal{M}=\prod_{x\in X}M_x,\;\;\Gamma(U,\mathcal{I})=\Big\{(m_x)_{x\in X};\;\;m_x=0,\quad\forall x\in X\backslash U\}$$

Then for every $\mathcal{S}\in\mathbf{Sh}_R(X)$ we have

$$\operatorname{Hom}(\mathcal{S},\mathcal{M})=\prod_{x\in X}\operatorname{Hom}(\mathcal{S}_x,M_x)$$

which shows that $\mathcal{M}$ is injective. Part (b) is elementary.

□

The *Godement resolution* of a sheaf $\mathcal{F}\in\mathbf{Sh}_R(X)$ is constructed inductively as follows. We set

$$G^0_{\mathcal{F}}:=\prod_{x\in X}\mathcal{F}_x,\;\;G^0_{\mathcal{F}}(U):=\prod_{u\in U}\mathcal{F}_u.$$

We get a natural inclusion

$$\mathcal{F}\to G^0_{\mathcal{F}}.$$

We set

$$\mathcal{F}_1:=\operatorname{coker}(\mathcal{F}\to G^0_{\mathcal{F}}).$$

Then $G^1_{\mathcal{F}}:=G^0_{\mathcal{F}_1}$. Iterating we obtain the Godement resolution

$$0\to\mathcal{F}\to G^0_{\mathcal{F}}\to G^1_{\mathcal{F}}\to\cdots.$$

For every sheaf $\mathcal{F}\in\mathbf{Sh}_R(X)$ we get two left exact functors

$$\operatorname{Hom}(-,\mathcal{F}):\mathbf{Sh}_R(X)^{op}\to {}_R\mathbf{Mod}$$

$$\underline{Hom}(-,\mathcal{F}):\mathbf{Sh}_R(X)^{op}\to\mathbf{Sh}_R(X).$$

Given a continuous map $f:X\to Y$ we get two functors

$$f_*:\mathbf{Sh}(X)\to\mathbf{Sh}(Y)\;\text{ and }\;f^{-1}:\mathbf{Sh}(Y)\to\mathbf{Sh}(X).$$

More explicitly, for every $\mathcal{S}\in\mathbf{Sh}(X)$ the *push-forward* $f_*\mathcal{S}$ is the *sheaf*

$$Y\supset U\longmapsto\mathcal{S}(f^{-1}(U)),$$

while for $\mathcal{T}\in\mathbf{Sh}(Y)$ we define $f^{-1}\mathcal{T}$ as the sheaf associated to the presheaf $(f^{-1})_0\mathcal{T}$ defined by

$$X\supset V\longmapsto\Gamma(V,(f^{-1})_0\mathcal{T}):=\varinjlim_{U\supset f(V)}\Gamma(U,\mathcal{T}).$$

$f_*$ is *left exact* while $f^{-1}$ is *exact*. For every sheaf $\mathcal{T}\in\mathbf{Sh}(Y)$ and every open set $U\subset Y$ we have

$$\Gamma(U,f_*f^{-1}\mathcal{T})=\Gamma(f^{-1}(U),f^{-1}\mathcal{T}).$$

Since $U \supset f(f^{-1}(U))$ we deduce from the universality property of inductive limit that there exists a natural map

$$\Gamma(U, \mathcal{T}) \to \Gamma(f^{-1}(U), (f^{-1})_0\mathcal{T}) \to \Gamma(U, f_*f^{-1}\mathcal{T}).$$

This defines a canonical morphism

$$\mathbb{A}_f : \mathcal{T} \longrightarrow f_*f^{-1}\mathcal{T}.$$

We regard $\mathbb{A}_f$ as a natural transformation between the functors $\mathbf{1}_{\mathbf{Sh}(Y)}$ and $f_*f^{-1}$. This is known as the *adjunction functor*.

Given a morphism

$$\phi \in \mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{T}, \mathcal{S})$$

we obtain a morphism

$$f_*\phi \in \mathrm{Hom}_{\mathbf{Sh}(X)}(f_*f^{-1}\mathcal{T}, f_*\mathcal{S})$$

and thus a morphism

$$f_*\phi \circ \mathbb{A}_f \in \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathcal{T}, f_*\mathcal{S}).$$

We obtain in this fashion a natural *isomorphism* i.e. we have natural isomorphisms

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{T}, \mathcal{S}) \cong \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathcal{T}, f_*\mathcal{S}), \;\; \forall \mathcal{S} \in \mathbf{Sh}(X), \;\; \mathcal{T} \in \mathbf{Sh}(Y). \tag{2.3}$$

which shows that $f_*$ is the *right adjoint* of $f^{-1}$. The adjunction morphism $\mathbb{A}_f$ corresponds to $\mathbb{1}_{f^{-1}\mathcal{T}}$ via (2.3).

**Example 2.5.** (a) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is the continuous map $(x, y) \mapsto x^2 + y^2$. We denote by $\mathcal{T}$ the sheaf of continuous functions on $\mathbb{R}$. Then $f^{-1}\mathcal{T}$ is the subsheaf of the sheaf of continuous functions on $\mathbb{R}^2$ whose sections are the continuous functions constant along the level sets of $f$.

Observe that for every open interval $I = (\ell, L) \subset \mathbb{R}$ we have

$$\Gamma(I, f_*f^{-1}\mathcal{T}) = \begin{cases} C\big((\ell, L)\big) & \text{if} \quad \ell > 0 \\ 0 & \text{if} \quad L < 0 \\ C\big([0, L)\big) & \text{if} \quad \ell < 0 < L \end{cases}.$$

(b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is the holomorphic map $z \mapsto z^3$. We denote by $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{C}$. Denote by $\rho$ a primitive cubic root of $1$. Then

$$(f_*f^{-1}\mathcal{O})_{z_0} \cong \begin{cases} \mathcal{O}^3_{z_0} & \text{if} \quad z_0 \neq 0 \\ \mathcal{O}_0 & \text{if} \quad z_0 = 0 \end{cases}.$$

Note that the holomorphic map $f$ induces a morphism of sheaves

$$f^{\#} : f^{-1}\mathcal{O} \to \mathcal{O}.$$

□

For every sheaf $\mathcal{S} \in \mathbf{Sh}(X)$ and every subset $i : A \hookrightarrow X$ we set

$$\mathcal{S}|_A := i^{-1}\mathcal{S} \in \mathbf{Sh}(A).$$

We set

$$\Gamma(A, \mathcal{S}) := \Gamma(A, \mathcal{S}|_A) = \varinjlim_{U \supset A} \Gamma(U, \mathcal{S}).$$

We have a tautological map

$$\Gamma(X, \mathcal{S}) \to \Gamma(A, \mathcal{S}).$$

A *family of supports*[2] on a paracompact topological space is a collection $\Phi$ of *closed subsets* of $X$ satisfying the following conditions.

[2]Traditionally, a family of supports with these properties is called a *paracompactifying* family of supports.

- Any finite union of sets in $\Phi$ is a set in $\Phi$.
- Any closed subset of a set in $\Phi$ is a set in $\Phi$.
- Every set in $\Phi$ admits a neighborhood which is a set in $\Phi$.

**Example 2.6.** (a) The collection of all closed subsets of $X$ is a family of supports. Usually we will not indicate this family by any symbol. Sometimes we will use the notation $cl$ for this family.
(b) If $X$ is a locally compact space then the collection of all compact subsets is a family of supports. We will denote this family by $c = c_X$.
(c) Suppose $Y \hookrightarrow X$ is a locally closed subset of the paracompact space $X$, and $\Phi$ is a family of supports on $X$. Then the collection

$$\Phi\mid_Y := \Bigl\{ S \in \Phi;\ \ S \subset Y \Bigr\}$$

is a family of supports on $Y$. For example if $X = \mathbb{C}$, $Y$ is the open unit disk in $\mathbb{C}$ and $\Phi$ consist of all the closed sets in $\mathbb{C}$ then $\Phi|_Y$ consists of all the compact subsets of the unit disk.
(d) Suppose $Y \hookrightarrow X$ is a locally closed subset of the paracompact space $X$, and $\Phi$ is a family of supports on $X$. Then the collection

$$\Phi \cap Y := \Bigl\{ S \cap Y;\ \ S \in \Phi \Bigr\}$$

is a family of supports on $Y$. □

*In the sequel all topological spaces will be tacitly assumed to be locally compact, unless otherwise indicated.*

Suppose $\Phi$ is a family of supports. For every open set $U \hookrightarrow X$ we set

$$\Gamma_\Phi(U, \mathcal{S}) := \bigl\{ s \in \Gamma(U, \mathcal{S});\ \ \operatorname{supp} s \in \Phi \bigr\}.$$

The resulting functor

$$\Gamma_\Phi(X, -) : \mathbf{Sh}_R(X) \to {}_R\mathbf{Mod}\,.$$

is left exact. In particular, $\Gamma_c$ denotes sections with compact support.

**Example 2.7.** Suppose $X \subset \mathbb{R}$ is the set of all integers equipped with the induced topology and $\underline{\mathbb{R}}$ is the constant sheaf on $X$. Then

$$\Gamma(X, \underline{\mathbb{R}}) = \prod_{n \in \mathbb{Z}} \mathbb{R},\ \ \Gamma_c(X, \underline{\mathbb{R}}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{R}.$$

□

Suppose $f : Y \to Z$ is continuous map between two spaces. Then $f$ defines a functor

$$f_! : \mathbf{Sh}(Y) \to \mathbf{Sh}(Z),$$

where for every $\mathcal{S} \in \mathbf{Sh}(Y)$ we define $f_!\mathcal{S}$ as the *sheaf*

$$U \mapsto \Gamma(U, f_!\mathcal{S}) = \Gamma_f(f^{-1}(U), \mathcal{S}) := \bigl\{ s \in \Gamma(f^{-1}(U), \mathcal{S});\ \ \operatorname{supp} s \xrightarrow{f} U \ \ \text{is proper}^{3} \bigr\}.$$

The resulting functor $f_! : \mathbf{Sh}_R(X) \to \mathbf{Sh}_R(Y)$ is left-exact.

A subset $W$ of a topological space is called *locally closed* if any point $w \in W$ admits an open neighborhood $V$ in $X$ such that $V \cap W$ is closed in $V$. Observe that the open subsets are locally closed. The closed subsets are locally closed. In fact a subset is locally closed if and only if it can be described as the intersection of a closed subset of $X$ with an open subset of $X$.

[3] This means that $f$ is closed (maps closed sets to closed sets) and its fibers are compact.

Consider the inclusion $i : Z \hookrightarrow X$ of a subset $Z$. We get an *exact* functor

$$i_! : \mathbf{Sh}_R(Z) \to \mathbf{Sh}_R(X).$$

Note that

$$\Gamma(U, i_!\mathcal{S}) = \Big\{ s \in \Gamma(U \cap Z, \mathcal{S});\ \ \operatorname{supp} s\ \text{ is closed in } U \Big\}.$$

For every sheaf $\mathcal{S}$ on $X$ we set

$$\mathcal{S}_Z := i_!\mathcal{S}|_Z = i_! i^{-1}\mathcal{S}.$$

We have the following result whose proof can be found in [5, II.2.9].

**Proposition 2.8.**

$$Z \quad \textit{locally closed} \Longleftrightarrow (\mathcal{S}_Z)_x \cong \begin{cases} \mathcal{S}_x & \text{if} \quad x \in Z \\ 0 & \text{if} \quad x \in X \backslash Z. \end{cases}$$

*Moreover $\mathcal{S}_Z$ is the unique sheaf on $X$ with the above property.* □

In the sequel we will assume $Z$ is locally closed.

For every locally closed set $Z$ the correspondence $\mathbf{Sh}_R(X) \to \mathbf{Sh}_R(X)$, $\mathcal{S} \mapsto \mathcal{S}_Z$ defines an *exact* functor, and we have a natural isomorphism

$$\mathcal{S}_Z \cong \underline{R}_Z \otimes \mathcal{S}.$$

Note that for every Abelian group $G$ we have

$$i_!({}_Z\underline{G}) = ({}_X\underline{G})_Z =: \underline{G}_Z.$$

Let us emphasize that ${}_Z\underline{G}$ is a sheaf on $Z$, while $\underline{G}_Z$ is a sheaf on the ambient space $X$. When $j : Z \hookrightarrow X$ is the inclusion of a closed subset we have

$$j_! = j_* : \mathbf{Sh}_R(Z) \to \mathbf{Sh}_R(X).$$

and thus, according to (2.3) a natural isomorphism

$$\operatorname{Hom}_{\mathbf{Sh}_R(Z)}(j^{-1}\mathcal{F}, \mathcal{S}) \cong \operatorname{Hom}_{\mathbf{Sh}_R(X)}(\mathcal{F}, j_*\mathcal{S}),\ \ \mathcal{F} \in \mathbf{Sh}_R(X),\ \ \mathcal{S} \in \mathbf{Sh}_R(Z).$$

In particular if we let $\mathcal{S} = j^{-1}\mathcal{F}$ we obtain a morphism

$$\mathcal{F} \to \mathcal{F}_Z = j_*\mathcal{S} \tag{2.4}$$

corresponding to $\mathbb{I}_{j^{-1}\mathcal{F}} \in \operatorname{Hom}_{\mathbf{Sh}_R(Z)}(j^{-1}\mathcal{F}, j^{-1}\mathcal{F})$.

Suppose $i : \mathcal{O} \hookrightarrow X$ is the inclusion of an open subset. Then $\mathcal{S}_{\mathcal{O}} = i_! i^{-1}\mathcal{S}$ is the sheaf described by

$$X \supset U \longmapsto \mathcal{S}_{\mathcal{O}}(U) = \big\{ s \in \mathcal{S}(\mathcal{O} \cap U);\ \ \operatorname{supp} s\ \text{ is closed in }\ U \big\}. \tag{2.5}$$

We have an isomorphism of Abelian groups

$$\operatorname{Hom}_{\mathbf{Sh}_R(X)}(i_!\mathcal{F}, \mathcal{S}) \cong \operatorname{Hom}_{\mathbf{Sh}_R(\mathcal{O})}(\mathcal{F}, i^{-1}\mathcal{S}) = \operatorname{Hom}_{\mathbf{Sh}_R(\mathcal{O})}(\mathcal{F}, \mathcal{S}|_{\mathcal{O}}) \tag{2.6}$$

which is natural in $\mathcal{F} \in \mathbf{Sh}_R(\mathcal{O})$ and $\mathcal{S} \in \mathbf{Sh}_R(X)$ . In particular, if we let $\mathcal{F} = i^{-1}\mathcal{S}$ we obtain a natural *extension by zero* morphism

$$\tau : \mathcal{S}_{\mathcal{O}} \to \mathcal{S} \tag{2.7}$$

corresponding to $\mathbb{I}_{i^{-1}\mathcal{S}} \in \operatorname{Hom}_{\mathbf{Sh}_R(\mathcal{O})}(i^{-1}\mathcal{S}, i^{-1}\mathcal{S})$.

Let us describe the isomorphism (2.6). Suppose $\Phi \in \operatorname{Hom}_{\mathbf{Sh}(\mathcal{O})}(\mathcal{F}, i^{-1}\mathcal{S})$ so that $\Phi$ is described by a family of morphisms

$$\Phi_U : \Gamma(U, \mathcal{F}) \to \Gamma(U, i^{-1}\mathcal{S}) = \Gamma(U, \mathcal{S}),$$

one morphism for each open subset $U \subset \mathcal{O}$. We need to produce a family of morphisms

$$\Psi_V : \Gamma(V, i_!\mathcal{F}) \to \Gamma(V, \mathcal{S}),$$

one morphism for each open subset $V \subset X$. To do this observe that

$$\Gamma(V, i_!\mathcal{F}) = \Big\{ u \in \Gamma(\mathcal{O} \cap V, \mathcal{F});\ \ \operatorname{supp} u \ \text{ is closed in } V \Big\}.$$

The morphism $\Phi_{\mathcal{O}\cap V}$ maps the sections in $\Gamma(\mathcal{O} \cap V, \mathcal{F})$ with support closed in $V$ to sections in $\Gamma(\mathcal{O} \cap V, \mathcal{S})$ with support closed in $V$, i.e. sections of $\mathcal{S}_{\mathcal{O}}$ over $V$. We get a morphism

$$\Psi_V : \Gamma(\mathcal{O} \cap V, \mathcal{F}) \to \Gamma(V, \mathcal{S}_{\mathcal{O}}) \xrightarrow{\tau_V} \Gamma(V, \mathcal{S}).$$

The last morphism coincides with the tautological morphism (2.7).

We define a *good neighborhood* of a locally closed set $S$ to be an open neighborhood $U$ of $S$ such that $S$ is closed with respect to the subspace topology on $U$. For any good neighborhood $U$ of $S$ we define

$$\Gamma_S(U, \mathcal{F}) := \big\{ s \in \Gamma(U, \mathcal{F});\ \ \operatorname{supp} s \subset S \big\} = \ker\big( \mathcal{F}(U) \to \mathcal{F}(U \backslash S) \big).$$

Note that if $V$ is another good neighborhood of $S$ then $U \cap V$ is a good neighborhood of $S$ and we deduce

$$\Gamma_S(U \cap V, \mathcal{F}) = \Gamma_S(U, \mathcal{F}) = \Gamma_S(V, \mathcal{F}).$$

The module $\Gamma_S(U, \mathcal{F})$ is thus independent of the good neighborhood $U$ and we will denote it by $\Gamma_S(X)$. We obtain in this fashion a *left exact* functor

$$\Gamma_S(X, -) : \mathbf{Sh}_R(X) \to {}_R\mathbf{Mod}\,.$$

The correspondence

$$U \mapsto \Gamma_{S\cap U}(U, \mathcal{F})$$

defines a sheaf on $X$ which we denote by $\Gamma_S(\mathcal{F})$. We obtain in this fashion a *left exact* functor

$$\Gamma_S(-) : \mathbf{Sh}_R(X) \to \mathbf{Sh}_R(X).$$

Let us point out that

$$\Gamma_S(\mathcal{F}) \cong \underline{\mathit{Hom}}(\underline{R}_S, \mathcal{F}). \tag{2.8}$$

When $U$ is open we have

$$\Gamma_U(\mathcal{F}) = i_* i^{-1}\mathcal{F}. \tag{2.9}$$

For closed subset $Z \subset X$ we obtain a left exact functor

$$\Gamma_Z(X, -) = \Gamma(X, -) \circ \Gamma_Z(-) : \mathbf{Sh}_R(X) \to {}_R\mathbf{Mod},\ \ \mathcal{F} \mapsto \Gamma(X, \Gamma_Z(\mathcal{F})).$$

Note that in general for $\mathcal{F}, \mathcal{G} \in \mathbf{Sh}_R(X)$ and any locally closed set $S \subset X$ we have

$$\operatorname{Hom}(\mathcal{F}_S, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, \Gamma_S(\mathcal{G})).$$

The last isomorphism follows from the adjunction isomorphism

$$\operatorname{Hom}(\mathcal{F} \otimes \underline{\mathbb{R}}_S, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, \underline{\mathit{Hom}}(\underline{\mathbb{R}}_S, \mathcal{G})).$$

Hence the functor $\mathcal{G} \to \Gamma_S(\mathcal{G})$ is the right adjoint of the exact functor $\mathcal{F} \to \mathcal{F}_S$. Let us observe that for any open set $U \subset X$ and any $\mathcal{S} \in \mathbf{Sh}_R(X)$ we have

$$\operatorname{Hom}_{\mathbf{Sh}_R}(\underline{R}_U, \mathcal{S}) \cong \operatorname{Hom}(\underline{R}, \Gamma_U(\mathcal{S})\,) \cong \Gamma_U(\mathcal{S}) = \Gamma(U, \mathcal{S}).$$

*Remark* 2.9 (Warning). There is a striking similarity between the sheaves $\mathcal{F}_S$ and $\Gamma_S(\mathcal{F})$. Are they really different?

If for example $\mathcal{F} = \underline{\mathbb{R}}$ and $S$ is an open disk in $\mathbb{R}^2$ and $B$ is a small open disk intersecting $S$ but not contained in $S$ then a section of $\mathcal{F}_S$ on $B$ is a section of $\mathcal{F}$ on $B \cap S$ with support closed in $B$. But any section of $\underline{\mathbb{R}}$ on $B \cap S$ is constant so its support is either the empty set or the entire $B \cap S$ which is not closed in $B$. We deduce

$$\Gamma(B, \mathcal{F}_S) = 0.$$

On the other hand

$$\Gamma\big(B,\Gamma_S(\mathcal{F})\,\big) = \Gamma_{S\cap B}(B,\mathcal{F}) = \Gamma(S\cap B,\mathcal{F}) \cong \mathbb{R}.$$

If $S$ is the horizontal axis in $\mathbb{R}^2$ and $\mathcal{F} = \underline{\mathbb{R}}$, then for any open disk $D$ intersecting this axis there is no section of $\underline{\mathbb{R}}$ on $D$ supported on the segment along which $D$ meets the horizontal axis. This shows that $\Gamma_S(\underline{\mathbb{R}}) = 0$. Clearly, $\underline{\mathbb{R}}_S$ is nontrivial.

□

We have constructed several (semi-exact) functors on the Abelian category of sheaves. To describe the associated derived functors it will be convenient to describe large families of sheaves adapted to these functors. Recall that a full *additive* subcategory $\mathcal{J}$ of an Abelian category is adapted to a left-exact additive functor $F : \mathcal{A} \to \mathcal{B}$ if the following conditions hold.

- Every object is a sub-object of an object in $\mathcal{J}$.
- If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$ such that $A, B \in \mathcal{J}$ then $C \in \mathcal{J}$.
- $F$ maps short exact sequences of objects in $\mathcal{J}$ to short exact sequences in $\mathcal{B}$.

**Definition 2.10.** Suppose $\Phi$ is a family of supports on $X$.
(a) A sheaf $\mathcal{S} \to X$ is called *flabby* if for every open set $U \hookrightarrow X$ the restriction map $\mathcal{S}(X) \to \mathcal{S}(U)$ is onto.
(b) A sheaf $\mathcal{S}$ is called $\Phi$-*soft* if for every set $S \in \Phi$ the tautological map

$$\Gamma_\Phi(X,\mathcal{S}) \to \Gamma(S,\mathcal{S}) = \Gamma(S,\mathcal{S}|_S)$$

is surjective.
(c) A sheaf $\mathcal{S}$ is called $\Phi$-fine if the sheaf $\underline{Hom}(\mathcal{S},\mathcal{S})$ is $\Phi$-soft.
(d) The sheaf $\mathcal{S}$ is called *flat* if the functor $\otimes\mathcal{S}$ is exact.

□

*Remark* 2.11. Flabby sheaves are also $\Phi$-soft. When $\Phi$ is the collection of all closed (resp. compact) subsets we will refer to the $\Phi$-soft sheaves simply as soft (resp. $c$-soft) sheaves. □

We have the following sequences of inclusions

$$\text{injective sheaves} \subset \text{ flabby sheaves} \subset \Phi\text{-soft sheaves},$$

$$\Phi\text{-fine sheaves} \subset \Phi\text{-soft sheaves}.$$

**Proposition 2.12** ([8])**.** *Suppose $X, Y$ are locally compact spaces, $Z \subset X$ is a locally closed set and $f : X \to Y$ is a continuous map.*
*(a) Let $\mathcal{S} \in \mathbf{Sh}_R(X)$. The class of injective sheaves is adapted to the functors*

$$\operatorname{Hom}_{\mathbf{Sh}_R(X)}(-,\mathcal{S}),\quad \underline{Hom}_{\mathbf{Sh}_R(X)}(-,\mathcal{S}),\quad f_*.$$

*(b) The class of flabby sheaves is adapted to the functors*

$$\Gamma_\Phi(X,-),\;\; \Gamma_Z(-),\;\; \Gamma_Z(X,-).$$

*(c) The class of c-soft sheaves is adapted to the functors*

$$\Gamma_c(X,-),\;\; f_!.$$

*(d) Let $\mathcal{S} \in \mathbf{Sh}_R(X)$. The class of flat sheaves is adapted to the functor $\otimes\mathcal{S}$.*

To study the compositions of such functors we need to know the behavior of these classes of sheaves with respect to these functors

**Proposition 2.13** ([8]). *Suppose $X, Y$ are locally compact spaces, $Z \subset X$ is a locally closed set and $f : X \to Y$ is a continuous map.*
*(a) If $\mathcal{S}$ is injective then $f_*\mathcal{S}$ and $\Gamma_Z(\mathcal{S})$ are injective, and $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{S})$ is flabby for any sheaf $\mathcal{F} \in \mathbf{Sh}_R(X)$.*
*(b) $f_*(\text{flabby}) = \text{flabby}$, $\Gamma_Z(\text{flabby}) = \text{flabby}$.*
*(c) $f_!(\text{c-soft}) = \text{c-soft}$. If $\mathcal{S}$ is c-soft then so is $\mathcal{S}|_Z$ and $\mathcal{S}_Z$.*

**Definition 2.14.** Suppose $\Phi$ is a family of supports on the space $X$.
(a) We denote by $H^*_\Phi(X, -)$ the cohomology with supports in $\Phi$, i.e. the derived functors of

$$\Gamma_\Phi(X, -) : \mathbf{Sh}_R(X) \to {}_R\mathbf{Mod}\,.$$

(b) If $G$ is an Abelian group we set

$$H^*(X, G) := H^*_{cl}(X, \underline{G}).$$

(c) If $X$ is locally compact then the collection $c$ of compact subsets defines a family of supports and we set

$$H^*_c(X, G) := H^*_c(X, \underline{G}).$$

□

**Lemma 2.15.** *Suppose $\Phi$ is an admissible family of supports on $X$. For every locally closed subset $i : W \hookrightarrow X$ and any sheaves $\mathcal{S} \in \mathbf{Sh}(X)$, $\mathcal{F} \in \mathbf{Sh}(W)$ we have natural isomorphisms*

$$H^*_{\Phi|_W}(W, \mathcal{F}) = H^*_\Phi(X, i_!\mathcal{F}),$$

$$H^*_{\Phi|_W}(W, i^{-1}\mathcal{S}) = H^*_\Phi(X, \mathcal{S}_W).$$

**Proof** The second equality follows from the first. To prove the first equality observe that the functor

$$i_! : \mathbf{Sh}(W) \to \mathbf{Sh}(X)$$

is exact and we have a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{Sh}(W) & \xrightarrow{\;i_!\;} & \mathbf{Sh}(X) \\ & {\scriptstyle \Gamma_{\Phi|_W}(W,-)} \searrow \quad \swarrow {\scriptstyle \Gamma_\Phi(X,-)} & \\ & \mathbf{Ab} & \end{array}$$

□

*Remark* 2.16. Suppose $X$ is a locally compact space. If $\mathcal{O} \hookrightarrow X$ is *open* then for every sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ we have

$$H^*_c(\mathcal{O}, \mathcal{F}|_\mathcal{O}) \cong H^*_c(X, \mathcal{F}_\mathcal{O}).$$

If additionally $\mathcal{O}$ is pre-compact we deduce

$$H^*_c(\mathcal{O}, \mathcal{F}|_\mathcal{O}) \cong H^*(X, \mathcal{F}_\mathcal{O}).$$

If $S \hookrightarrow X$ is *closed* then

$$H^*(X, \mathcal{F}_S) \cong H^*(S, \mathcal{F}|_S), \;\; H^*_c(X, \mathcal{F}_S) \cong H^*_c(S, \mathcal{F}|_S).$$

□

For every closed subset $Z \subset X$ and *every* sheaf $\mathcal{S} \in \mathbf{Sh}(X)$ we have a short exact *excision sequence* in $\mathbf{Sh}(X)$

$$0 \to \mathcal{S}_{X \setminus Z} \to \mathcal{S} \to \mathcal{S}_Z \to 0 \tag{2.10}$$

More generally, given a locally closed subset $Z$, a *closed subset* $Z' \subset Z$ and an *arbitrary* sheaf $\mathcal{F} \in \mathbf{Sh}(X)$ we get a short exact excision sequence

$$0 \to \mathcal{S}_{Z \setminus Z'} \to \mathcal{S}_Z \to \mathcal{S}_{Z'} \to 0 \tag{2.11}$$

Given open sets $U_1, U_2 \subset X$, closed sets $Z_1, Z_2 \subset X$ and *arbitrary* sheaf $\mathcal{F}$ we have the following short exact *Mayer-Vietoris sequences* of sheaves

$$0 \to \mathcal{F}_{U_1 \cap U_2} \to \mathcal{F}_{U_1} \oplus \mathcal{F}_{U_2} \to \mathcal{F}_{U_1 \cup U_2} \to 0, \tag{2.12a}$$

$$0 \to \mathcal{F}_{Z_1 \cup Z_2} \to \mathcal{F}_{Z_1} \oplus \mathcal{F}_{Z_2} \to \mathcal{F}_{Z_1 \cap Z_2} \to 0 \tag{2.12b}$$

The sequence (2.12a) induces the well known Mayer-Vietoris sequences in the cohomology with *compact supports* while (2.12b) induces the Mayer-Vietoris sequence for the usual cohomology.

*Remark* 2.17. If $Z$ is a closed subset of the locally compact space $X$, and $\mathcal{O} = X \setminus Z$, the sequence (2.10) shows that we can interpret $H^*(X, \mathcal{S}_{\mathcal{O}})$ as a relative cohomology

$$H^*(X, \mathcal{S}_{\mathcal{O}}) = H^*(X, Z; \mathcal{S}) = H^*(X, X \setminus \mathcal{O}, \mathcal{S}).$$

This statement can be made quite rigorous.

Suppose $\Phi$ is the set of all closed subsets of the paracompact space $X$, $Z$ is a closed subset of $X$, and $G$ is an Abelian group. Then we have an isomorphism

$$H^*_{\Phi|_{X\setminus S}}(X \setminus Z, G) \cong H^*(X, Z; G),$$

where in the right had side is the relative Alexander-Spanier cohomology. When $X$ and $Z$ are not too wild this coincides with the singular cohomology. For more details we refer to [3, II.12] or [5, II.4.10]. Note also that Lemma 2.15 implies that for every sheaf $\mathcal{S}$ we have and isomorphism

$$H^*_{\Phi|_{X\setminus Z}}(X \setminus Z, G) \cong H^*(X, \underline{G}_{X \setminus Z}) = H^*(X, \underline{G}_{\mathcal{O}}).$$

□

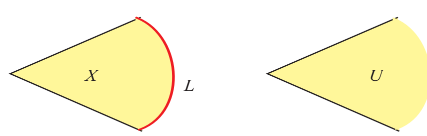



FIGURE 1. *A closed cone and an open cone over $L$.*

**Example 2.18.** Suppose that $L$ is a "nice" compact space (e.g. a $CW$-complex). Form the cones (see Figure 1)

$$X := [0,1] \times L / \{0\} \times L, \;\; U = X \setminus \big( \{1\} \times L \big).$$

For any Abelian group $G$ we have a short exact sequence

$$0 \to \underline{G}_U \to \underline{G}_X \to \underline{G}_L \to 0.$$

From Remark 2.16 we deduce

$$H^\bullet(X, \underline{G}_U) \cong H^\bullet_c(U, G), \;\; H^\bullet(X, \underline{G}_L) \cong H^\bullet(L, \underline{G}).$$

Hence we obtain a long exact sequence

$$\cdots \to H_c^k(U,G) \to H^k(X,G) \to H^k(L,G) \to H_c^{k+1}(U,G) \to \cdots .$$

For $m \geqslant 2$ we get isomorphisms

$$H_c^m(U,G) \cong H^{m-1}(L,G).$$

For $m = 1$ we get a short exact sequence

$$0 \to H^0(X,G) \xrightarrow{i} H^0(L,G) \to H_c^1(U,G) \to 0$$

If $b_0(L)$ denotes the number of components of $L$ then

$$H^0(L,G) \cong G^{b_0(L)}$$

and the map $i$ has the form

$$G \ni g \mapsto \underbrace{(g,\cdots,g)}_{b_0(L)} \in G^{b_0(L)}.$$

The image of this group is the diagonal subgroup of $G^{b_0(L)}$. We deduce

$$H_c^1(U,G) \cong G^{b_0(L)}/\Delta(G^{b_0(L)}) \cong G^{b_0(L)-1}.$$

In particular

$$\chi_c(U) := \chi(H_c^*(U,\mathbb{Q})) = 1 - \chi(L).$$

□

Finally we would like to say a few words about the *local cohomology* modules supported by a locally closed set. These are the derived functors of $\Gamma_Z(X,-)$ and they are denoted by $H_Z^\bullet(X,\mathcal{F})$.

For every locally closed set and every *closed* set $Z' \subset Z$ we have a long exact sequence

$$\cdots \to H^j_{Z'}(X,\mathcal{F}) \to H^j_Z(X,\mathcal{F}) \to H^j_{Z\setminus Z'}(X,\mathcal{F}) \xrightarrow{+1} \cdots \tag{2.13}$$

To understand the origin of this sequence let us observe that if $\mathcal{F}$ is a *flabby* sheaf then we have a short exact sequence of *flabby* sheaves

$$0 \to \Gamma_{Z'}(\mathcal{F}) \to \Gamma_Z(\mathcal{F}) \to \Gamma_{Z\setminus Z'}(\mathcal{F}) \to 0.$$

The injectivity of the first arrow is tautological, while the surjectivity of the second arrow follows from the flabiness of $\Gamma_Z(\mathcal{F})$ and the isomorphism

$$\Gamma(U,\Gamma_{Z\setminus Z'}(\mathcal{F})) \cong \Gamma(U\setminus Z',\Gamma_Z(\mathcal{F})).$$

We obtain a distinguished triangle in the derived category of sheaves

$$\Gamma_{Z'}(\mathcal{F}^\bullet) \to \Gamma_Z(\mathcal{F}^\bullet) \to \Gamma_{Z\setminus Z'}(\mathcal{F}^\bullet) \xrightarrow{[1]} .$$

In particular if $Z = X$ and $Z' = S \subset X$ is closed so $U = X\setminus S$ is open we obtain the distinguished triangle in the derived category of $R$-modules

$$R\Gamma_S(X,\mathcal{F}^\bullet) \to R\Gamma(X,\mathcal{F}^\bullet) \xrightarrow{a} \Gamma(X\setminus S,\mathcal{F}^\bullet) \xrightarrow{[1]}$$

The morphism $a$ is called the *attaching map*. We obtain the long exact sequence

$$\cdots \to H^j_S(X,\mathcal{F}) \to H^j(X,\mathcal{F}) \to H^j(X\setminus S,\mathcal{F}) \xrightarrow{+1} \cdots \tag{2.14}$$

Given a cohomology class $u \in H^j(X\setminus S,\mathcal{F})$ we can ask when it extends to a cohomology class on $X$, i.e. it is the image of an element $\hat{u} \in H^j(X,\mathcal{F})$ via the natural morphism. We see that this happens if and only if the element $\delta u \in H^{j+1}_S(X, \mathcal{F})$ is trivial. Thus the local cohomology groups can be viewed as collecting the obstructions to extension problems.

Let us point out that for any *flabby* sheaf $\mathcal{F}$, any open subsets $U_1, U_2$ and any closed sets $Z_1, Z_2$ we have short exact sequences

$$0 \to \Gamma_{U_1 \cup U_2}(\mathcal{F}) \to \Gamma_{U_1}(\mathcal{F}) \oplus \Gamma_{U_2}(\mathcal{F}) \to \Gamma_{U_1 \cap U_2}(\mathcal{F}) \to 0 \tag{2.15a}$$

$$0 \to \Gamma_{Z_1 \cap Z_2}(\mathcal{F}) \to \Gamma_{Z_1}(\mathcal{F}) \oplus \Gamma_{Z_2}(\mathcal{F}) \to \Gamma_{Z_1 \cup Z_2}(\mathcal{F}) \to 0. \tag{2.15b}$$

In (2.15a) the arrows are induced by the restriction maps $\Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$, $U \supset V$ open subsets. In (2.15b) the arrows are "extension by zero" morphisms. The exactness of these sequences is due to the flabiness of $\mathcal{F}$.

The local cohomology sheaves, i.e. the homology of $R\Gamma_Z(\mathcal{F})$ is are the sheaves associated to the presheaves

$$\mathcal{H}^j_Z(U) := H^j_{Z \cap U}(U, \mathcal{F}).$$

*Remark* 2.19. Suppose $X = \mathbb{R}^2$ and $Z$ is a line in this plane. Note that the sequence (2.14) implies that the local cohomology of $\underline{\mathbb{R}}$ supported on $Z$ *is in general not zero* although, as shown in Remark 2.9, $\Gamma_Z(\underline{\mathbb{R}})$ is zero!!! The local cohomology is of $\mathcal{F}$ along $Z$ is *not* the cohomology of $\Gamma_Z(\mathcal{F})$. The long exact sequence (2.14) suggests that we can interpret the local cohomology as the relative cohomology of the pair $(X, X\backslash Z)$. For more on this interpretation we refer to [3, II§12].

□

**Proposition 2.20.** *If $Z$ is a closed subset of a locally compact space $X$ and $i : Z \hookrightarrow X$ denotes the canonical inclusion then for any field $\mathbb{K}$ and any sheaf $\mathcal{F}$ of $\mathbb{K}$-vector spaces on $X$ we have*

$$H^p_Z(X, \mathcal{F}) \cong \mathrm{Ext}^p(i_* \underline{\mathbb{K}}, \mathcal{F}) \cong \mathrm{Ext}^p(\underline{\mathbb{K}}_Z, \mathcal{F}).$$

**Proof** We have an isomorphism of functors $\mathbf{Sh}_{\mathbb{K}}(X) \to \mathbf{Vect}_{\mathbb{K}}$ (see [8, Prop. 2.3.10])

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(i_{*Z} \underline{\mathbb{K}}, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\underline{\mathbb{K}}_Z, \mathcal{F}) \cong \Gamma_Z(X, \mathcal{F}).$$

Their derived functors must be isomorphic as well whence the desired conclusion.

□

**Example 2.21.** Consider again the cone $X$ discussed in Example 2.18. We denote by $x$ its vertex. We would like to compute the local cohomology $H^\bullet_{\{x\}}(X, \mathbb{R})$. We have a long exact sequence

$$\cdots \to H^\bullet_{\{x\}}(X) \to H^\bullet(X) \to H^\bullet(X\backslash\{x\}) \to H^{\bullet+1}_{\{x\}}(X) \to \cdots$$

Note that we have a natural morphism $H^\bullet_{\{x\}}(X) \to H^\bullet_c(U)$, $U = X\backslash\big(\{1\} \times L\big)$. Observing that $X\backslash\{x\}$ deformation retracts to $\{1\} \times L$. Comparing the above sequence with

$$\cdots \to H^\bullet_c(U) \longrightarrow H^\bullet(X) \longrightarrow H^\bullet(C) \longrightarrow H^{\bullet+1}_c(U) \longrightarrow \cdots.$$

we deduce from the five-lemma that

$$H^\bullet_{\{x\}}(X) \cong H^\bullet_c(U) \cong H^\bullet_c(B_r(x)),$$

where $B_r(x)$ is a small open ball in $X$ centered at $x$. □

**Example 2.22** (Baby micro-local Morse theory)**.** Consider the function

$$\phi(x,y) = |x|^2 - |y|^2$$

defined on a small open ball $B^{p+q}$ centered at the origin of $\mathbb{R}^{p+q} = \mathbb{R}^p_x \oplus \mathbb{R}^q_y$, $p > 0$. We would like to compute $H^\bullet_{\{\phi \leqslant 0\}}(B, \underline{\mathbb{R}})$. For simplicity we set

$$Z := \{\phi \leqslant 0\} \cap B^{p+q}.$$

This region is shaded in gray in Figure 2. Observe that

$$B^{p+q} \setminus Z \simeq B^p \setminus 0 \simeq S^{p-1},$$

where $\simeq$ stands for the homotopy. Using (2.14) we deduce

$$\cdots \to H^j(B^{p+q}, \mathbb{R}) \to H^j(S^{p-1}, \mathbb{R}) \to H^{j+1}_Z(B^{p+q}, \mathbb{R}) \to H^{j+1}(B^{p+q}, \mathbb{R}) \to \cdots$$

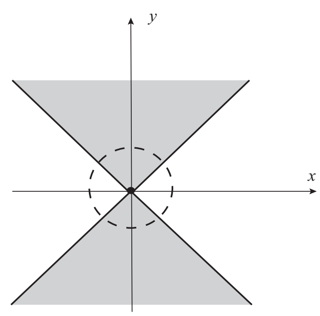



FIGURE 2. *The region* $y^2 - x^2 \geqslant 0$.

We deduce that

$$H^{j+1}_Z(B^{p+q}, \mathbb{R}) \cong H^j(S^{p-1}, \mathbb{R}) \cong H^{j+1}(S^p, \mathbb{R}), \quad \forall j > 0.$$

For $j = 0$ we have a short exact sequence

$$0 \to H^0(B^{p+q}, \mathbb{R}) \to H^0(S^{p-1}, \mathbb{R}) \to H^1_S(B^{p+q}, \mathbb{R}) \to 0.$$

This proves that

$$H^j_{\{\phi \leqslant 0\}}(B^{p+q}, \mathbb{R}) \cong H^j(B^p, \partial B^p; \mathbb{R}) \cong H^j(\phi^{-1}((0, \varepsilon]), \phi^{-1}(\varepsilon); \mathbb{R}).$$

Similarly

$$H^j_{\{\phi \geqslant 0\}}(B^{p+q}, \mathbb{R}) \cong H^j(B^q, \partial B^q; \mathbb{R})$$

so that

$$\chi\big( H^\bullet_{\{\phi \geqslant 0\}}(B, \mathbb{R}) \big) = (-1)^q = (-1)^{m(\phi,0)},$$

where $m(\phi, 0)$ denotes the Morse index of $\phi$ at $0$. □

Let $\mathcal{J}$ denote one of the full additive subcategories of injective, flabby, or $c$-soft sheaves, or the *opposite* of the full subcategory of flat sheaves. .

A sheaf $\mathcal{F}$ on a locally compact space $X$ is said to have $\mathcal{J}$*-dimension* $\leqslant r$ if it admits a resolution of length $\leqslant r$ by objects in $\mathcal{J}$, i.e. there exists a long exact sequence in $\mathbf{Sh}(X)$

$$0 \to \mathcal{F} \to \mathcal{S}_0 \to \cdots \to \mathcal{S}_r \to 0, \quad \mathcal{S}_i \in \mathcal{J}.$$

We write this

$$\dim_{\mathcal{J}} \mathcal{F} \leqslant r.$$

We define the $\mathcal{J}$-dimension as the smallest $r$ with this property. Observe that $\mathcal{F} \in \mathcal{J}$ iff $\dim_{\mathcal{J}} \mathcal{F} \leqslant 0$. We can regard the ring $R$ as a sheaf over the space consisting of a single point. Clearly $R$ is flabby and hence soft. On the other hand we can speak of the flat, or injective dimension. We have the following result.

**Theorem 2.23** ([13]). *Suppose $R$ is a commutative Noetherian ring with $1$ and $r \in \mathbb{Z}_{\geqslant 0}$. Then the following statements are equivalent.*
*(a)* $\dim_{inj} R = r$.
*(b)* $\dim_{flat} R = r$
*(c)* $\dim_{proj} R = r$.
*When any of these conditions is satisfied we write*

$$\operatorname{gldim} R = r.$$

The space $X$ is said to have $\mathcal{J}$*-dimension* $\leqslant r$ if every sheaf on $X$ has $\mathcal{J}$-dimension $\leqslant r$. We write this

$$\dim_{\mathcal{J}} X \leqslant r.$$

All sheaves have finite flat dimension provided that the coefficient ring has finite global dimension. We have the following result whose proof could be found in [2, §6].

**Proposition 2.24.** *For any $\mathcal{F} \in \mathbf{Sh}_R(X)$ we have*

$$\dim_{flat} \mathcal{F} \leqslant \operatorname{gldim} R.$$

When $\mathcal{J}$ is one of the categories, injective, flabby, soft, then the notion of $\mathcal{J}$-dimension depends on the algebraic topology of the space $X$ and captures some of our intuition of dimension. For a proof of the following results we refer to [11, Exposé 2]. In particular, they give an algebraic-topologic description of the sheaves in $\mathcal{J}$ since they are the sheaves of $\mathcal{J}$-dimension $0$.

**Proposition 2.25.** *Assume $X$ is a locally compact space. Then the following statements are equivalent.*
*(a)*

$$\dim_{soft} \mathcal{F} \leqslant r.$$

*(b)*

$$H_c^{r+1}(X, \mathcal{F}) = 0.$$

$$H_c^q(X, \mathcal{F}) = 0, \;\; \forall q > p.$$

*(d) For any resolution*

$$0 \to \mathcal{F} \to \mathcal{S}_0 \to \mathcal{S}_1 \to \cdots \to \mathcal{S}_{r-1} \to \mathcal{S}_r \to 0$$

*where $\mathcal{S}_0, \cdots, \mathcal{S}_{r-1}$ are c-soft, then $\mathcal{S}_r$ is c-soft as well.*
*(e) Every point $x \in X$ has an open neighborhood $V_x$ such that $H_c^{r+1}(U, \mathcal{F}) = 0$, for all open subsets $U \hookrightarrow V_x$.*

**Proposition 2.26.** *Assume $X$ is a locally compact space. Then the following statements are equivalent.*
*(a)*

$$\dim_{flabby} \mathcal{F} \leqslant r.$$

*(b) For every closed set $S \subset X$ we have*

$$H^{r+1}_S(X, \mathcal{F}) = 0.$$

*(c) For every closed set $S \subset X$ we have*

$$\mathcal{H}^{r+1}_S(\mathcal{F}) = 0 \in \mathbf{Sh}_R(X).$$

There are some relations between these notions of dimensions. We state them under a simplifying assumption on the topological space $X$.

**Proposition 2.27.** *Assume $X$ is a subset of an Euclidean space and $\mathcal{F} \in \mathbf{Sh}_R(X)$. Then the following hold*

*(a)*

$$\dim_{flabby} \mathcal{F} \leqslant \dim_{soft} \mathcal{F} \leqslant \dim_{flabby} \mathcal{F} + 1.$$

*(b) If $R$ is a regular ring*[4] *of dimension $p$ then*

$$\dim_{flabby} X \leqslant \dim_{inj} X \leqslant \dim_{soft} X + p + 1.$$

*In particular, if $R = \mathbb{C}$ we have $p = 0$ and*

$$\dim_{flabby} X \leqslant \dim_{inj} X \leqslant \dim_{soft} X + 1.$$

**Example 2.28** (The bi-functor $R\,\underline{Hom}$ and $\overset{\mathrm{L}}{\otimes}$)**.** Fix a Noetherian commutative ring with 1. For any injective sheaf $\mathcal{S} \in \mathbf{Sh}_R$ the functor

$$\mathbf{Sh}_R(X)^{op} \to \mathbf{Sh}_R(X), \;\; \mathcal{F} \mapsto \underline{Hom}(\mathcal{F}, \mathcal{S})$$

is exact and thus we obtain a derived functor

$$D^-(\mathbf{Sh}_R\, X)^{op} \times D^+(\mathbf{Sh}_R\, X) \to D^+(\mathbf{Sh}_R\, X), \;\; (f^\bullet, \mathcal{G}^\bullet) \mapsto R\,\underline{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Every sheaf admits a resolution by a bounded from above complex of flat sheaves. We obtain in this fashion a (left) derived bi-functor

$$\overset{\mathrm{L}}{\otimes} : D^-(\mathbf{Sh}_R\, X) \times D^-(\mathbf{Sh}_R\, X) \to D^-(\mathbf{Sh}_R\, X), \;\;\; (\mathcal{F}^\bullet, \mathcal{G}^\bullet) \mapsto \mathcal{F}^\bullet \overset{\mathrm{L}}{\otimes}_R \mathcal{G}^\bullet.$$

For any $\mathcal{F}, \mathcal{G} \in D^-(\mathbf{Sh}_R)$ and any $\mathcal{S} \in D^+(\mathbf{Sh}_R\, X)$ we have the *adjunction isomorphism*

$$R\,\underline{Hom}(\mathcal{F} \overset{\mathrm{L}}{\otimes} \mathcal{G}, \mathcal{S}) \cong R\,\underline{Hom}\big(\mathcal{F}, R\,\underline{Hom}(\mathcal{G}, \mathcal{S})\big). \tag{2.16}$$

When $\mathcal{F}, g$ are sheaves (i.e. complexes of sheaves concentrated in dimension 0 then we set

$$\underline{Ext}^\bullet(\mathcal{F}, \mathcal{G}) := R\,\underline{Hom}^\bullet(\mathcal{F}, \mathcal{G}).$$

□

[4] This means that all its local rings are regular of dimension $p$.

## 3. THE DERIVED FUNCTOR $Rf_!$

Fix a commutative noetherian ring $\mathcal{R}$ with 1. For every topological space $\mathcal{T}$ and $* \in \{b, +, -\}$ we set

$$D^*(\mathcal{R}_{\mathcal{T}}) := D^*(\mathbf{Sh}_{\mathcal{R}}(\mathcal{T})).$$

Suppose $f : X \to S$ is a continuous map between locally compact spaces. Then the full subcategory of $\mathbf{Sh}_{\mathcal{R}}(X)$ consisting of $c$-soft sheaves is adapted to the functor

$$f_! : \mathbf{Sh}_{\mathcal{R}}(X) \to \mathbf{Sh}_{\mathcal{R}}(S).$$

In particular we have a derived functor

$$Rf_! : D^+(\mathcal{R}_X) \to D^+(\mathcal{R}_S)$$

We list below some of its most frequently used properties. For proofs we refer to [7, 8].

For every sheaf $\mathcal{F}$ on $X$, $j \in \mathbb{Z}$, and every $s \in S$ we have a natural isomorphism

$$(R^j f_! \mathcal{F})_s := \mathcal{H}^j(Rf_!\mathcal{F})_s \cong H^j_c(f^{-1}(s), \mathcal{F}), \tag{3.1}$$

where $\mathcal{H}^j(Rf_!\mathcal{F})$ denote the cohomology sheaves of the complex $Rf_!\mathcal{F}$. This shows that we can view $Rf_!$ as a sort of integration-along-fibers functor. Observe that if $X \xrightarrow{f} S \xrightarrow{g} T$ then we have a natural isomorphism

$$R(g \circ f)_! \cong Rg_! \circ Rf_!.$$

To formulate the other natural properties of $Rf_!$ we need to discuss Cartesian diagrams.

Suppose $f : X \to S$ and $g : Y \to S$ are two continuous maps of finite dimensional, locally compact spaces. We set

$$X \times_S Y := \Big\{ (x, y) \in X \times Y;\ \ f(x) = g(y) \Big\}.$$

A *Cartesian diagram* associated to $f$ and $g$ is a diagram of the form

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

**Example 3.1.** Here are a few interesting examples of cartesian diagrams. Given a continuous map

(a) If $g : X \to S$ is a locally trivial fibration then $X \times_S Y \to X$ is the pull back of $X \xrightarrow{g} S$ via $f$.

(b) $S = \{pt\}$ and $f : X \to S$ and $g : Y \to S$ are the constant maps then $X \times_S Y$ coincides with the Cartesian product of $X$ and $Y$.

(c) Suppose $f : X \to S$ is continuous, $Y := \{s\} \subset S$ and $g$ is the inclusion $\{s\} \hookrightarrow S$ Then

$$X \times_S Y \cong f^{-1}(f(x))$$

and the corresponding Cartesian diagram is

$$\begin{array}{ccc} f^{-1}(s) & \xrightarrow{f} & \{s\} \\ {\scriptstyle j_s}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ X & \xrightarrow[f]{} & S \end{array}$$

where $j_s$ denotes the inclusion of the fiber

$$f^{-1}(s) \hookrightarrow X.$$

(d) If $f : X \to S$ and $g : Y \to S$ are inclusions of subsets $X, Y \subset S$ then $X \times_S Y \cong X \cap Y$.

□

**Proposition 3.2** (Base Change Formula)**.** *Given a Cartesian diagram*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

*we have a natural isomorphism of functors* $D^+(\mathcal{R}_Y) \to D^+(\mathcal{R}_X)$

$$f^{-1}Rg_! \cong R(\pi_X)_!\pi_Y^{-1}.$$

*and a natural isomorphism of functors* $D^+(\mathcal{R}_X) \to D^+(\mathcal{R}_Y)$

$$g^{-1}Rf_! \cong R(\pi_Y)_!\pi_X^{-1}.$$

Observe that the base change formula coupled with Example 3.1(b) implies (3.1).

The key fact behind the base change formula is the following.

**Lemma 3.3.** *Given a Cartesian diagram*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

*we have a natural isomorphism of functors* $\mathbf{Sh}_{\mathcal{R}}(Y) \longrightarrow \mathbf{Sh}_{\mathcal{R}}(Y)$

$$g^{-1} \circ f_! \cong (\pi_Y)_! \circ \pi_X^{-1}. \tag{3.2}$$

**Proof** First we construct a natural morphism

$$f_! \circ (\pi_X)_* \to g_* \circ (\pi_Y)_!. \tag{3.3}$$

Let $\mathcal{G} \in \mathbf{Sh}_{\mathcal{R}}(X \times_S Y)$ and $V$ an open subset of $S$. Then

$$f_! \circ (\pi_X)_*\mathcal{G}(V) = \Gamma_f(f^{-1}(V), (\pi_X)_*\mathcal{G})$$

where we recall that $\Gamma_f$ signifies sections with compact vertical (with respect to $f$) support. Next observe that

$$\pi_X^{-1}(f^{-1}(V)) = \big\{(x, y) \in X \times Y;\ \ f(x) = g(y) \in V\big\} = \pi_Y^{-1}(g^{-1}(V)) = X \times_V Y.$$

Thus $u \in f_! \circ (\pi_X)_*\mathcal{G}(V)$ if and only if $u \in \Gamma(X \times_V Y, \mathcal{G})$ and there exists $T \subset X$ proper over $S$ such that

$$\operatorname{supp} u \subset \pi_X^{-1}(T).$$

Then

$$\pi_Y : \operatorname{supp} u \to Y$$

is proper and $u$ defines a section of $g_* \circ (\pi_Y)_!$.

*Remark* 3.4. There may not exist a natural morphism $g_* \circ (\pi_Y)_! \to f_! \circ (\pi_X)_*$. As an example suppose $S = \mathbb{R}$, $X = (0, 2)$, $Y = [1, 2)$ and $f, g$ are the canonical inclusions. Then $X \times_S Y = X \cap Y = Y = [1, 2)$ so that the map $\pi_X$ is tautologically proper. Let $\mathcal{G}$ be the constant sheaf $\underline{\mathbb{R}}$ over $X \cap Y$. Then the constat section $1 \in \Gamma(X \cap Y, \underline{\mathbb{R}})$ is proper over $Y = X \cap Y$. However, there is no subset $C$ of $X$ proper over $S$ (i.e. a closed subset of $\mathbb{R}$ contained in $X$ such that $X \cap Y \subset C$. Thus this section defines a section of $g_* \circ (\pi_Y)_!$ but does not define a section of $f_! \circ (\pi_X)_*$. □

To construct the morphism (3.2) we observe that since $g_*$ is the right adjoint of $g^{-1}$ we have

$$\operatorname{Hom}(g^{-1} \circ f_!, (\pi_Y)_! \circ \pi_X^{-1}) \cong \operatorname{Hom}(f_!, g_*(\pi_Y)_! \circ \pi_X^{-1})$$

Using the morphism (3.3) we obtain a morphism

$$\operatorname{Hom}(f_!, g_*(\pi_Y)_! \circ \pi_X^{-1}) \longleftarrow \operatorname{Hom}(f_!, f_! \circ (\pi_X)_* \pi_X^{-1})$$

The natural morphism $\mathbb{I} \to (\pi_X)_* \pi_X^{-1}$ defines a canonical morphism in

$$f_! \to f_! \circ (\pi_X)_* \pi_X^{-1}$$

which via the above chain of morphisms defines a canonical morphism

$$g^{-1} \circ f_! \to (\pi_Y)_! \circ \pi_X^{-1}.$$

We claim that this is an isomorphism. Let $\mathcal{S} \in \mathbf{Sh}_{\mathcal{R}}(X)$ and $y \in Y$. Then

$$(g^{-1} \circ f_! \mathcal{S})_y = (f_! \mathcal{S})_{g(y)} = \Gamma_c(f^{-1}(g(y)), \mathcal{S}).$$

The map $\pi_X$ induces a homeomorphism

$$\pi_Y^{-1}(y) \to f^{-1}(g(y)).$$

and an isomorphism

$$\Gamma_c(f^{-1}(g(y)), \mathcal{S}) \xrightarrow{\cong} \Gamma_c(\pi_Y^{-1}(y), \pi_X^{-1}\mathcal{S}) \cong ((\pi_Y)_! \pi_X^{-1} \mathcal{S})_y$$

□

The functor $Rf_!$ interacts nicely with the functors $R\operatorname{Hom}$ and $\overset{\mathrm{L}}{\otimes}$. For several reasons listed below we need to make some additional assumptions on the maps $X \xrightarrow{f} S$.

- We need to make sure the derived functor $\overset{\mathrm{L}}{\otimes}$ is defined. Thus we need to require that each bounded (resp. from above) complex on $X$ and $S$ has resolution by a bounded (resp. from above) complex of flat sheaves. This is the case if both spaces $X$ and $S$ have finite dimension. In this case $\overset{\mathrm{L}}{\otimes}$ is defined on $D^- \times D^-$.
- The functor $R\operatorname{Hom}$ is defined on $D \times D^+$. Thus the common domain of definition of $R\operatorname{Hom}$ and $\overset{\mathrm{L}}{\otimes}$ is $D^- \times D^b$.

To summarize, in the sequel we will assume that *all space are admissible, i.e they are locally compact and finite dimensional. Will regard $Rf_!$ as a functor*

$$Rf_! : D^b(\mathcal{R}_X) \to D^b(\mathcal{R}_S).$$

**Proposition 3.5** (Projection Formula)**.** *Suppose $f : X \to S$ is a continuous between admissible spaces. Then we have an isomorphism*

$$(Rf_!\mathcal{F}^\bullet)\overset{\mathrm{L}}{\otimes}_{\mathcal{R}}\mathcal{G}^\bullet \cong Rf_!(\mathcal{F}^\bullet\overset{\mathrm{L}}{\otimes}_{\mathcal{R}} f^{-1}\mathcal{G}^\bullet)$$

*natural in $\mathcal{F}^\bullet \in D^b(\mathcal{R}_X)$ and $\mathcal{G}^\bullet \in D^b(\mathcal{R}_S)$.*

Let us describe the key fact behind the projection formula. Given a sheaf $\mathcal{F} \in \mathbf{Sh}_{\mathcal{R}}(X)$ and a *flat* sheaf $\mathcal{G} \in \mathbf{Sh}_{\mathcal{R}}(S)$ there exists a canonical isomorphism

$$f_!(\mathcal{F})\otimes_{\mathcal{R}}\mathcal{G} \to f_!(\mathcal{F}\otimes_{\mathcal{R}} f^{-1}(\mathcal{G}))$$

induced by a morphism

$$f_*(\mathcal{F})\otimes_{\mathcal{R}}\mathcal{G} \to f_*(\mathcal{F}\otimes_{\mathcal{R}} f^{-1}(\mathcal{G})).$$

Put it differently, the above morphism is some canonical element

$$\mathrm{Hom}(f_*(\mathcal{F})\otimes\mathcal{G}, f_*(\mathcal{F}\otimes f^{-1}(\mathcal{G}))$$

where for simplicity we omitted any reference of the ring $\mathcal{R}$. This element is the image of $\mathbb{I}_{\mathcal{F}\otimes f^{-1}\mathcal{G}}$ via the the following sequence of morphisms

$$\mathbb{I} \in \mathrm{Hom}_X(\mathcal{F}\otimes f^{-1}\mathcal{G}, \mathcal{F}\otimes f^{-1}\mathcal{G})$$

(use the adjunction morphism $(f^{-1}f_*)\mathcal{F} \to \mathcal{F}$)

$$\to \mathrm{Hom}_X\big(\,(f^{-1}f_*\mathcal{F})\otimes f^{-1}\mathcal{G}, \mathcal{F}\otimes f^{-1}\mathcal{G}\,\big)$$

$$\xrightarrow{\cong} \mathrm{Hom}\big(\,f^{-1}(f_*\mathcal{F}\otimes\mathcal{G}), \mathcal{F}\otimes f^{-1}\mathcal{G}\,\big)$$

(use the fact that $f_*$ is the right adjoint of $f^{-1}$)

$$\xrightarrow{\cong} \mathrm{Hom}\big(\,f_*\mathcal{F}\otimes\mathcal{G}, f_*(\mathcal{F}\otimes f^{-1}\mathcal{G})\,\big).$$

At stalk level this map can be described as follows. Given $s \in S$ we have

$$f_!(\mathcal{F})_s\otimes\mathcal{G}_s \xrightarrow{\cong} \Gamma_c\big(\,f^{-1}(s), \mathcal{F}\,|_{f^{-1}(s)}\,\big)\otimes\mathcal{G}_s$$

$$\xrightarrow{\tau} \Gamma_c\big(\,f^{-1}(s), \mathcal{F}\,|_{f^{-1}(s)}\otimes\underline{\mathcal{G}_s}\,\big)$$

$$\xrightarrow{\cong} f_!(\mathcal{F}\otimes f^{-1}\mathcal{G})_s$$

The tautological morphism $\tau$ is an isomorphism when $\mathcal{G}$ is flat. This follows from the following fact, [8, Lemma 2.5.12].

**Lemma 3.6.** *Suppose $\mathcal{R}$ is a commutative ring with $1$ and $M$ is a flat $\mathcal{R}$-module. Suppose $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules over the locally compact space $X$. Then there exists a canonical isomorphism*

$$\Gamma_c(X,\mathcal{F})\otimes M \longrightarrow \Gamma_c(X,\mathcal{F}\otimes M).$$

*In particular, if $\mathcal{F}$ is c-soft then so is $\mathcal{F}\otimes M$.*

**Proposition 3.7** (Künneth Formula)**.** *Given a Cartesian diagram of admissible spaces and continuous maps*

$$\begin{array}{ccc} X\times_S Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle\pi_X}\downarrow & \overset{\delta}{\searrow} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}\qquad ,$$

*where* $\delta = f\pi_X = g\pi_Y$. *We have a isomorphism*

$$R\delta_!(\pi_X^{-1}\mathcal{F}\overset{\mathrm{L}}{\otimes}\pi_Y^{-1}\mathcal{G}) \cong Rf_!\mathcal{F}\overset{\mathrm{L}}{\otimes}Rg_!\mathcal{G}. \tag{3.4}$$

*natural in* $\mathcal{F} \in D^b(\mathcal{R}_X)$ *and* $\mathcal{G} \in D^b(\mathcal{R}_Y)$.

**Proof** Using the projection formula we deduce

$$(R\pi_X)_!\big(\pi_X^{-1}\mathcal{F}\overset{\mathrm{L}}{\otimes}\pi_Y^{-1}\mathcal{G}\big) \cong \mathcal{F}\overset{\mathrm{L}}{\otimes}(R\pi_X)_!\pi_Y^{-1}\mathcal{G}$$

Using the base change formula we deduce

$$(R\pi_X)_!\pi_Y^{-1}\mathcal{G} \cong f^{-1}Rg_!\mathcal{G}$$

so that

$$(R\pi_X)_!\big(\pi_X^{-1}\mathcal{F}\overset{\mathrm{L}}{\otimes}\pi_Y^{-1}\mathcal{G}\big) \cong \mathcal{F}\overset{\mathrm{L}}{\otimes}f^{-1}Rg_!\mathcal{G}.$$

Using the projection formula once again we deduce

$$R\delta_!\big(\pi_X^{-1}\mathcal{F}\overset{\mathrm{L}}{\otimes}\pi_Y^{-1}\mathcal{G}\big) \cong Rf_!(R\pi_X)_!\big(\pi_X^{-1}\mathcal{F}\overset{\mathrm{L}}{\otimes}\pi_Y^{-1}\mathcal{G}\big) \cong Rf_!\mathcal{F}\overset{\mathrm{L}}{\otimes}Rg_!\mathcal{G}.$$

□

Künneth formula together with Proposition 2.25 implies that the product of two finite dimensional spaces is a finite dimensional space.

**Corollary 3.8.** *Given two admissible spaces* $X, Y$ *we denote by* $\pi_Y$ *the natural projection*

$$\pi_Y : X \times Y \twoheadrightarrow Y.$$

*Then for any Noetherian ring* $\mathcal{R}$ *we have an isomorphism in* $D^+(\mathcal{R}_X)$

$$(R\pi_Y)_!\underline{\mathcal{R}} \cong c_Y^{-1}R\Gamma_c(X, \underline{\mathcal{R}})$$

*In particular, the higher derived sheaves of* $(R\pi_Y)_!\underline{\mathcal{R}}$ *are constant on* $Y$.

**Proof** Consider the Cartesian diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle c_X} \\ Y & \xrightarrow[c_Y]{} & pt \end{array}$$

The base change formula implies

$$(R\pi_Y)_!\underline{\mathcal{R}} \cong R(\pi_Y)_!\pi_X^{-1}\underline{\mathcal{R}} \cong c_Y^{-1}(c_X)_!\underline{R} \cong c_Y^{-1}R\Gamma_c(X, \underline{\mathcal{R}}).$$

□

## 4. LIMITS

We would like to survey and generalize the classical concepts of inductive/projective limits. We start with the classical notions.

Suppose $(I, <)$ is an ordered set. We identify $I$ with a category, where we have an object $O_i$ for each $i \in I$ and a exactly one morphism $O_i \to O_j$ for each $i < j$. If $\mathcal{C}$ is a category, then an *inductive family* in $\mathcal{C}$ parametrized by $I$ is a functor

$$X : I \to \mathcal{C}, \;\; i \mapsto X_i, \;\; i < j \mapsto \phi_{ji} : X_i \to X_j.$$

A *projective family* in $\mathcal{C}$ is then a functor

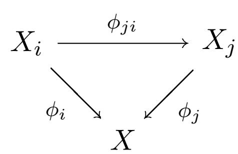
$$Y : I^{op} \to \mathcal{C}, \;\; i \mapsto Y_i, \;\; i < j \mapsto \psi_{ij} : Y_j \to Y_i.$$

**Definition 4.1.** (a) Suppose $(X_i, \phi_{ji})$ is an inductive family of objects in a category $\mathcal{C}$. We say that

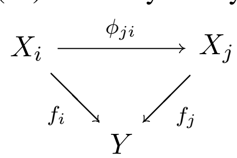
$$X = \varinjlim_I X_i$$

if there exists a collection of morphisms $\phi_i : X_i \to X$ satisfying the following conditions.

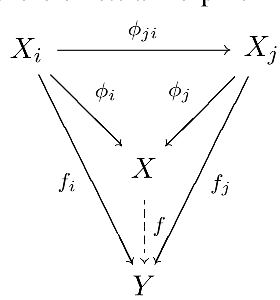
(a1) For every $i < j$ we have a commutative diagram




(a2) For every family of morphisms $X_i \xrightarrow{f_i} Y$ such that




there exists a morphism $f : X \to Y$ such that the diagram below is commutative




(b) Suppose $(Y_i, \psi_{ij})$ is a projective system in $\mathcal{C}$. The definition of $Y = \varprojlim_I Y_i$ is obtained from the definition of $\varinjlim$ by reversing all the arrows.

(c) Suppose $(X_i, \phi_{ji})$ is an inductive family of objects in a category $\mathcal{C}$. We say that

$$\underrightarrow{\mathbf{Lim}}_I X_i = X$$

if there exists a collection of morphisms $\phi_i : X_i \to X$ satisfying (a1), (a2) and there exists $i_0 \in I$ and a morphism $e_0 : X \to X_{i_0}$ such that

$$\phi_{i_0} \circ e_0 = \mathbb{1}_X$$

and for every $i$ there exists $j > i, i_0$ such that the diagram below is commutative

$$\begin{array}{ccc} X_j & \xleftarrow{\phi_{ji_0}} & X_{i_0} \\ {\scriptstyle\phi_{ji}}\uparrow & & \uparrow{\scriptstyle e_0} \\ X_i & \xrightarrow[\phi_i]{} & X \end{array} \tag{4.1}$$

(d) The definition of $\underleftarrow{\mathbf{Lim}}$ is the same but with all the arrows reversed. □

*Remark* 4.2 (Mnemonic device). $\varinjlim$ produces an *initial object* while $\varprojlim$ produces a *terminal object*. □

**Example 4.3.** If $I$ is equipped with the trivial order relation, i.e. not two different elements in $I$ are comparable then

$$\varinjlim_I X_i = \bigoplus_{i\in I} X_i$$

and

$$\varprojlim_{I^{op}} Y_i = \prod_{i\in I} Y_i.$$

□

**Example 4.4.** Suppose $(A_n, \phi_{nm})_{n\geqslant m\geqslant 0}$ is an inductive sequence of $R$-modules, $R$-commutative ring with 1, such that $\underrightarrow{\mathbf{Lim}}_n A_n$ exists. We denote this limit by $A$. In particular

$$A \cong \varinjlim_n A_n \cong \bigoplus_n A_n/\Delta_\phi, \quad \Delta_\phi = \sum_{i\leqslant j} \Delta_{ij},$$

where

$$\Delta_{ij} = \Big\{(c_1, \cdots, c_k, \cdots) \in \bigoplus_n A_n, \;\; a_n = 0, \;\; \forall n \neq i, j, \;\; a_j = -\phi_{ji}(a_i)\Big\}.$$

For example

$$\Delta_{12} = \Big\{(a, -\phi_{12}(a), 0, 0\cdots); \;\; a \in A_1\Big\} \text{ etc.}$$

and we have

$$\begin{aligned}(a_1, a_2, \cdots, a_n, 0, \cdots) &= (a_1, a_2, \cdots, a_n, 0, \cdots) - (a_1, -\phi_{21}(a_1), 0\cdots) = \cdots \\ &= \big(\underbrace{0, \cdots, 0}_{n-1}, a_n + \phi_{n1}(a_1) + \cdots + \phi_{n,n-1}(a_{n-1}), 0, \cdots\big) \in \varinjlim_n A_n.\end{aligned}$$

Since $A = \underrightarrow{\mathbf{Lim}}_n A_n$ there exists $s_0 : A \to A_0$ such that $\phi_0 s_0 = \mathbb{1}_A$, and for any $k \geqslant 0$ there exists $n = n_k \geqslant k$ such that the diagram below is commutative

$$\begin{array}{ccc} A_k & \xrightarrow{\phi_k} & A \\ {\scriptstyle\phi_{nk}}\downarrow & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle s_0} \\ A_n & \xleftarrow[\phi_{n0}]{} & A_0. \end{array}$$

Note that $\phi_n s_n = \phi_n \phi_{n0} s_0 = \phi_0 s_0 = \mathbb{1}_A$ so that $s_n$ must be $1-1$. This shows in particular that

$$\ker(A_0 \xrightarrow{\phi_{n0}} A_n) = \ker(A_0 \xrightarrow{\phi_0} A).$$

In other words the increasing sequence of submodules $\ker(A_0 \xrightarrow{\phi_{n0}} A_n)$ stabilizes. We can visualize this condition in a different way. Form

$$\bar{A}_n = A_n / \ker(A_n \to \varinjlim_n A_n).$$

the morphisms $\phi_{nm}$ induce monomorphisms

$$\bar{\phi}_{nm} : \bar{A}_m \to \bar{A}_m$$

and

$$\bigcup_n \bar{A}_n = \varinjlim_n \bar{A}_n \cong \varinjlim_n A_n = A.$$

If $\varinjlim_n A_n = \underrightarrow{\mathbf{Lim}}_n A_n$ then the ascending chain $(\bar{A}_1 \subset \bar{A}_2 \subset \cdots)$ stabilizes. This happens automatically if $A_n$ are modules over a Noetherian ring $R$ and $A = \varinjlim A_n$ is a finitely generated $R$-module. Conversely, if the limit $A = \varinjlim_n A_n$ is a finitely generated *projective* module over the Noetherian ring $R$ then

$$A = \varinjlim_n A_n = \underrightarrow{\mathbf{Lim}}_n A_n$$

For example if $\mathcal{A}$ is a sheaf of Abelian groups on a metric space $X$ and $A_n = \Gamma(B_{2^{-n}}(x), \mathcal{A})$ then

$$\varinjlim_n A_n = \mathcal{A}_x$$

then $\underrightarrow{\mathbf{Lim}}_n A_n = \mathcal{A}_x$ iff and only there exists a natural method of extending a germ $f \in \mathcal{A}_x$ to the ball $B_1(x)$. Moreover the sections of the sheaf satisfy a weak form of the unique continuation principle: there exists $r_0 > 0$ such that if the germ at $x$ of $f \in \Gamma(B_1(x), \mathcal{A})$ is zero then the restriction of $f$ to $B_{r_0}(x)$ must be trivial. □

**Definition 4.5.** Suppose $(A_n, \psi_{mn})_{n \geqslant m \geqslant 0}$ is a projective system. We say that it satisfies the *Mittag-Leffler condition* if for every $k \geqslant 0$ there exists $m_k \geqslant k$ such that the morphisms

$$\psi_{km} : A_m \to A_k$$

have the same image for all $m \geqslant m_k$. We will refer to the projective systems satisfying the Mittag-Lefler condition as *Mittag-Leffler systems*. □

**Example 4.6.** Suppose $(A_n)_{n \geqslant 0}$ is a projective sequence of Abelian groups, i.e. for every $n > 0$ we are given a morphism $A_n \xrightarrow{\psi_n} A_{n-1}$ and for $n > m$ we denote by $\psi_{mn}$ the composition

$$\psi_{mn} : A_n \xrightarrow{\psi_m} A_{m-1} \to \cdots \xrightarrow{\psi_{n+1}} A_n.$$

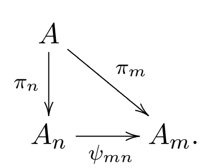
Suppose

$$A = \underleftarrow{\mathbf{Lim}}_n A_n.$$

Then

$$A = \varprojlim_n A_n = \Big\{ \vec{a} = (a_0, a_1, \cdots, ) \in \prod A_n; \;\; a_i = \psi_{ij}(a_j), \;\; \forall j > i \Big\}.$$

The natural projections $\pi_n : A \to A_n$ lie in commutative diagrams



Additionally, there exists $\rho_0 : A_0 \to A$ such that $\rho_0\pi_0 = \mathbb{1}_A$ such that, for every $m \geqslant 0$ there exists $n = n_m > m$ for which the diagram below is commutative

$$\begin{array}{ccc} A_n & \xrightarrow{\psi_{0n}} & A_0 \\ {\scriptstyle \psi_{mn}}\downarrow & \overset{\rho_n}{\searrow} & \downarrow{\scriptstyle \rho_0} \\ A_m & \xleftarrow[\pi_m]{} & A. \end{array} \tag{4.2}$$

We set $\rho_n = \rho_0 \circ \psi_{0n} : A_n \to A$. If we choose $m = 0$ in the above diagram and we set $B_0 = \psi_{0n_0}(A_{n_0}) \subset A_0$ we obtain a commutative diagram

$$\begin{array}{ccc} A_{n_0} & & \\ {\scriptstyle \psi_{0n_0}}\downarrow & \overset{\rho_{n_0}}{\searrow} & \\ B_0 & \xleftarrow[\pi_0]{} & A. \end{array}$$

We deduce that $\pi_0$ is onto $B_0$ and its inverse is $\rho_0$. $\rho_0$ is thus an isomorphism

$$\rho_0 : B_0 \to A, \;\; B_0 \ni b_0 \mapsto (b_0, t_1(b_0), \cdots, t_n(b_0), \cdots) \in A$$

$$t_m : B_0 \to A_m, \;\; t_0 = \mathbb{1}_{B_0}, \;\; t_m = \psi_{mn} t_n, \;\; \forall m \leqslant n.$$

We conclude that

$$\psi_{0n} \circ t_n = \mathbb{1}_{B_0} \Longrightarrow \psi_{0n}(A_n) \supset B_0 = \psi_{0n_0}(A_{n_0}), \;\; \forall n \geqslant 0.$$

In other words, the decreasing sequence of subgroups $\operatorname{Im}(A_n \xrightarrow{\phi_{0n}} A_0)$ stabilizes.

Now observe that

$$\rho_m \circ \pi_m = \rho_0 \circ \underbrace{\psi_{0m} \circ \pi_m}_{\pi_0} = \mathbb{1}_A.$$

We can now play the same game with 0 replaced by $m$ and $n_0$ replaced by $n_m$. We denote by $B_m$ the image of $\psi_{mn_m} : A_{n_m} \to A_m$. Using the diagram (4.2) we deduce in a similar fashion that

$$\psi_{mn}(A_n) = B_m, \;\; \forall n \geqslant m.$$

This shows that $(A_n, \psi_{mn})$ is a Mittag-Leffler system.

□

Let $\mathbb{N} := \mathbb{Z}_{\geqslant 0}$ and denote by $\operatorname{Hom}(\mathbb{N}^{op}, \mathbf{Ab})$ the category of projective systems of Abelian groups. This is an Abelian category and $\varprojlim$ defines an additive covariant functor

$$\varprojlim : \operatorname{Hom}(\mathbb{N}^{op}, \mathbf{Ab}) \to \mathbf{Ab}\,.$$

Let us observe that for every Abelian group $G$ we have

$$\varprojlim_n \operatorname{Hom}(G, A_n) \cong \operatorname{Hom}(G, \varprojlim_n A_n)$$

This is a left exact functor, i.e. transforms injective morphisms into injective morphisms. However it is not an exact functor.

**Example 4.7.** Let $R$ be a principal ideal domain ($R = \mathbb{Z}, \mathbb{Q}[t]$), and $p \in R$ a prime element. Set $A_n = R$, $B_n = R/p^n$ Denote by $\pi_n$ the natural projection $A_n \to B_n$ so that we have a commutative diagram

$$\begin{array}{ccc} A_n = R & \xrightarrow{\pi_n} & B_n = R/p^n \\ {\scriptstyle p\cdot}\downarrow & & \downarrow \\ A_{n-1} = R & \xrightarrow[\pi_{n-1}]{} & R/p^{n-1} \end{array}$$

Observe that $\varprojlim_n R/p^nR$ is the ring of $p$-adic integers while $\varprojlim_n A_n = 0$.

□

**Proposition 4.8.** *Suppose we are given a short exact sequence of projective sequences of Abelian groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & {\scriptstyle \alpha_n}\downarrow & & {\scriptstyle \beta_n}\downarrow & & \downarrow{\scriptstyle \gamma_n} & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow[f_{n-1}]{} & B_{n-1} & \xrightarrow[g_{n-1}]{} & C_{n-1} & \longrightarrow & 0 \end{array}$$

*(a) If $(A_n)$ and $(C_n)$ satisfy the Mittag-Leffler condition then so does $(B_n)$.*
*(b) If $(B_n)$ satisfies the Mittag-Leffler condition then so does $(C_n)$.*
*(c) If $(A_n)$ satisfies the Mittag-Leffler condition then the sequence*

$$0 \to \varprojlim_n A_n \xrightarrow{f} \varprojlim_n B_n \xrightarrow{g} \varprojlim_n C_n \to 0$$

*is exact.*

**Proof** Diagram chasing and soul searching. □

Consider a projective sequence of complexes of Abelian groups $(A_n^\bullet, d)$ so that for every $n \geqslant 1$ we have a commutative diagram

$$\begin{array}{ccc} A_n^\bullet & \xrightarrow{d} & A_n^{\bullet+1} \\ {\scriptstyle \alpha_n}\downarrow & & \downarrow{\scriptstyle \alpha_n} \\ A_{n-1}^\bullet & \xrightarrow{d} & A_{n-1}^{\bullet+1} \end{array}$$

We get projective sequences $Z^\bullet(A_n)$, $B^\bullet(A_n)$, $H^\bullet(A_n)$. The inverse limit

$$A_\infty^\bullet := \varprojlim_n A_n^\bullet,$$

is a complex and the canonical morphisms $A_\infty^\bullet \to A_n^\bullet$ are morphisms of complexes. We get morphisms

$$\phi_n^k : H^k(A_\infty) \to H^k(A_n)$$

and by passing to the limit we obtain morphisms

$$\phi^k : H^k(A_\infty) \to \varprojlim_n H^k(A_n).$$

**Proposition 4.9.** *Assume that for each $k$ the projective system $(A_n^k)_{n\geqslant 0}$ satisfies the Mittag-Leffler condition. Then the following hold.*

*(a) The morphism $\phi^k$ is surjective for every $k$.*
*(b) If for some $k$ the projective system $\big(H^{k-1}(A_n)\big)_{n\geqslant 0}$ satisfies the Mittag-Leffler condition then $\phi^k$ is bijective.*

**Proof** (a) For every $k$ we have short exact sequences

$$0 \to Z_n^k \longrightarrow A_n^k \longrightarrow B_n^{k+1} \to 0, \tag{4.3}$$

$$0 \to B_n^k \to Z_n^k \to H^k(A_n) \to 0. \tag{4.4}$$

They induce a short sequence

$$0 \to \varprojlim_n B_n^k \to \varprojlim_n Z_n^k \to \varprojlim_n H^k(A_n) \to 0. \tag{4.5}$$

Since $(A_n^\bullet)$ satisfies $ML$ we deduce from (4.3) and Proposition 4.8(b) that $(B_n^\bullet)$ satisfies $ML$ so that the sequence (4.5) is exact.

Now observe that

$$Z_\infty^k := \ker(A_\infty^k \xrightarrow{d_\infty} A_\infty^{k+1}) \cong \varprojlim_n Z_n^k$$

and the canonical map

$$B_\infty^k := \operatorname{Im}(A_\infty^{k-1} \xrightarrow{d_\infty} A_\infty^k) \longrightarrow \varprojlim_n B_n^k$$

is one-to-one. Hence we have a natural surjection

$$Z_\infty^k / B_\infty^k \to \varprojlim_n Z_n^k / \varprojlim_n B_n^k \overset{(4.5)}{\cong} \varprojlim_n H^k(A_n).$$

(b) Using the short exact sequence (4.4) with $k$ replaced by $k-1$ and using the fact that $B(B_n^{k-1})_{n\geqslant 0}$, $(H^{k-1}(A)n))_{n\geqslant 0}$ satisfy $ML$ we deduce from Proposition 4.8(a) that $(Z_n^{k-1})_{n\geqslant 0}$ satisfies $ML$. We deduce that the sequence

$$0 \to \varprojlim_n Z_n^{k-1} \longrightarrow \varprojlim_n A_n^k \xrightarrow{d_\infty} \varprojlim_n B_n^k \to 0$$

is exact so that the canonical map

$$B_\infty^k := \varprojlim_n \operatorname{Im}(A_\infty^{k-1} \xrightarrow{d_\infty} A_\infty^k) \longrightarrow \varprojlim_n B_n^k$$

is a bijection.

□

**Proposition 4.10.** *Let $X$ be a topological space and $\mathcal{F} \in D^+(\mathbf{Sh}_{\mathbb{Z}}(X))$. Consider an increasing sequence of open subsets $(U_n)_{n\geqslant 1}$ and a decreasing sequence of closed subsets. We set*

$$U = \bigcup_{n\geqslant 1} U_n, \quad Z_n = \bigcap_{n\geqslant 1} Z_n.$$

*Then the following hold.*

*(a) For any $j$ the natural map*

$$\phi_j : H_Z^j(X, \mathcal{F}) \to \varprojlim_n H_{Z_n}^j(U_n, \mathcal{F})$$

*is surjective.*

*(b) Assume that for a given $j$ the projective system $\{H^{j-1}_{Z_n}(U_n, \mathcal{F})\}_{n\geqslant 1}$ satisfies $ML$ Then $\phi_j$ is bijective.*
*(c) Suppose now that $(X_n)$ is an increasing family of subsets of $X$ satisfying*

$$X = \bigcup_n X_n, \quad X_n \subset \operatorname{Int}(X_{n+1}).$$

*If for some $j$ the projective system $\{H^{j-1}(X_n, \mathcal{F})$ satisfies $ML$ then the natural map*

$$H^j(X, \mathcal{F}) \to \varprojlim_n H^j(X_n, \mathcal{F})$$

*is bijective.*

**Proof** We can assume that $\mathcal{F}$ is a complex of flabby sheaves. Denote by $E^\bullet_n$ the simple complex associated to the double complex

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Gamma(U_n\backslash Z_n, \mathcal{F}^{j-1}) & \longrightarrow & \Gamma(U_n\backslash Z_n, \mathcal{F}^{j}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \Gamma(U_n, \mathcal{F}^{j-1} & \longrightarrow & \Gamma(U_n, \mathcal{F}^{j}) & \longrightarrow & \cdots
\end{array}$$

Then

$$H^j_{Z_n}(U_n, \mathcal{F}) \cong H^j(E^\bullet_n), \quad H^j_Z(U, \mathcal{F}) \cong H^j(\varprojlim_n E_n).$$

(a) and (b) follows from Proposition 4.9. Part (c) also follows from Proposition 4.9 using the fact that the projective system $\{H^{j-1}(\operatorname{Int} X_n, \mathcal{F})\}_{n\geqslant 1}$ satisfies $ML$ and

$$\varprojlim_n H^j(X_n, \mathcal{F}) \cong \varprojlim_n H^j(\operatorname{Int} X_n, \mathcal{F}).$$

□

It is very easy yet very profitable to enlarge the notion of projective, injective limit. We will achieve this by replacing the directed index family $I$ in the definition of a projective(inductive) family by a more general object.

Suppose $I, \mathcal{C}$ are categories. Then an *$I$-inductive* (resp. *$I$-projective*) family in $\mathcal{C}$ is a functor $X : I \to \mathcal{C}$ (resp. $Y : I^{op} \to \mathcal{C}$). We denote by $\mathbf{Ind}\,(I, \mathcal{C})$ (resp. $\mathbf{Proj}\,(I, \mathcal{C})$) the collection of inductive (resp. projective) families in $\mathcal{C}$. These collections are categories in a natural fashion.

Suppose $\mathcal{C} = \mathbf{Set}$ and $Y \in \mathbf{Proj}\,(I, \mathbf{Set})$. We define

$$\varprojlim_I Y = \Big\{ (y_i;\ i \in I) \in \prod_{i\in I} Y_i;\ \ y_i = Y(f) y_j,\ \ \forall i \xrightarrow{f} j \Big\}.$$

We have a natural functor

$$\mathbf{Const} : \mathcal{C} \to \mathbf{Proj}\,(I, \mathcal{C}), \quad \mathcal{C} \ni C \mapsto \mathbf{Const}_C,$$

where

$$\mathbf{Const}_C(i) = C, \ \ \mathbf{Const}_C(f) = \mathbb{1}_C, \ \ \forall i \xrightarrow{f} j.$$

Given $Y \in \mathbf{Proj}\,(I, \mathcal{C})$ and $C \in \mathcal{C}$ we can identify a morphism

$$\mathbf{Const}_C \to Y$$

with a collection of maps $C \stackrel{\psi_i}{\to} Y_i$ such that for every $i \stackrel{f}{\to} j$ we have a commutative diagram

$$\begin{array}{ccc} C & & \\ {\scriptstyle \psi_j}\downarrow & \searrow^{\psi_i} & \\ Y_j & \xrightarrow[X(f)]{} & Y_i \end{array}$$

Hence

$$\mathrm{Hom}_{\mathbf{Proj}}(\mathbf{Const}_C, Y) = \varprojlim_I \mathrm{Hom}_{\mathcal{C}}(C, Y_i).$$

Define

$$\varprojlim_I Y : C^{op} \to \mathbf{Set}, \quad C \mapsto \mathrm{Hom}_{\mathbf{Proj}}(\mathbf{Const}_C, Y) = \varprojlim_I \mathrm{Hom}_{\mathcal{C}}(C, Y_i).$$

Similarly, given $X \in \mathbf{Ind}\,(I, \mathcal{C})$

$$\varinjlim_I X : C \to \mathbf{Set}, \quad C \mapsto \mathrm{Hom}_{\mathbf{Ind}}(X, \mathbf{Const}_C) = \varprojlim_I \mathrm{Hom}_{\mathcal{C}}(X_i, C).$$

**Definition 4.11.** (a) If $X : I \to \mathcal{C}$ is an inductive family we say that $\varinjlim_I X_i$ exists in $\mathcal{C}$ if the functor

$$\varinjlim_I X : C \to \mathbf{Set}$$

is representable, i.e there exists an object $X_\infty \in \mathcal{C}$ and bijection

$$\mathrm{Hom}_{\mathcal{C}}(X_\infty, C) \cong \mathrm{Hom}_{\mathbf{Ind}}(X, \mathbf{Const}_C) \cong \varprojlim_I \mathrm{Hom}_{\mathcal{C}}(X_i, C)$$

natural in $C \in \mathcal{C}$.
(b) If $Y : I^{op} \to C$ is a projective limit in $\mathcal{C}$ then we say that the limit $\varprojlim_I Y_i$ exists in $\mathcal{C}$ if the functor $\varprojlim_I Y : C^{op} \to \mathbf{Set}$ is representable, i.e. there exists an object $Y_\infty \in \mathcal{C}$ and bijections

$$\mathrm{Hom}_{\mathcal{C}}(C, Y_\infty) \cong \mathrm{Hom}_{\mathbf{Proj}}(\mathbf{Const}_C, Y) \cong \varprojlim_I \mathrm{Hom}_{\mathcal{C}}(C, Y_i),$$

natural in $C \in \mathcal{C}$.

Observe that when $I$ is the category associated to a directed ordered set we obtain the old definitions of $\varinjlim$ and $\varprojlim$.

**Definition 4.12.** A (nonempty) category $I$ is called *directed* (or *filtrant*) if it satisfies the following conditions.

(i) For every $i, j \in I$ there exists $k \in I$ and morphisms $i \to k$ and $j \to k$.
(ii) For any morphisms $f, g : i \rightrightarrows j$ there exists a morphism $h : j \to k$ such that

$$hf = fg.$$

Suppose $I$ is a directed small category. Consider an inductive family of sets

$$X : I \to \mathbf{Set}, \quad i \mapsto X_i.$$

Observe that $\varinjlim_I X_i$ exists in $\mathbf{Set}$ and we have a natural isomorphism

$$\varinjlim_I X_i = \bigsqcup_{i \in I} X_i / \sim, \quad x_i \sim x_j \Longleftrightarrow \exists i \stackrel{f}{\to} k, \quad j \stackrel{g}{\to} k : \quad X(f)x_i = X(g)x_j$$

In the sequel we will work exclusively with directed small categories $I$ so that $\varinjlim_I$ exists in $\mathbf{Set}$.

Suppose we are given a category $\mathcal{C}$, an inductive family $X : I \to \mathcal{C}$, and a projective family $Y : I^{op} \to \mathcal{C}$. We obtain a contravariant functor

$$"\varinjlim_I X_i" : \mathcal{C}^{op} \ni C \mapsto \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(C, X_i) \in \mathbf{Set}$$

and a covariant functor

$$"\varprojlim_I "Y_i : \mathcal{C} \ni C \mapsto \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(Y_i, C).$$

A functor $\mathcal{C}^{op} \to \mathbf{Set}$ isomorphic to some $"\varinjlim_I X_i"$ is called an *IND-object* while a functor $\mathcal{C} \to \mathbf{Set}$ isomorphic to some $"\varprojlim_I "Y_i$ is called a *PRO-object*.

*Remark* 4.13. The reason for introducing such concepts is that the homological properties of the inductive (projective) limit constructions depend only on the IND (resp. PRO) objects they define (see [11, Exp. 6], [12]). More precisely two inductive families $(X_i)_{i\in I}$, $(X_j)_{j\in J}$ (resp two projective families $(Y_i)_{i\in I}$, $(Y_j)_{j\in J}$) are called *essentially equivalent* if they define isomorphic IND-objects (resp. PRO-objects). When working with inductive (projective) families in Abelian categories which admit arbitrary inductive (projective) limits then essentially equivalent families will have isomorphic higher derived limits $\varinjlim^q$ (resp. $\varprojlim^q$).

□

We would like to understand when an IND-object (resp. a PRO-object) is representable. Differently put, an inductive (projective) family defines a representable IND-object (resp. PRO-object) when it is essentially equivalent to a constant family. We consider only the case of $"\varinjlim_I X_i"$.

Let us first observe that every element $\phi \in \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(C, X_i)$ is described by some morphism

$$\phi_i : C \to X_i.$$

Two morphisms $\phi_i : C \to X_i$ and $\phi_j : C \to X_j$ describe the same element if there exists $k \in I$ and morphisms $i \xrightarrow{f} k$, $j \xrightarrow{g} k$ such that the diagrams below are commutative

$$\begin{array}{ccc} & & X_k \\ & \overset{\phi_k}{\nearrow} & \uparrow{\scriptstyle f} \\ C & \xrightarrow[\phi_i]{} & X_i, \end{array} \qquad \begin{array}{ccc} & & X_k \\ & \overset{\phi_k}{\nearrow} & \uparrow{\scriptstyle g} \\ C & \xrightarrow[\phi_j]{} & X_j. \end{array}$$

Suppose there exists an object $X_\infty \in C$ and natural isomorphisms

$$\varinjlim_I \mathrm{Hom}_{\mathcal{C}}(C, X_i) \cong \mathrm{Hom}_{\mathcal{C}}(C, X_\infty).$$

If we let $C = X_{i_0}$, for some $i_0 \in I$ we obtain

$$\varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_i) \cong \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_\infty)$$

The morphism $\mathbb{1}_{X_{i_0}} \in \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_0}) \subset \bigsqcup_I \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_i)$ defines an element in $\varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_i)$ and thus a morphism

$$\phi_{i_0} \in \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_\infty).$$

Suppose we have a morphism $i_0 \xrightarrow{f} j_0$. Then $f$ and $\mathbb{1}_{X_{i_0}}$ determine the same element in $\varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_i)$. We have a commutative diagram

$$\begin{array}{ccc} \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_{j_0}, X_i) & \longleftrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_{j_0}, X_\infty) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_i) & \longleftrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_\infty) \end{array}$$

which implies that

$$f^*(\phi_{j_0}) = \phi_{i_0} \Longleftrightarrow \phi_{i_0} = \phi_{j_0} \circ f, \quad \forall i_0 \xrightarrow{f} j_0.$$

If we take $C = X_\infty$ we conclude that the element $\mathbb{1}_{X_\infty} \in \mathrm{Hom}_{\mathcal{C}}(X_\infty, X_\infty)$ determines an element

$$\rho \in \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_\infty, X_i).$$

This is represented by an element in $\bigsqcup_I \mathrm{Hom}_{\mathcal{C}}(X_\infty, X_i)$ so that there exists $k_0 \in I$ and a morphism

$$\rho_{k_0} : X_\infty \to X_{k_0}$$

which corresponds to $\mathbb{1}_{X_\infty}$. Now consider a morphism $\phi_j : X_j \to X_\infty$. We get a commutative diagram

$$\begin{array}{ccc} \rho_{k_0} \in \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_\infty, X_i) & \longleftrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_\infty, X_\infty) \ni \mathbb{1}_{X_\infty} \\ {\scriptstyle \phi_j^*}\downarrow & & \downarrow{\scriptstyle \phi_j^*} \\ \rho_{k_0} \circ \phi_j \in \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_j, X_i) \ni \mathbb{1}_{X_j} & \longleftrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_j, X_\infty) \ni \phi_j. \end{array}$$

This shows that the morphisms $\rho_{k_0} \circ \phi_j$ and $\mathbb{1}_{X_j}$ define the same element in $\varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_j, X_i)$. Hence, there must exists $k > k_0, j$ and morphisms $k_0 \xrightarrow{f} k$, $j \xrightarrow{g}$ such that

$$\mathcal{F} \circ \rho_{k_0} \circ \phi_j = g \Longleftrightarrow \begin{array}{ccc} X_\infty & \xrightarrow{\rho_{k_0}} & X_{k_0} \\ {\scriptstyle \phi_j}\uparrow & & \downarrow{\scriptstyle f} \\ X_j & \xrightarrow[g]{} & X_k. \end{array}$$

If we now assume that $I$ is the category associated to a directed ordered set we obtain the following characterization of the functors $\mathbf{Lim}$. For more details we refer to [8, I.11] or [11, Exp. 6].

**Proposition 4.14.** *Suppose $I$ is a directed ordered set and $(A_i, \phi_{ji})_{j>i}$ is an inductive family and $(B_i, \psi_{ij})_{j>i}$ is a projective family in a category $\mathcal{C}$. Then the following hold.*

*(a) $A_\infty \cong \varinjlim_I A_i$ iff $A_\infty$ represents the functor*

$$\mathcal{C}^{op} \ni C \mapsto \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(C, A_i) \in \mathbf{Set}.$$

*natural in $X \in \mathcal{C}$.*
*(b) $B_\infty \cong \varprojlim_I B_i$ iff represents the functor*

$$\mathcal{C} \ni \mapsto \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(B_i, C) \cong \mathrm{Hom}_{\mathcal{C}}(B_\infty, X) \in \mathbf{Set}.$$

□

**Proposition 4.15.** *If $F : \mathcal{C} \to \mathcal{C}'$ is a functor and*

$$\underrightarrow{\mathbf{Lim}}_I X_i = X$$

*then*

$$\underrightarrow{\mathbf{Lim}}_I F(X_i) = F(\underrightarrow{\mathbf{Lim}}_I X_i) = F(X).$$

**Proof** Let $\rho_i = F(\phi_i) : F(X_i) \to F(X)$, $\rho_{ji} = F(\phi_{ji}) : F(X_i) \to F(X_j)$. Suppose we have and object $Z \in \mathcal{C}'$ and morphisms $g_i : F(X_i) \to Z$ such that for every $i < j$ the diagram below is commutative

$$\begin{array}{ccc} F(X_i) & \xrightarrow{\rho_{ji}} & F(X_j) \\ & {\scriptstyle g_i}\searrow \quad \swarrow{\scriptstyle g_j} & \\ & Z & \end{array}$$

We define $s_0 = F(e_0)$, $g := g_{i_0} s_0$. We have to show that for every $i \in I$ we have a commutative diagram

$$\begin{array}{ccc} F(X_i) & \xrightarrow{\rho_i} & F(X) \\ & {\scriptstyle g_i}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

Choose $j > i, i_0$ as in (4.1). Then

$$g_i = g_j \rho_{ji}, \;\; g_j \rho_{j i_0} = g_{i_0}.$$

On the other hand

$$\rho_{ji} \stackrel{(4.1)}{=} \rho_{j i_0} s_0 \rho_i \Longrightarrow g_i = g_j \rho_{j i_0} s_0 \rho_i = g_{i_0} s_0 \rho_i = g \rho_i.$$

□

**Definition 4.16.** Suppose $\mathcal{A}$ is an Abelian category, $\mathcal{C} \subset \mathcal{A}$ is a subcategory (not necessarily Abelian). We say that an inductive family $(X_i, \phi_{ji})_{i \in I}$ in $\mathcal{A}$ is *essentially of type* $\mathcal{C}$ if for every $i$ there exists $j > i$ such that the morphism $\phi_{ji}$ factorizes through an object in $\mathcal{C}$, i.e. for every $i < j$ there exists an object $C_{ji}$ in $\mathcal{C}$ and morphisms

$$f_{ji} : X_i \to C_{ji}, \;\; g_{ji} : C_{ji} \to X_j$$

such that the diagram below is commutative.

$$\begin{array}{ccc} X_i & & \\ {\scriptstyle f_{ji}}\downarrow & \searrow{\scriptstyle \phi_{ji}} & \\ C_{ji} & \xrightarrow[g_{ji}]{} & X_j \end{array}$$

For a proof of the following result we refer to [11, Exp. 6, Prop.5.4].

**Proposition 4.17.** *Suppose $(X_i, \phi_{ji})_{i\in I}$ is an inductive family in $\mathcal{A}$ such that $\varinjlim_I X_i$ exists. Then the family $(X_i)$ is essentially of type $\mathcal{C}$ if and only if the following holds.*

*There exists an inductive family $(C_s, f_{ts})_{s\in S}$ in $\mathcal{C}$ such that*

$$\varinjlim_S C_s \cong \varinjlim_I X_i.$$

**Example 4.18.** Suppose the category $\mathcal{C}$ consists of a single object $\mathcal{O}$ and a single morphism, $\mathbb{1}_{\mathcal{O}}$. An inductive family of type $\mathcal{C}$ is called an *essentially constant* family. We see that an essentially constant family is an inductive family $(X_i, \phi_{ji})_{i\in I}$ for which $\varinjlim_I X$ exists. □

## 5. Cohomologically constructible sheaves

We need a brief algebra interlude. $R$ will denote a Noetherian ring with 1. We assume that $A$ has *finite global dimension*, $\operatorname{gldim} R < \infty$.

In geometric applications it is desirable to associate numerical invariants to various objects in a derived category of complexes. Clearly this cannot happen without some finiteness assumption. The perfect complexes are precisely those from which we can extract numerical invariants.

When working with sheaves, it is convenient and necessary to impose certain rigidity assumptions, much like the requirement of coherence for the sheaves in algebraic geometry. The right notion of rigidity will be that of constructibility.

**Definition 5.1.** An object $C^\bullet \in D^b({}_R\mathbf{Mod})$ is called *perfect* if it is isomorphic (in $D^b$) to a bounded complex of finitely generated projective $R$-modules.

We denote by ${}_R\mathbf{Mod}^f$ the Abelian category of finitely generated $R$-modules and by $D^b({}_R\mathbf{Mod}^f)$ the associated derived category of bounded complexes of finitely generated $R$-modules.

**Proposition 5.2.** *Every object in $D^b({}_R\mathbf{Mod}^f)$ is perfect.*

**Proof** We have to prove that any bounded complex of finitely generated $R$-modules is quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules. The finitely generated projective modules form a full additive category of the Abelian category of finitely generated $R$-modules. Moreover, every finitely generated $R$-module is the quotient of a finitely generated $R$-module. If we choose $n \geqslant \operatorname{gldim} R$ then we deduce that if

$$X \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is an exact sequence with $M$ and $X$ finite and $P_i$ finite projective then we deduce $X$ must be projective. We can now conclude using [8, Cor. 1.7.8].

□

We denote by $D^b_f({}_R\mathbf{Mod})$ the full subcategory of $D^b({}_R\mathbf{Mod})$ consisting of complexes with finitely generated cohomology. Clearly $D^b({}_R\mathbf{Mod}^f)$ is a full subcategory of $D^b_f({}_R\mathbf{Mod})$.

**Proposition 5.3.** *The inclusion*

$$i : D^b({}_R\mathbf{Mod}^f) \hookrightarrow D^b_f({}_R\mathbf{Mod})$$

*is an equivalence of categories. In other words, to every bounded complex with finite cohomology we can associate in a natural way a quasi-isomorphic bounded complex of finite projective modules.* □

**Proof** The category of finitely generated $R$-modules is a *thick* Abelian subcategory of ${}_R\mathbf{Mod}$ (see [8, §1.7] for a definition of thickness). Observe that for any finitely generated $R$-module $M$ and every epimorphism of $R$-modules $f : N \to M$ there exists a finitely generated $R$-module $X$ and a morphism $g : X \to N$ such that the composition

$$g \circ f : X \to M$$

is an epimorphism. We can now apply [8, Prop. 1.7.11][5].

□

*Remark* 5.4. Putting together the above results we deduce that every bounded complex with finitely generated cohomology is quasi-isomorphic to a perfect complex, i.e. finite length complex of finitely generated projectives. □

Given $R$-modules $A$, $B$, $C$ we have natural morphisms

$$\mathbf{adj} : \operatorname{Hom}_R(A \otimes B, C) \to \operatorname{Hom}_R(A, \operatorname{Hom}(B, C)),$$

$$\operatorname{Hom}_R(A \otimes B, C) \ni T \mapsto \mathbf{adj}_T \in \operatorname{Hom}_R(A, \operatorname{Hom}(B, C)), \;\; \mathbf{adj}_T(a)(b) = T(a \otimes b).$$

and

$$\mathbf{ev} : \operatorname{Hom}_R(A, B) \otimes_R C \to \operatorname{Hom}_R(A, B \otimes_R C),$$

$$\operatorname{Hom}_R(A, B) \otimes_R C \ni T \otimes c \mapsto \mathbf{ev}_T \in \operatorname{Hom}_R(A, B \otimes_R C), \;\; \mathbf{ev}_T(a) = T(a) \otimes c.$$

The first morphism is an isomorphism and it shows that the functor $\operatorname{Hom}(B, -)$ is the right adjoint of the functor $\otimes_R B$. The second morphism is in general not an isomorphism. For example, if we take $B = R$ so that $\operatorname{Hom}(A, B) = \operatorname{Hom}(A, R) =: A^*$ then the image of $\mathbf{ev}$ consists of morphisms $A \to C$ with finitely generated image. Still, there is a more subtle obstruction. We have the following result, [1, §1, Thm. 1].

**Proposition 5.5.** *The following are equivalent.*

*(a) The $R$-module $C$ is flat.*
*(b)* $\mathbf{ev} : \operatorname{Hom}_R(A, B) \otimes_R C \to \operatorname{Hom}_R(A, B \otimes_R C)$ *is an isomorphism for any finitely* **presented** *$R$-module $A$ and any $R$-module $B$.*
*(c) The natural morphism $A^* \otimes C \to \operatorname{Hom}_R(A, C)$ is an isomorphism for any finitely presented $R$-module $A$.*

Any finitely generated projective module over a Noetherian ring is automatically finitely presented. We deduce that for a flat module $F$ and a finitely generated projective module we have an isomorphism

$$P^* \otimes F \cong \operatorname{Hom}(P, F).$$

Note that for every $R$-module $M$ have a tautological morphism

$$J_M : M \to M^{**}$$

which corresponds to the tautological linear map

$$M \otimes M^* \to R, \;\; m \otimes m^* \mapsto m^*(m),$$

via the adjunction isomorphism

$$\operatorname{Hom}(M \otimes M^*, R) \to \operatorname{Hom}(M, \operatorname{Hom}(M^*, R)).$$

[5] We are actually using the dual statement, with all arrows reversed, and monomorphisms replaced by epimorphisms

Observe that if any morphism of modules $M \xrightarrow{f} N$ induces a morphism $f^{**} : M^{**} \to N^{**}$ and the diagram

$$\begin{array}{ccc} M & \xrightarrow{J_M} & M^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ N & \xrightarrow[J_N]{} & N^{**} \end{array}$$

**Proposition 5.6.** *Assume $R$ is Noetherian with $1$. Suppose $P$ is a finitely generated projective module. Then the tautological morphism*

$$J : P \to P^{**}$$

*is an isomorphism.*

**Proof** We have a short exact sequence

$$0 \to K \to R^n \xrightarrow{\pi} P \to 0.$$

Since $P$ is projective the sequence is split. Fix a section $s : P \to R^n$, $\pi \circ s = \mathbb{1}_P$. Using the tautological identifications $R^* = R$ and

$$(R^n)^* = \left(\oplus_{j=1}^n R\right)^n \cong \prod_{j=1}^n R^* \cong \oplus_{j=1}^n R^*$$

we get a split exact sequence

$$0 \to K^{**} \to (R^n)^{**} \to P^{**} \to 0$$

and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R^n & \underset{s}{\overset{\pi}{\rightleftarrows}} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle J_K} & & \downarrow{\scriptstyle J_R} & & \downarrow{\scriptstyle J_P} & & \\ 0 & \longrightarrow & K & \longrightarrow & R^n & \underset{s^{**}}{\overset{\pi^{**}}{\rightleftarrows}} & P & \longrightarrow & 0 \end{array}$$

Since $\pi^{**}$ is onto we deduce $J_P$ is onto. On the other hand

$$s^{**} J_P = J_R s.$$

and since $J_R s$ is one-to-one we deduce that $J_P$ is one-to-one.

□

**Proposition 5.7** (Special adjunction formula)**.** [6] *Suppose $R$ is a Noetherian ring with $1$ such that* $\operatorname{gldim} R < \infty$. *Then we have a natural isomorphism*

$$\operatorname{RHom}(A^\bullet, B^\bullet) \overset{\mathrm{L}}{\otimes} C^\bullet \cong \operatorname{RHom}(A^\bullet, \operatorname{RHom}(B^\bullet, C^\bullet)) \tag{5.1}$$

*which is natural in $A^\bullet, B^\bullet \in D_f^b({}_R\mathbf{Mod})$, $C^\bullet \in D^b({}_R\mathbf{Mod})$.*

**Proof** This is clearly true for $A^\bullet, B^\bullet \in D^b({}_R\mathbf{Mod}^f)$, $C \in D^b({}_R\mathbf{Mod})$. Next use the equivalence of categories

$$D^b({}_R\mathbf{Mod}^f) \hookrightarrow D_f^b({}_R\mathbf{Mod}).$$

□

[6]"Formule d'adjonction cher à Cartan"

**Definition 5.8.** Suppose $X$ is a locally compact space of finite soft dimension and $R$ is a Noetherian ring of finite global dimension. An object $\mathcal{F} \in D^b(\mathbf{Sh}_R(X))$ is called *cohomologically constructible* (c.c. for brevity) if for any point $x \in X$ the following conditions are satisfied.
(a) $\underrightarrow{\mathbf{Lim}}_{U \ni x} R\Gamma(U, \mathcal{F})$ and $\underleftarrow{\mathbf{Lim}}_{U \cap x} R\Gamma_c(U, \mathcal{F})$ exist and the canonical maps

$$\underrightarrow{\mathbf{Lim}}_{U \ni x} R\Gamma(U, \mathcal{F}) \to \mathcal{F}_x, \tag{5.2a}$$

$$R\Gamma_{\{x\}}(X, \mathcal{F}) \to \underleftarrow{\mathbf{Lim}}_{U \ni x} R\Gamma_c(U, \mathcal{F}) \tag{5.2b}$$

are isomorphisms (in the derived category).
(b) The complexes $\mathcal{F}_x$ and $R\Gamma_{\{x\}}(X, \mathcal{F})$ are perfect.

The cohomologically constructible complexes form a full subcategory of $D^b(\mathbf{Sh}_R(X))$ which we denote by $D_{cc}(\mathbf{Sh}_R(X))$.

*Remark* 5.9. Observe that condition (a) can be rephrased as saying that for every $x \in X$ the inductive system

$$\big\{ R\Gamma(U, \mathcal{F});\;\; U \text{ neighborhood of } x \,\big\}$$

and the projective system

$$\big\{ R\Gamma_c(U, \mathcal{F});\;\; U \text{ neighborhood of } x \,\big\}$$

are essentially constant and

$$\varinjlim_{U \ni x} R\Gamma(U, \mathcal{F}) \cong \mathcal{F}_x, \;\; \varprojlim_{U \ni x} R\Gamma_c(U, \mathcal{F}) \cong R\Gamma_{\{x\}}(X, \mathcal{F}).$$

The constructibility condition has some obvious co-homological consequences. It implies for example that the inductive system

$$\big\{ \mathbb{H}^\bullet(U, \mathcal{F});\;\; U \text{ neighborhood of } x \,\big\}$$

and the projective system

$$\big\{ \mathbb{H}^\bullet_K(X, \mathcal{F});\;\; K \text{ compact neighborhood of } x \,\big\}$$

are essentially constant in ${}_R\mathbf{Mod}$, and

$$\varinjlim_{U \ni x} \mathbb{H}^\bullet(U, \mathcal{F}) \cong \mathbb{H}^\bullet(\mathcal{F}_x) \in {}_R\mathbf{Mod}^f, \;\; \varprojlim_{K \ni x} \mathbb{H}^\bullet_K(X, \mathcal{F}) \cong \mathbb{H}^\bullet_{\{x\}}(X, \mathcal{F}) \in {}_R\mathbf{Mod}^f \,.$$

Often all one needs in applications are these cohomological statements (see [2, Chap. V], [11, Exp. 8, 9] for more details). In particular, in [2, Chap. V,§3] it is shown that if

$$\mathbb{H}^\bullet(\mathcal{F}_x), \quad \mathbb{H}^\bullet_{\{x\}}(X, \mathcal{F}) \in {}_R\mathbf{Mod}^f$$

then the condition

$$\varprojlim_{K \ni x} \mathbb{H}^\bullet_K(X, \mathcal{F}) \cong \mathbb{H}_{\{x\}}(X, \mathcal{F})$$

implies

$$\varinjlim_{U \ni x} \mathbb{H}^\bullet(U, \mathcal{F}) \cong \mathbb{H}^\bullet(\mathcal{F}_x).$$

□

We will discuss later on how to produce a large supply of c.c. complexes of sheaves.

## 6. Duality with coefficients in a field. The absolute case

Suppose $\mathbb{K}$ is a field and $X$ is a locally compact space. For every $\mathbb{K}$-vector space $E$ we denote by $E^\vee$ its dual

$$E^\vee := \operatorname{Hom}_{\mathbb{K}}(E, \mathbb{K}).$$

Suppose $\mathcal{S}$ is a sheaf $\mathbb{K}$-vector spaces on $X$. The inclusion $j : V \hookrightarrow U$ of two open sets induces an "extension by zero" map

$$j_! : \Gamma_c(V, \mathcal{S}) \to \Gamma_c(U, s)$$

and by duality a map

$$j_!^\vee : \Gamma_c(U, \mathcal{S})^\vee \to \Gamma_c(V, \mathcal{S})^\vee.$$

We obtain a presheaf $\mathcal{S}^\vee$ by setting

$$\Gamma(U, \mathcal{S}^\vee) := \Gamma_c(U, \mathcal{S})^\vee.$$

*Remark* 6.1 (Food for thought). The presheaf $\mathcal{S}^\vee$ should not be confused with the presheaf $\underline{Hom}(\mathcal{S}, \underline{\mathbb{K}})$. Suppose $\mathbb{K} = \mathbb{R}$ and $\mathcal{S}$ is the constant sheaf $\underline{\mathbb{R}}$ on a compact smooth manifold $X$. Then $\underline{Hom}(\mathcal{S}, \underline{\mathbb{K}}) \cong \underline{\mathbb{R}}$. On the other hand $\mathcal{S}^\vee = 0$ since $\Gamma_c(V, \underline{\mathbb{R}}) = 0$ for every open ball $V \hookrightarrow X$.

As a different type of example consider $X = \mathbb{R}$. For each open set $U \subset \mathbb{R}$ denote by $\Omega^1$ the sheaf of 1-forms on $U$.

Let $U = (-1, 1)$ and set $\eta = \frac{1}{1-x^2}dx$. The segment $I := (-1, 1)$ can be viewed as a linear functional $\Omega^1_c(U) \to \mathbb{R}$ by

$$\Omega^1_c(U) \ni \varphi \mapsto (\varphi) := \int_I \varphi.$$

Although the linear functional $\int_I : \Omega^1_c(U) \to \mathbb{R}$ admits many extensions to a linear functional $\Omega^1(U) \to \mathbb{R}$, the "natural choice" is not defined for the 1-form $\eta$.

□

**Proposition 6.2.** *If $\mathcal{S}$ is a $c$-*soft *sheaf on the locally compact space $X$ then the presheaf $\mathcal{S}^\vee$ is a* sheaf.

**Proof** Suppose $U, V$ are two open subsets of $X$. Applying the functor $\Gamma_c$ to the Mayer-Vietoris sequence (2.12a) we obtain an exact sequence

$$0 \to \Gamma_c(U \cap V, \mathcal{S}) \to \Gamma_c(U, \mathcal{S}) \oplus \Gamma_c(V, \mathcal{S}) \to \Gamma_c(U \cup V, \mathcal{S}) \to H^1_c(U \cap V, \mathcal{S}).$$

Since the restriction to an open set of a $c$-soft sheaf is also $c$-soft we deduce $H^1_c(U \cap V, \mathcal{S}) = 0$ so that we have a short exact sequence

$$0 \to \Gamma_c(U \cap V, \mathcal{S}) \to \Gamma_c(U, \mathcal{S}) \oplus \Gamma_c(V, \mathcal{S}) \to \Gamma_c(U \cup V, \mathcal{S}) \to 0.$$

Applying $\operatorname{Hom}_{\mathbb{K}}(-, \underline{\mathbb{K}})$ to this sequence we obtain an exact sequence

$$0 \to \Gamma(U \cup V, \mathcal{S}^\vee) \to \Gamma(U, \mathcal{S}^\vee) \oplus \Gamma(V, \mathcal{S}^\vee) \to \Gamma(U \cap V, \mathcal{S}^\vee). \tag{6.1}$$

This shows that given two sections $s_U \in \Gamma(U, \mathcal{S}^\vee)$ and $s_V \in \Gamma(V, \mathcal{S}^\vee)$ which agree on the overlap there exists a unique section $s_{U \cup V} \in \Gamma_c(U \cup V, \mathcal{S}^\vee)$ which extends both $s_U$ and $s_V$.

Consider now a *directed family* $\mathcal{U}$ of open subsets of $X$, i.e. for any $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \supset U \cup V$. Denote by $\mathcal{O}$ the union of all open sets in $\mathcal{U}$. We see immediately from the *compactness* of supports that

$$\Gamma_c(\mathcal{O}, \mathcal{S}) = \varinjlim_{U \in \mathcal{U}} \Gamma_c(U, \mathcal{S}).$$

Using the natural isomorphisms

$$\operatorname{Hom}_{\mathbb{K}}(\varinjlim_{U \in \mathcal{U}} \Gamma_c(U, \mathcal{S}), \mathbb{K}) \cong \varprojlim_{U \in \mathcal{U}} \operatorname{Hom}_{\mathbb{K}}(\Gamma_c(U, \mathcal{S}), \mathbb{K}) = \varprojlim_{U \in \mathcal{U}} \Gamma(U, \mathcal{S}^\vee)$$

We deduce that given a *directed family* $s_U \in \operatorname{Hom}_{\mathbb{K}}(\Gamma_c(U,\mathcal{S}),\mathbb{K}) = \Gamma_c(U,\mathcal{S}^\vee)$, $U \in \mathcal{U}$ (i.e. a family satisfying $s_U|_{\Gamma_c(V,\mathcal{S})} = s_V$, $\forall U \supset V$) there exists a unique $s_{\mathcal{O}} : \Gamma_c(\mathcal{O},\mathcal{S}) \to \mathbb{K}$ such that

$$s_{\mathcal{O}}|_{\Gamma_c(U,\mathcal{S})} = s_U, \quad \forall U \in \mathcal{U}.$$

Given an arbitrary open cover $\mathcal{V}$ of an open subset $\mathcal{O} \subset X$ we can enlarge it to the *directed* cover $\tilde{\mathcal{V}}$ of $\mathcal{O}$ consisting of all the finite union of members of $\mathcal{V}$. A family of sections $\{s_V;\ \ V \in \mathcal{V}\}$ which agree on the overlaps, extends using (6.1) to a *directed* family of sections $\{s_U;\ \ U \in \tilde{\mathcal{V}}\}$ which defines a unique section $s_{\mathcal{O}} \in \Gamma(\mathcal{O},\mathcal{S}^\vee)$. This implies that $\mathcal{S}^\vee$ defines a sheaf.

□

**Proposition 6.3.** *Suppose $X$ is a* finite dimensional *locally compact space. Then for every $c$-soft sheaf $\mathcal{S} \in \mathbf{Sh}_{\mathbb{K}}(X)$ and any $\mathcal{F} \in \mathbf{Sh}_{\mathbb{K}}(X)$ the sheaf $\mathcal{S} \otimes_{\mathbb{K}} \mathcal{F}$ is $c$-soft.*

**Proof** The proof is carried in two steps. First we notice that if $j : U \to X$ denotes the inclusion of an open set then we have a natural isomorphism

$$\mathcal{S} \otimes j_!({}_U\underline{\mathbb{K}}) \cong j_! j^{-1}\mathcal{S} = \mathcal{S}_U.$$

Next we construct a resolution of $F$

$$\cdots \to \mathcal{P}_r \to \cdots \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{F} \to 0,$$

where each $\mathcal{P}_j$ is a direct sum of sheaves of the form $\underline{\mathbb{K}}_U$, $U \stackrel{j}{\hookrightarrow} X$ open. Since we are working with sheaves of $\mathbb{K}$-vector spaces the functor

$$\mathbf{Sh}_{\mathbb{K}}(X) \stackrel{\mathcal{S}\otimes_{\mathbb{K}}}{\longrightarrow} \mathbf{Sh}_{\mathbb{K}}(X)$$

is exact and thus we get another resolution

$$\cdots \to \mathcal{S} \otimes \mathcal{P}_r \stackrel{d_r}{\to} \cdots \to \mathcal{S} \otimes \mathcal{P}_1 \stackrel{d_1}{\to} \mathcal{S} \otimes \mathcal{P}_0 \stackrel{d_0}{\to} \mathcal{S} \otimes \mathcal{F} \to 0,$$

In particular we get a resolution of $\ker d_r$

$$0 \to \ker d_r \to \mathcal{S} \otimes \mathcal{P}_{r-1} \to \cdots \to \mathcal{S} \otimes \mathcal{P}_1 \to \mathcal{S} \otimes \mathcal{P}_0 \stackrel{d_0}{\to} \mathcal{S} \otimes \mathcal{F} \to 0$$

where $\mathcal{S} \otimes \mathcal{P}_j$ are $c$-soft for any $0 \leqslant j < r$. If $r > \dim_s X$ we deduce $\mathcal{F} \otimes \mathcal{S}$ is $c$-soft.

□

**Proposition 6.4.** *Let $X$ be a finite dimensional locally compact space and $\mathcal{S}$ a $c$-soft sheaf. Then there is an isomorphism of functors*

$$\mathbf{Sh}_{\mathbb{K}}(X) \ni \mathcal{F} \to \Gamma_c(X, \mathcal{F} \otimes \mathcal{S})^\vee \in \mathbf{Vect}_{\mathbb{K}}$$

*and*

$$\mathbf{Sh}_{\mathbb{K}}(X) \ni \mathcal{F} \to \operatorname{Hom}(\mathcal{F}, \mathcal{S}^\vee) \in \mathbf{Vect}_{\mathbb{K}}.$$

*In other words, the functor*

$$\mathbf{Sh}_{\mathbb{K}}(X)^{op} \ni \mathcal{F} \to \Gamma_c(X, \mathcal{F} \otimes \mathcal{S})^\vee \in \mathbf{Vect}_{\mathbb{K}}$$

*is represented by the sheaf $\mathcal{S}^\vee$.*

**Idea of proof** First we need to define the morphism of functors. In other words, for any sheaf $\mathcal{F}$ we need to construct linear map

$$T_{\mathcal{F}} : \Gamma_c(X, \mathcal{F} \otimes \mathcal{S})^\vee \to \operatorname{Hom}(\mathcal{F}, \mathcal{S}^\vee).$$

Equivalently, this means that we have to associate to each open set $U \hookrightarrow X$ a linear map

$$T_{\mathcal{F},U} : \Gamma_c(X, \mathcal{F} \otimes \mathcal{S})^\vee \longrightarrow \operatorname{Hom}_{\mathbb{K}}(\Gamma(U,F), \Gamma_c(U,\mathcal{S})^\vee).$$

which is compatible with the restriction maps. Given a linear functional

$$L : \Gamma_c(X, \mathcal{F} \otimes \mathcal{S}) \to \mathbb{K},$$

we define

$$L_U : \Gamma(U, \mathcal{F}) \to \Gamma_c(U, \mathcal{S})^{\vee}$$

as follows. For $f_U \in \Gamma(U, \mathcal{F})$ and $s_U \in \Gamma_c(U, \mathcal{S})$ we have $f_U \otimes s_U \in \Gamma_c(U, \mathcal{F} \otimes \mathcal{S}) \subset \Gamma_c(X, \mathcal{F} \otimes \mathcal{S})$ and we set

$$\langle L_U(f_U), s_U \rangle = \langle L, (f_U \otimes s_U) \rangle,$$

where $\langle -, - \rangle$ denotes the natural pairing between a vector space and its dual.

We observe first that $T_{\mathcal{F}}$ is an isomorphism when $\mathcal{F}$ is a sheaf of the form $\mathbb{K}_U$, $U \hookrightarrow X$ open subset. In general, any $\mathbb{K}$-sheaf $\mathcal{F}$ admits a presentation of the form

$$R \to G \to \mathcal{F} \twoheadrightarrow 0$$

where $G$ and $R$ are direct sums of sheaves of the form $\underline{\mathbb{K}}_U$. The two functors are left exact and we conclude using the five-lemma.

□

*Remark* 6.5. Proposition 6.4 result is the heart of the Verdier duality. When the space $X$ is a point it coincides with the adjunction formula

$$(U \otimes_{\mathbb{K}} V^{\vee})^{\vee} \cong \operatorname{Hom}_{\mathbb{K}}(U, V),$$

for every $\mathbb{K}$-vector spaces $U$ and $V$. □

**Corollary 6.6.** *If $X$ and $\mathcal{S}$ are as in Proposition 6.4 then the sheaf $\mathcal{S}^{\vee}$ is injective.*

**Proof** We have to show that the functor

$$\mathbf{Sh}_{\mathbb{K}}(X) \ni \mathcal{F} \mapsto \operatorname{Hom}(\mathcal{F}, \mathcal{S}^{\vee}) = \Gamma_c(X, \mathcal{F} \otimes \mathcal{S})^{\vee}$$

is exact.

Indeed, suppose we are given a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

Since we work with sheaves of *vector spaces* we get a short *exact* sequence of *c-soft sheaves*

$$0 \to \mathcal{F}' \otimes \mathcal{S} \to \mathcal{F} \otimes \mathcal{S} \to \mathcal{F}'' \otimes \mathcal{S} \to 0$$

The conclusion follows from the fact that the $c$-soft sheaves are $\Gamma_c$-injective objects.

□

Let $X$ denote a locally compact space of finite dimension and let $\mathbb{K}$ denote a fixed field. We denote by $D^+_{\mathbb{K}}(X)$ the derived category of bounded from below complexes of sheaves of $\mathbb{K}$-vector spaces. We denote by $D^+_{\mathbb{K}}(\mathcal{I}_X)$ subcategory of $K^+(\mathbf{Sh}_{\mathbb{K}} X)$ of complexes of injective sheaves. We know that $D^+_{\mathbb{K}}(\mathcal{I}_X)$ is equivalent to the derived category $D^+_{\mathbb{K}}(X)$. When $X = *$ is the space consisting of a single vector space then $D^+_{\mathbb{K}}(*)$ is equivalent to the category of bounded from below complexes of $\mathbb{K}$-vector spaces with trivial differentials.

To every object $I^{\bullet} \in D^+_{\mathbb{K}}(\mathcal{I}_X)$ we associate the vector space

$$[\Gamma_c(X, I^{\bullet}), \mathbb{K}] \cong [\Gamma_c(X, I^{\bullet}), [\mathbb{K}]].$$

We get a (contravariant) functor

$$D^+(\mathcal{I}_X)^{op} \to \mathbf{Vect}_{\mathbb{K}}, \quad I^{\bullet} \mapsto [\Gamma_c(X, I^{\bullet}), \mathbb{K}].$$

We will prove that this functor is representable. More precisely we have the following result.

**Theorem 6.7** (Poincaré-Verdier. The absolute case)**.** *Suppose $X$ is a finite dimensional, locally compact space. Then there exists $\mathcal{D}^\bullet \in D^+_{\mathbb{K}}(\mathcal{I}_X)$ and a natural isomorphism of vector spaces*

$$[I^\bullet, \mathcal{D}^\bullet](= \mathrm{Hom}_{D^+_{\mathbb{K}}(\mathcal{I}_X)}(I^\bullet, \mathcal{D}^\bullet)) \cong [\Gamma_c(X, I^\bullet), \mathbb{K}]$$

*as $I^\bullet$ varies through $\mathcal{D}^+(\mathcal{I}_X)$.*

**Proof** Fix a finite soft resolution $(\mathcal{S}^\bullet, d)$ of $\mathbb{K}_X$

$$0 \to \underline{\mathbb{K}} \to \mathcal{S}^0 \to \cdots \to \mathcal{S}^n \to 0, \;\; n = \dim X.$$

Define the complex $(\mathcal{S}^{\vee\bullet}, d^\vee)$, where

$$\Gamma\big(\, U, (\mathcal{S}^\vee)^m \,\big) = \mathrm{Hom}(\Gamma_c(U, \mathcal{S}^{-m}), \mathbb{K}),$$

$$d^\vee = (-1)^{m+1} d^* : \mathrm{Hom}(\Gamma_c(U, \mathcal{S}^{-m}), \mathbb{K}) \to \mathrm{Hom}(\Gamma_c(U, \mathcal{S}^{-m-1}), \mathbb{K})$$

For every integers $p$ and $q$ we have according to Proposition 6.4 natural isomorphisms

$$\Gamma_c(X, I^p \otimes \mathcal{S}^q)^\vee \cong \mathrm{Hom}(I^p, (\mathcal{S}^\vee)^q).$$

Taking the direct sum over all $p + q = -m$ we obtain a canonical isomorphism of vector spaces

$$\Big\{\Gamma_c(\, X, I^\bullet \otimes \mathcal{S}^\bullet\,)^\vee\Big\}^m \cong \mathrm{Hom}^m(I^\bullet, (\mathcal{S}^\vee)^\bullet).$$

**Lemma 6.8.** *The above isomorphisms induce isomorphisms of* complexes

$$\Psi : \Big(\Big\{\Gamma_c(\, X, I^\bullet \otimes \mathcal{S}^\bullet\,)^\vee\Big\}^\bullet, \delta\Big) \xrightarrow{\cong} \Big(\mathrm{Hom}^\bullet(I^\bullet, (\mathcal{S}^\vee)^\bullet), d\Big).$$

**Proof** Suppose $L : \Gamma_c(\, X, I^p \otimes \mathcal{S}^q\,) \to \mathbb{K}$, $p + q = -m$. We have to show that

$$\Psi\delta(L) = d\Psi(L) \;\text{ in }\; \mathrm{Hom}^{m+1}(I^\bullet, (\mathcal{S}^\vee)^\bullet).$$

We have

$$\Psi(L) \in \mathrm{Hom}(I^p, (\mathcal{S}^\vee)^q), \quad d\Psi(L) = u \oplus v \in \mathrm{Hom}(I^{p+1}, (\mathcal{S}^\vee)^q) \oplus \mathrm{Hom}(I^p, (\mathcal{S}^\vee)^{q+1}).$$

More precisely

$$d\Psi(L) = d^\vee_{\mathcal{S}^\bullet} \circ \Psi(L) + (-1)^{m+1}\Psi(L) \circ d_{I^\bullet} = (-1)^{m+1}\Big(\Psi(L) \circ d_{I^\bullet} + (-1)^p d^*_{\mathcal{S}^\bullet}\Psi(L)\Big)$$

$$\Psi(\delta L) = (-1)^{m+1}\Psi\big(L \circ d_{I^\bullet \otimes \mathcal{S}^\bullet}\big)$$

$$= (-1)^{m+1}\Big(\Psi\big(L \circ (d_{I^\bullet} \otimes \mathbb{1}_{\mathcal{S}^\bullet})\big) + (-1)^p \Psi\big(L \circ (\mathbb{1}_{I^\bullet} \otimes d_{\mathcal{S}^\bullet})\big)\Big).$$

This proves the desired claim.

□

From the quasi-isomorphism ${}_X\underline{\mathbb{K}} \rightsquigarrow \mathcal{S}^\bullet$ we get a quasi-isomorphism $I^\bullet \rightsquigarrow I^\bullet \otimes \mathcal{S}^\bullet$[7] and since the complexes are $c$-soft we obtain quasi-isomorphisms

$$\Gamma_c(X, I^\bullet) \rightsquigarrow \Gamma_c(X, I^\bullet \otimes \mathcal{S}^\bullet) \Longrightarrow \Gamma_c(X, I^\bullet)^\vee \leftsquigarrow \Gamma_c(X, I^\bullet \otimes \mathcal{S}^\bullet)^\vee.$$

Using the isomorphism in Lemma 6.8 we obtain a quasi-isomorphism

$$\mathrm{Hom}^\bullet(I^\bullet, (\mathcal{S}^\vee)^\bullet) \rightsquigarrow \Gamma_c(X, I^\bullet)^\vee.$$

[7] We are tacitly using the fact that $I^\bullet$ consists of $\mathbb{K}$-flat objects.

Taking $H^0$ of both sides we obtain an isomorphism

$$[I^\bullet, (\mathcal{S}^\vee)^\bullet] \to [\Gamma_c(X, I^\bullet), \mathbb{K}].$$

We have obtained the duality theorem with $\mathcal{D}^\bullet = (\mathcal{S}^\vee)^\bullet$.

□

The complex $\mathcal{D}^\bullet$ is called the *dualizing complex*. It is a bounded complex of injective objects uniquely determined up to homotopy. Moreover, the above proof shows that we can choose $\mathcal{D}^\bullet$ such $\mathcal{D}^p = 0$ if $p \notin [-\dim X, 0]$. The cohomology sheaves $\mathcal{H}^p(\mathcal{D}^\bullet)$ are uniquely determined up to isomorphism. We have the following result.

**Proposition 6.9.** *For any integer $p$, the cohomology sheaf $\mathcal{H}^{-p}(\mathcal{D}^\bullet)$ is the sheaf associated to the presheaf*

$$U \mapsto \operatorname{Hom}(H^p_c(U, \underline{\mathbb{K}}), \mathbb{K}).$$

**Proof** We want to prove that we have an isomorphism

$$H^p_c(X, \mathcal{F})^\vee \cong H^{-p} \operatorname{Hom}(\mathcal{F}, \mathcal{D}^\bullet) \tag{6.2}$$

natural in $\mathcal{F} \in \mathbf{Sh}(X)$. To do this, we choose an injective resolution $\mathcal{F} \rightsquigarrow I^\bullet$. Notice that

$$H^p_c(X, \mathcal{F}) = H^p\Gamma_c(X, I^\bullet) = H^0\Gamma_c(X, I[p]^\bullet).$$

From the duality formula we have

$$H^p_c(X, \mathcal{F})^\vee \cong [\Gamma_c(X, I[p]^\bullet), \mathbb{K}] = [I[p]^\bullet, \mathcal{D}^\bullet] \cong [I^\bullet, \mathcal{D}^\bullet[-p]]$$

Using the quasi-isomorphism $\mathcal{F} \rightsquigarrow I^\bullet$ we deduce from Theorem 1.2 that we have an isomorphism

$$[I^\bullet, \mathcal{D}^\bullet[-p]] \cong [\mathcal{F}, \mathcal{D}^\bullet[-p]] = H^{-p} \operatorname{Hom}(\mathcal{F}, \mathcal{D}^\bullet).$$

This concludes the proof of (6.2).

Let us now choose $\mathcal{F} = \underline{\mathbb{K}}_U$ in (6.2), $U \hookrightarrow X$ open subset. If we denote by $i$ the natural inclusion then $\underline{\mathbb{K}}_U = i_! i^{-1}(\underline{\mathbb{K}})$. We deduce

$$H^p_c(X, \underline{\mathbb{K}}_U) \cong H^{-p} \operatorname{Hom}(i_! i^{-1}\underline{\mathbb{K}}, \mathcal{D}^\bullet).$$

The computations in Example 2.18(a) show that

$$H^p_c(X, \underline{\mathbb{K}}_U) \cong H^p_c(U, \mathbb{K}).$$

Hence

$$\begin{aligned} H^p_c(U, \mathbb{K})^\vee &\cong H^{-p} \operatorname{Hom}(i_! i^{-1}\underline{\mathbb{K}}, \mathcal{D}^\bullet) \overset{(2.6)}{\cong} H^{-p} \operatorname{Hom}(i^{-1}\underline{\mathbb{K}}, i^{-1}\mathcal{D}^\bullet) \\ &= H^{-p} \operatorname{Hom}({}_U\underline{\mathbb{K}}, \mathcal{D}^\bullet|_U) = \mathcal{H}^{-p}(U, \mathcal{D}^\bullet). \end{aligned}$$

Since the isomorphism (6.2) is natural in $\mathcal{F}$ we deduce that the isomorphism

$$H^p_c(U, \mathbb{K})^\vee \cong \mathcal{H}^{-p}(U, \mathcal{D}^\bullet)$$

is compatible with the inclusions of open sets.

□

From now on we will use the notation $\omega_X = \omega_{X,\mathbb{K}}$ for the dualizing complex of the finite dimensional locally compact space $X$. We will always assume that $\omega_X$ is concentrated in dimensions $-\dim X, \cdots, -1, 0$.

Fix a sheaf $\mathcal{F} \in \mathbf{Sh}_{\mathbb{K}}(X)$. We get a left exact functor

$$\operatorname{Hom}(\mathcal{F}, -) : \mathbf{Sh}_{\mathbb{K}}(X) \to \mathbf{Vect}_{\mathbb{K}}.$$

This is the composition of two functors

$$\underline{Hom}(\mathcal{F},-):\mathbf{Sh}_{\bar{k}}(X)\to\mathbf{Sh}_{\mathbb{K}}(X),\ \ \Gamma(X,-):\mathbf{Sh}_{K}(X)\to\mathbf{Vect}_{\mathbb{K}}\,.$$

For any injective sheaf $\mathcal{I}\in\mathbf{Sh}_{\mathbb{K}}(X)$ the sheaf $\underline{Hom}(\mathcal{F},\mathcal{I})$ is flabby (see [5, Lemme II.7.3.2] or [8, Prop. 2.4.6]), i.e. $\Gamma(X,-)$ acyclic.

Thus the derived functor of $\operatorname{Hom}(\mathcal{F},-)$ exists and we have an isomorphism

$$R\operatorname{Hom}(\mathcal{F},-)\cong R\Gamma(X,-)\circ R\,\underline{Hom}(\mathcal{F},-).$$

For any (bounded) complex $\mathcal{S}^\bullet$ we can compute the $\operatorname{Hom}(\mathcal{F},-)$-hypercohomology of $\mathcal{S}^\bullet$ using the second hypercohomology spectral sequence whose $E_2$-term is

$$E_2^{p,q}=\operatorname{Ext}^q(\mathcal{F},\mathcal{H}^p(\mathcal{S})).$$

If we let $\mathcal{S}^\bullet=\omega_X^\bullet$ and we use the dualizing complex $\omega_X$ we obtain a complex whose $E_2$ term is

$$E_2^{p,q}=\operatorname{Ext}^q(\mathcal{F},\mathcal{H}^p(\omega_X))$$

and converges to

$$H^m\operatorname{Hom}_{D^b(\mathbf{Vect})}(\Gamma_c(X,\mathcal{F}),\mathbb{K})=H_c^{-m}(X,\mathcal{F})^\vee.$$

Suppose $X$ is a topological manifold. In this case we deduce that $\mathcal{H}^p(\omega_X)=0$ for $p\neq-n$. The sheaf $\mathcal{H}^{-n}(\omega_X)$ is called the *orientation sheaf* of $X$. It is a locally constant sheaf which we denote by $\mathbf{or}_X$. This sheaf, viewed as a complex concentrated in dimension $0$ is quasi-isomorphic to $\omega_X[-n]$ so that

$$\omega_X\cong\mathbf{or}_X[n].$$

In this case we have

$$E_2^{p,q}(X)=\begin{cases}\operatorname{Ext}^q(\mathcal{F},\mathbf{or}_X) & \text{if}\quad p=-n\\ 0 & \text{if}\quad p\neq-n\end{cases}.$$

The spectral sequence degenerates at $E_2$ and we obtain a natural in $\mathcal{F}$ isomorphism

$$H_c^{n-q}(X,\mathcal{F})^\vee=H_c^{-(q-n)}(X,\mathcal{F})^\vee\cong\operatorname{Ext}^q(\mathcal{F},\mathbf{or}_X).\tag{6.3}$$

*Remark* 6.10 (Yoneda's trace). Using the isomorphism

$$[\omega_X,\omega_X]\cong[\Gamma_c(X,\omega_X),\mathbb{K}]$$

we obtain a linear map

$$\int_X:\Gamma_c(X,\omega_X)\to\mathbb{K}$$

corresponding to the identity map $\mathbf{1}\in[\omega_X,\omega_X]$. This is called *Yoneda's trace*. Let us disect this construction.

Fix a soft resolution $\mathcal{S}$ of $\underline{\mathbb{K}}$. Then

$$\omega_X\cong\mathcal{S}^\vee\cong\mathcal{S}\otimes\mathcal{S}^\vee.$$

Then

$$[\Gamma_c(X,\omega_X),\mathbb{K}]\cong[\Gamma_c(X,\mathcal{S}\otimes\mathcal{S}^\vee),\mathbb{K}]$$

The trace $\int_X$ is uniquely determined by the requirement

$$\int_X u\otimes L=L(u),\ \ \forall u\in\Gamma_c(X,\mathcal{S}),\ \ L\in\mathcal{S}^\vee(X)=\operatorname{Hom}\big(\Gamma_c(X,\mathcal{S}),\mathbb{K}\big).$$

Given

$$\Phi\in[I^\bullet,\omega_X]$$

we obtain $\mathbb{Y}_\Phi \in [\Gamma_c(X, I^\bullet), \mathbb{K}]$ as the composition

$$\Gamma_c(X, I^\bullet) \xrightarrow{\Phi} \Gamma_c(X, \omega_X) \xrightarrow{\int_X} \mathbb{K}.$$

The correspondence

$$[I^\bullet, \omega_X] \ni \Phi \mapsto \mathbb{Y}_\Phi \in [\Gamma_c(X, I^\bullet), \mathbb{K}]$$

is precisely the duality isomorphism.

□

**Example 6.11** (Classical Poincaré duality)**.** Suppose $I^\bullet$ is an injective resolution of the constant sheaf $\underline{\mathbb{K}}$. Then

$$[I^\bullet[-p], \omega_X] = [I^\bullet, \omega_X[p]] = R\operatorname{Hom}(\underline{\mathbb{K}}, \omega_X[p]) \cong R\Gamma(X, \omega_X[p]).$$

On the other hand we have

$$[I^\bullet[-p], \omega_X] \cong [\Gamma_c(X, I^\bullet[-p]), \mathbb{K}]$$

Passing to (hyper)cohomology (i.e. applying the functor $H_0$) we deduce

$$\mathbb{H}^{-p}(X, \omega_X) \cong H^p_c(X, \mathbb{K})^\vee.$$

Now suppose $X$ is a manifold so that $\omega_X = \mathbf{or}_X[n]$. We deduce the classical Poincaré duality

$$H^{n-p}(X, \mathbf{or}_X) \cong H^p_c(X, \mathbb{K})^\vee.$$

In this case we have an integration map

$$\int_X : \Gamma_c(X, \mathbf{or}_X[n]) \to \mathbb{K}.$$

Every element in $H^{n-p}(X, \mathbf{or}_X)$ can be represented as an element of

$$\Phi \in \big[\, I^\bullet[p]\,,\, \mathbf{or}_X[n] \,\big] = \big[\, I^\bullet\,,\, \mathbf{or}_X[n-p] \,\big].$$

As explained in Example 1.18 we get a cup product map

$$\Phi\cup : H^p_c(X, \mathbb{K}) \to H^0_c(X, \mathbf{or}_X[n]), \;\; \alpha \mapsto \Phi \cup \alpha,$$

and then Yoneda's map $\mathbb{Y}_\Phi : H^p_c(X, \mathbb{K}) \to \mathbb{K}$ has the form

$$H^p_c(X, \mathbb{K}) \ni \alpha \mapsto \int_X \Phi \cup \alpha \in \mathbb{K}.$$

Now set $p = n - k$. The above discussion shows that we have a perfect pairing

$$H^k(X, \mathbf{or}_X) \times H^{n-k}_c(X, \mathbb{K}) \to \mathbb{K}, \;\; (\Phi, \alpha) \mapsto \int_X \Phi \cup \alpha.$$

□

**Example 6.12** (Alexander duality)**.** Suppose $Z$ is a closed subset of an oriented topological manifold $X$. From Proposition 2.20 we deduce that for every sheaf $\mathcal{F}$ on $X$ we have

$$H^p_Z(X, \mathcal{F}) \cong \operatorname{Hom}_{D^b(\mathbf{Sh}(X))}[\underline{\mathbb{K}}_Z, \mathcal{F}[p]].$$

The shifted dualizing complex $\omega^\bullet_X[-n]$ is a resolution of $\underline{\mathbb{K}}$ and thus letting $\mathcal{F} = \underline{\mathbb{K}}$

$$H^p_Z(X, \underline{\mathbb{K}}) \cong \operatorname{Hom}_{D^b(\mathbf{Sh}(X))}[\underline{\mathbb{K}}_Z, \omega_X[p-n]] \cong H^{n-p}_c(X, \underline{\mathbb{K}}_Z)^\vee \cong H^{n-p}_c(Z, \underline{\mathbb{K}})^\vee.$$

An element $\Phi \in H^p_Z(X, \underline{\mathbb{K}})$ is represented by a homotopy class of morphisms

$$\Phi : \underline{\mathbb{K}}_Z[n-p] \to \omega_X$$

and we get a cup product map
$$\Phi\cup : H_c^{n-p}(Z,\underline{\mathbb{K}}) = H_c^{n-p}(X,\underline{\mathbb{K}}_Z) \to H^0(X,\omega_X).$$
The isomorphism
$$H^p_Z(X,\underline{\mathbb{K}}) \cong H_c^{n-p}(Z,\underline{\mathbb{K}})^\vee$$
can be given the Yoneda description
$$H^p_Z(X,\underline{\mathbb{K}}) \ni \Phi \mapsto \int_X \circ\Phi\cup : H_c^{n-p}(Z,\underline{\mathbb{K}}) \to \mathbb{K}.$$
We deduce that if $Z$ has soft dimension $\leqslant r$ then
$$H^p_Z(X,\underline{\mathbb{K}}) \neq 0 \Longrightarrow r \leqslant p \leqslant n.$$
□

## 7. The general Poincaré-Verdier duality

The Poincaré-Verdier duality discussed in the previous section has a relative version, which deals with continuous families of locally compact spaces. In the sequel $\mathcal{R}$ will denote a commutative noetherian ring with 1. We assume $\operatorname{gldim}\mathcal{R} < \infty$. For every locally compact space $X$ we set
$$\operatorname{Hom}_X := \operatorname{Hom}_{\mathbf{Sh}_\mathcal{R}(X)},\ \ D^*(\mathcal{R}_X) := D^*(\mathbf{Sh}_\mathcal{R}(X)),$$
and by $\mathcal{I}_X = \mathcal{I}_{X,\mathcal{R}}$ the full subcategory of $\mathbf{Sh}_\mathcal{R}(X)$ consisting of injective sheaves of $\mathcal{R}$-modules.

**Theorem 7.1** (Relative Poincaré-Verdier duality)**.** *Suppose $f : X \to S$ is a continuous map between two finite dimensional, locally compact spaces. Then there exists a functor of triangulated categories*[8]
$$f^! : D^+(\mathcal{R}_S) \to D^+(\mathcal{R}_X)$$
*and an isomorphism*
$$\operatorname{Hom}_{D^+(\mathcal{R}_S)}(Rf_!\mathcal{F}\,,\,\mathcal{G}) \cong \operatorname{Hom}_{D^+(\mathcal{R}_X)}(\mathcal{F}\,,\,f^!\mathcal{G})$$
*natural in $\mathcal{F} \in D^+(\mathcal{R}_X)$ and $\mathcal{G} \in D^+(\mathcal{R}_S)$. Briefly, $f^!$ is the right adjoint of $Rf_!$.*

To understand the strategy, let us first think naively and forget that $\mathcal{F},\mathcal{G}$ are complexes of sheaves and think of them as genuine sheaves. Then for every open set $U \subset X$ we have
$$\Gamma(U, f^!\mathcal{G}) = \operatorname{Hom}(\mathcal{R}_U, f^!\mathcal{G}) \cong \operatorname{Hom}(f_!\mathcal{R}_U, \mathcal{G}).$$
We see a first difficulty: the sheaf $f_!\underline{\mathcal{R}}_U$ is often trivial so that the above construction would produce the trivial sheaf. Take for example the canonical projection $\pi : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$. Then for every $x \in \mathbb{R}$ and every open set $x \in V \subset \mathbb{R}$ the sheaf $\mathcal{R}_U$ will not have sections on $\pi^{-1}(V)$ with compact vertical support. The problem is that the sheaf $\underline{\mathcal{R}}$ is "too rigid". We need to "soften it up".

Thus we would have to replace $\underline{\mathcal{R}}$ with a soft resolution, $\mathcal{R} \approx \mathcal{L}$ which we can think of as an "approximation of 1". This operation is very similar to the regularization procedure in analysis. In that context one chooses a sequence of compactly supported functions $(\varphi_n)$ converging as distributions to the Dirac $\delta$, which is a unit with respect to the convolution. Then the operators $\varphi_n*$ approximate the identity, but their ranges consist of better behaved objects. It would be nice if given any sheaf $\mathcal{F}$ the tensor product $\mathcal{F}\otimes\mathcal{L}$ is a soft resolution of $\mathcal{F}$. This will not happen but, if we replace $\mathcal{F}$ by a soft resolution $\tilde{\mathcal{F}}$ and if by any chance $\mathcal{L}$ is also flat then the tensor product $\tilde{\mathcal{F}}\otimes\mathcal{L}$ would be a resolution of $\mathcal{F}$. If $\tilde{\mathcal{F}}\otimes\mathcal{L}$ were soft as well then we have produced our soft approximation of any sheaf. This heuristic discussion[9] may perhaps shed some light on the significance of the following result.

[8]In particular, $f^!$ commutes with the shift and maps distinguished triangles to distinguished triangles.

[9]In hindsight all things make perfect sense.

**Lemma 7.2.** *Suppose $K$ is a flat c-soft sheaf of Abelian groups on $X$. Then the following hold.*
*(i) For any sheaf $\mathcal{G} \in \mathbf{Sh}_{\mathcal{R}}(X)$ the tensor product $\mathcal{G} \otimes_{\mathbb{Z}} K \in \mathbf{Sh}_{\mathcal{R}}(X)$ is c-soft.*
*(ii) The functor*

$$\mathbf{Sh}_{\mathcal{R}}(X) \ni \mathcal{G} \mapsto f_!^K(\mathcal{G}) = f_!(\mathcal{G} \otimes_{\mathbb{Z}} K) \in \mathbf{Sh}_{\mathcal{R}}(S)$$

*is exact.*

**Proof** (i) If $\mathcal{G} = \mathbb{Z}_U$, $U \subset X$ open then $\mathcal{G} \otimes K = K_U$ is $c$-soft. In general we choose a resolution of $G$

$$\cdots \to \mathcal{G}^{-j} \to \mathcal{G}^{-j+1} \to \cdots \to \mathcal{G}^0 \to \mathcal{G} \to 0,$$

where each $\mathcal{G}^{-j}$ is a direct sum of sheaves of the form $\mathbb{Z}_U$, $U$ open. Taking $\otimes K$ we get a resolution

$$\cdots \to \mathcal{G}^{-j} \otimes K \to \mathcal{G}^{-j+1} \otimes K \to \cdots \to \mathcal{G}^0 \otimes K \to \mathcal{G} \otimes K \to 0,$$

where each $\mathcal{G}^{-j} \otimes K$ is $c$-soft. The finite dimensionality of $X$ will then force $\mathcal{G} \otimes K$ to be soft.

For (ii) observe that $\otimes K$ is exact, and maps sheaves to $c$-soft sheaves. Next, $f_!$ is exact on $c$-soft sheaves.

□

As suggested in the above heuristic discussion functor $f_!$ can be "approximated" by functors of the type $f_!^K$. We will prove that

$$f_!^K : \mathbf{Sh}_{\mathcal{R}}(X) \to \mathbf{Sh}_{\mathcal{R}}(S)$$

has a right adjoint. In particular, we have to prove that for every $\mathcal{G} \in \mathbf{Sh}_{\mathcal{R}}(S)$ the functor

$$\mathbf{Sh}_{\mathcal{R}}(X) \ni \mathcal{F} \mapsto \operatorname{Hom}_S(f_!^K(\mathcal{F}), \mathcal{G}) \in \mathbf{Ab}^{op}$$

is representable. The next result is the heart of the proof of the duality theorem. It explains when a contravariant functor form the category of sheaves on a space to the category of Abelian groups is representable. It is in essence a more sophisticated version of the acyclic models theorem.

**Theorem 7.3** (Representability Theorem)**.** *Suppose we are given a functor*

$$F : \mathbf{Sh}_{\mathcal{R}}(X) \to \mathbf{Ab}^{op}.$$

*Then $F$ is representable if and only if it is continuous, i.e. it transforms injective limits in $\mathbf{Sh}_{\mathcal{R}}$ to projective limits in $\mathbf{Ab}$.*

**Proof** The necessity follows from the discussion on limits in §4.

Assume now that $F$ transforms inductive limits to projective limits. For simplicity we take $\mathcal{R} = \mathbb{Z}$. We want to show it is representable. Let us first guess what could be its representative. It has to be a sheaf $\mathcal{G}$ on $X$ such that for any other sheaf $\mathcal{F}$ we have a natural isomorphism

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \cong F(\mathcal{F}).$$

If we take $\mathcal{F} = \underline{\mathbb{Z}}_U$ we deduce

$$\mathcal{G}(U) \cong F(\underline{\mathbb{Z}}_U).$$

For any open sets $V \subset U$ we have a natural morphism $\underline{\mathbb{Z}}_V \to \underline{\mathbb{Z}}_U$ and thus an induced morphism

$$|_V : F(\underline{\mathbb{Z}}_U) \to F(\underline{\mathbb{Z}}_V).$$

In particular, the correspondence $U \longmapsto F(\underline{\mathbb{Z}}_U)$ is a presheaf which we denote by $\mathcal{G}$. By construction

$$\mathcal{G}(U) = F(\underline{\mathbb{Z}}_U).$$

Let us show that $\mathcal{G}$ is in fact a sheaf.

Consider a collection of open sets $(U_\alpha)$ and set $U = \bigcup_\alpha U_\alpha$. Suppose $f_\alpha \in F(\underline{\mathbb{Z}}_{U_\alpha})$ are such that

$$f_\alpha |_{U_{\alpha\beta}} = f_\beta |_{U_{\alpha\beta}} .$$

Observe that we have an exact sequence of sheaves

$$\bigoplus_{\alpha,\beta} \underline{\mathbb{Z}}_{U_{\alpha\beta}} \longrightarrow \bigoplus_{\alpha} \underline{\mathbb{Z}}_{U_\alpha} \longrightarrow \underline{\mathbb{Z}}_U \to 0.$$

Since $F$ transforms inductive limits into projective limits it will transform direct sums into direct products, and kernels into cokernels. Hence we obtain a short exact sequence

$$0 \to \mathcal{G}(U) \longrightarrow \prod_{\alpha} \mathcal{G}(U_\alpha) \longrightarrow \prod_{\alpha,\beta} \mathcal{G}(U_{\alpha\beta}).$$

which shows that $\mathcal{G}$ is a sheaf.

Next, we need to construct an isomorphism

$$T_{\mathcal{F}} : \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to F(\mathcal{F})$$

functorial in $\mathcal{F}$. Let us explain the strategy. We first construct the isomorphism $T_{\mathcal{F}}$ for sheaves $\mathcal{F}$ in a full subcategory $\mathcal{M}$ of $\mathbf{Sh}(X)$. We will refer to $\mathcal{M}$ as the *category of models*. Next we will show that every sheaf $\mathcal{F}$ can be described as an inductive limit

$$\mathcal{F} = \varinjlim_{I} \mathcal{F}_i, \quad \mathcal{F}_i \in \mathcal{M}.$$

Then we use the natural isomorphisms

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\varinjlim_{I} \mathcal{F}_i, \mathcal{G}) \cong \varprojlim_{I} \mathrm{Hom}(\mathcal{F}_i, \mathcal{G})$$

$$\cong \varprojlim_{I} F(\mathcal{F}_i) \cong F(\varinjlim_{I} \mathcal{F}_i) \cong F(\mathcal{F}).$$

The models will be the sheaves $\underline{\mathbb{Z}}_U^n$, $U$ open, $n \geqslant 0$. We define

$$T_U : \mathrm{Hom}(\underline{\mathbb{Z}}_U^n, \mathcal{G}) \cong \mathcal{G}(U)^n \cong F(\underline{\mathbb{Z}}_U)^n$$

to be the tautological isomorphism.

For an arbitrary sheaf $\mathcal{F}$ we define a category

$$\Sigma = \Sigma_{\mathcal{F}} = \Big\{ (\underline{\mathbb{Z}}_U^n, f_U); \;\; U \;\text{ open}, \;\; f_U \in \mathrm{Hom}(\underline{\mathbb{Z}}_U^n, \mathcal{F}) \Big\},$$

where

$$\mathrm{Hom}_\Sigma\big( (\underline{\mathbb{Z}}_U^n, f_U), (\underline{\mathbb{Z}}_V^m, f_V) \big) = \left\{ \tau \in \mathrm{Hom}(\underline{\mathbb{Z}}_U^n, \underline{\mathbb{Z}}_V^m); \;\; \begin{array}{ccc} \underline{\mathbb{Z}}_U^n & \xrightarrow{\tau} & \underline{\mathbb{Z}}_V^n \\ & {\scriptstyle f_U}\searrow & \downarrow {\scriptstyle f_V} \\ & & \mathcal{F} \end{array} \;\; , \;\; f_V \circ \tau = f_U \right\}.$$

For every $t = (\underline{\mathbb{Z}}_U^n, f_U) \in \Sigma$ we set $\mathcal{F}_t = \underline{\mathbb{Z}}_U^n$ and

$$\phi_t := f_U \in \mathrm{Hom}(\mathcal{F}_t, \mathcal{F}).$$

Clearly if $s = (\underline{\mathbb{Z}}_V^m, f_V) \in \Sigma$ and

$$\tau \in \mathrm{Hom}_\Sigma(s, t) \subset \mathrm{Hom}(\underline{\mathbb{Z}}_V^m, \underline{\mathbb{Z}}_U^n)$$

we have a tautological morphism $\tau : \mathcal{F}_s \to \mathcal{F}_t$ and a tautological commutative diagram

$$\begin{array}{ccc} \mathcal{F}_s = \underline{\mathbb{Z}}_V^m & \xrightarrow{\tau} & \underline{\mathbb{Z}}_U^n = \mathcal{F}_t \\ & {\scriptstyle \phi_s = f_V}\searrow & \downarrow {\scriptstyle f_U = \phi_t} \\ & & \mathcal{F}. \end{array}$$

We want to show that

$$\mathcal{F} \cong \varinjlim_{\Sigma} \mathcal{F}_s,$$

that is, for any sheaf $\mathcal{S}$ and any morphisms $\sigma_s \in \mathrm{Hom}(\mathcal{F}_s, \mathcal{S})$, $s \in \Sigma$ such that

$$\sigma_t \circ \varphi = \sigma_s \Longleftrightarrow \begin{array}{ccc} \mathcal{F}_s & \xrightarrow{\varphi} & \mathcal{F}_t \\ & \searrow^{\sigma_s} & \downarrow{\sigma_t} \\ & & \mathcal{S} \end{array} \quad , \quad \forall \varphi \in \mathrm{Hom}_{\Sigma}(s,t), \tag{7.1}$$

there exists a unique morphism $\sigma : \mathcal{F} \to \mathcal{S}$ such that the diagrams below are commutative

$$\begin{array}{ccc} \mathcal{F} & \overset{\sigma}{\dashrightarrow} & \mathcal{S} \\ \uparrow{\phi_s} & \nearrow_{\sigma_s} & \\ \mathcal{F}_s & & \end{array} \tag{7.2}$$

The definition of $\sigma$ is tautological. We need to describe a family of morphisms

$$\sigma_U : \mathcal{F}(U) \to \mathcal{S}(U),$$

one for each open set $U$.

For every $f_U \in \mathcal{F}(U)$ we get an element $s = (\underline{\mathbb{Z}}_U, f_U) \in \Sigma_{\mathcal{F}}$ and the commutativity of (7.2) forces us to set

$$\sigma_U(f_U) = \sigma_s(1), \;\; 1 \in \Gamma(U, \underline{\mathbb{Z}}_U) = \mathrm{Hom}(\underline{\mathbb{Z}}_U, \underline{\mathbb{Z}}_U).$$

This proves the *uniqueness* of $\sigma$. The commutativity of the diagrams (7.1) implies that $\sigma$ is a morphism of sheaves of *sets*. To prove that $\sigma$ is a morphism of sheaves of *groups* consider

$$f_U^1, f_U^2 \in \mathcal{F}(U).$$

We obtain objects

$$s_k = (\underline{\mathbb{Z}}_U, f_U^k) \in \Sigma, \;\; k = 1, 2, \;\; s = (\underline{\mathbb{Z}}_U^2, (f_U^1, f_U^2)) \in \Sigma, \;\; t = (\underline{\mathbb{Z}}_U, f_U^1 + f_U^2) \in \Sigma.$$

Consider the morphisms

$$\delta \in \mathrm{Hom}_{\Sigma}(t, s), \;\; \tau_k \in \mathrm{Hom}_{\Sigma}(s_k, s),$$

defined by

$$\underline{\mathbb{Z}}_U \xrightarrow{\delta} \underline{\mathbb{Z}}_U^2, \;\; x \mapsto (x, x),$$

and

$$\underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}}_U^2, \;\; x \overset{\tau_1, \tau_2}{\longmapsto} (x, 0), (0, x).$$

We obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_{s_k} = \underline{\mathbb{Z}}_U & \xrightarrow{\tau_k} & \underline{\mathbb{Z}}_U^2 = \mathcal{F}_s & \xleftarrow{\delta} & \underline{\mathbb{Z}}_U = \mathcal{F}_t \\ & \searrow^{\sigma_{s_k}} & \downarrow{\sigma_s} & \swarrow_{\sigma_t} & \\ & & \mathcal{S}(U) & & \end{array} .$$

Hence

$$\sigma(f_U^1) = \sigma_{s_1}(1) = \sigma_s(1, 0), \;\; \sigma(f_U^2) = \sigma_{s_2}(1) = \sigma_s(0, 1)$$
$$\Longrightarrow \sigma(f_U^1) + \sigma(f_U^2) = \sigma_s(1, 0) + \sigma_s(0, 1) = \sigma_s(1, 1).$$

On the other hand

$$\sigma_s(1, 1) = \sigma_t(1) = \sigma(f_U^1 + f_U^2).$$

We can now use the additivity of $\sigma$ to prove that all the diagrams (7.2) are commutative so that

$$\mathcal{F} = \varinjlim_{s\in\Sigma} \mathcal{F}_s.$$

This completes the proof of the representability theorem.

□

Consider a flat soft sheaf $K \in \mathbf{Sh}_{\mathbb{Z}}(X)$ and the functor

$$F_{\mathcal{G},K} : \mathbf{Sh}_{\mathcal{R}}(X) \ni \mathcal{F} \mapsto \mathrm{Hom}_S(f_!^K(\mathcal{F}), \mathcal{G}) \in \mathbf{Ab}^{op}.$$

Since the functor $f_!^K$ is exact, we deduce that the functor $F_{\mathcal{G},K}$ maps cokernels into kernels. To prove that it maps injective limits to projective limits it suffices to show that it maps direct sums to direct products which is obvious because both $\otimes K$ and $f_!$ map direct sums to direct sums. This proves that $F_{\mathcal{G},K}$ is representable and we denote by $f_K^!(\mathcal{G})$ its representative.

Observe that since $f_!^K$ is exact we deduce from the isomorphism

$$\mathrm{Hom}_S\big( f_!^K(\mathcal{F}), \mathcal{G}\big) \cong \mathrm{Hom}_X\big(\mathcal{F}, f_K^!(\mathcal{G})\big)$$

that $\mathcal{G}$ is injective iff $f_K^!(\mathcal{G})$ is injective.

**Lemma 7.4.** *The constant sheaf $\underline{\mathbb{Z}}_X$ admits a resolution*

$$0 \to \underline{\mathbb{Z}}_X \to K^0 \to \cdots \to K^r \to 0$$

*where all $K^j$-s are flat, c-soft sheaves.*

This resolution will be a suitable truncation of the Godement resolution of ${}_X\underline{\mathbb{Z}}$, [7, Prop. VI.1.3].

Let $\mathbf{K}$ be a complex as in Lemma 7.4. For $\mathcal{G}^\bullet \in K^+(\mathcal{I}_{S,\mathcal{R}})$ we denote by $f_{\mathbf{K}}^!\mathcal{G} \in K^+(\mathcal{I}_{X,\mathcal{R}})$ the total complex associated to the double complex $f_{K^{-q}}^!\mathcal{G}^p$

$$f_{\mathbf{K}}^!(\mathcal{G}^\bullet) = \mathbf{Tot}(f_{K^{-q}}^!\mathcal{G}^p).$$

The functor

$$K^+(\mathcal{I}_{S,\mathcal{R}}) \ni \mathcal{G} \mapsto f_{\mathbf{K}}^!\mathcal{G} \in K^+(\mathcal{I}_{X,\mathcal{R}})$$

induces a morphism of triangulated categories

$$f^! : D^+(\mathcal{R}_S) \to D^+(\mathcal{R}_X)$$

which by its very construction is a right adjoint of $Rf_! : D^+(\mathcal{R}_X) \to D^+(\mathcal{R}_S)$.

□

**Example 7.5** (Absolute Verdier duality with integral coefficients)**.** Suppose $X$ is admissible, $\mathrm{gldim}\,\mathcal{R} < \infty$ and $c$ denotes the collapse map $c : X \to \{pt\}$. We deduce that for every $\mathcal{S} \in D^b(\mathcal{R}_X)$ we have an isomorphism

$$R\,\mathrm{Hom}_{\mathcal{R}}(\Gamma_c(X, \mathcal{S}), \mathcal{R}) \cong Rc_*\mathbb{D}_X\mathcal{S} = \mathbb{R}H(X, \mathbb{D}_X\mathcal{S}).$$

Suppose $\mathcal{R} \cong \mathbb{Z}$, and $\mathcal{S}$ is an injective resolution of the constant sheaf $\underline{\mathbb{Z}}$. Using the injective resolution

$$D_{\mathbb{Z}} : \quad 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

we deduce that

$$H^\bullet(R\,\mathrm{Hom}_{\mathbb{Z}}(\Gamma_c(X, \mathbb{Z}), \mathbb{Z})) \cong \mathbb{E}\mathrm{xt}^\bullet(\Gamma_c(X, \mathbb{Z}), D_{\mathbb{Z}}).$$

The hyper-Ext terms can be computed using a spectral sequence which degenerates at its $E_2$-term

$$E_2^{p,q} = \mathrm{Ext}^p(H_c^{-q}(X, \mathbb{Z}), \mathbb{Z}),$$

or in tabular form,

$$\begin{array}{c|cccc} & p=0 & p=1 & p=2 & \\ \hline q=0 & \mathrm{Hom}(H^0_c,\mathbb{Z}) & \mathrm{Ext}(H^0_c,\mathbb{Z}) & 0 & \cdots \\ q=-1 & \mathrm{Hom}(H^1_c,\mathbb{Z}) & \mathrm{Ext}(H^1_c,\mathbb{Z}) & 0 & \cdots \\ q=-2 & \mathrm{Hom}(H^2_c,\mathbb{Z}) & \mathrm{Ext}(H^2_c,\mathbb{Z}) & 0 & \cdots \\ & \vdots & \vdots & 0 & \vdots \end{array}.$$

This leads to the split short exact sequence

$$0 \to \mathrm{Ext}(H^{p+1}_c(X,\mathbb{Z}),\mathbb{Z}) \to H^{-p}(X,\mathbb{D}_X\underline{\mathbb{Z}}) \to \mathrm{Hom}(H^p_c(X,\mathbb{Z}),\mathbb{Z}) \to 0.$$

Since $\mathbb{D}_X\underline{\mathbb{Z}} \cong \omega_X$ we can use the above short exact sequence to compute the hypercohomology sheaves of $\omega_X$. We can rewrite the above results in a form similar to Proposition 6.9. More precisely, for any sheaf $\mathcal{F}$ on $X$ we have an isomorphism

$$\mathrm{RHom}(\Gamma_c(X,\mathcal{F}),\mathbb{Z}) \cong \mathrm{RHom}(\mathcal{F},\omega_X).$$

If we take $\mathcal{F} = \underline{\mathbb{Z}}_U$ we deduce

$$\mathrm{RHom}(R\Gamma_c(U,\underline{\mathbb{Z}}),\underline{\mathbb{Z}}) \cong R\Gamma(U,\omega_X).$$

In particular

$$H^\bullet(U,\omega_X) \cong \mathbb{E}\mathrm{xt}^\bullet(R\Gamma_c(U,\underline{\mathbb{Z}}),\underline{\mathbb{Z}}).$$

The last hyper-Ext can be computed via the above short exact sequence.

The middle term in the above sequence is known as the *Borel-Moore* homology with coefficients in $\mathbb{Z}$ an it is denoted by $H_p(X,\underline{\mathbb{Z}})$. Observe that if $X$ is a $n$-dimesional manifold then

$$H_p(X,\underline{\mathbb{Z}}) = H^{-p}(X,\omega_X) \cong H^{-p}(X,\mathbf{or}_X[n]) \cong H^{n-p}(X,\mathbf{or}_X).$$

In particular

$$H_n(X,\underline{\mathbb{Z}}) \cong H^0(X,\mathbf{or}_X) \cong R\,\mathrm{Hom}(\underline{\mathbb{Z}},\mathbf{or}_X).$$

An *orientation* on a manifold is a choice of an isomorphism $\underline{\mathbb{Z}} \to \mathbf{or}_X$. This determines an element $[X]$ in the top dimensional Borel-Moore cohomology group. We can identify it with the manifold itself. Note that $X$ *need not be compact*. For example, for $X = \mathbb{R}$ with the canonical orientation we obtain a cycle with non-compact support $[\mathbb{R}] \in H_1(\mathbb{R},\underline{\mathbb{Z}})$.

□

## 8. Some basic properties of $f^!$

Suppose $f : X \to S$ is a continuous map between two finite dimensional, locally compact spaces and $\mathcal{R}$ is a commutative noetherian ring with 1 such that $\mathrm{gldim}\,\mathcal{R} < \infty$. The functor $f^!$ enjoys several properties dual to the properties of the functor $Rf_!$. For simplicity we will restrict ourself to the derived categories of bounded complexes.

Observe first that there exist two natural morphisms

$$\mathbb{I} \to f^!Rf_!, \quad Rf_!f^! \to \mathbb{I}$$

The first is obtained from the isomorphism

$$\mathrm{Hom}(Rf_!,Rf_!) \cong \mathrm{Hom}(\mathbb{I},f^!Rf_!)$$

while the second is obtained from the isomorphism

$$\mathrm{Hom}(Rf_! f^!, \mathbb{I}) \cong \mathrm{Hom}(f^!, f^!),$$

Observe next that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a sequence of continuous maps between finite dimensional, locally compact spaces then

$$(g \circ f)^! \cong f^! \circ g^!.$$

**Proposition 8.1** (Base Change Formula)**.** *Suppose we are given a Cartesian square*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

*of continuous maps and finite dimensional, locally compact spaces. Then the following hold.*
*(i) There is an isomorphism of functors*

$$f^! \circ Rg_* \cong R(\pi_X)_* \circ \pi_Y^!. \tag{8.1}$$

*(ii) There is a morphism of functors*

$$\pi_X^{-1} \circ f^! \to \pi_Y^! \circ g^{-1}. \tag{8.2}$$

**Proof** The isomorphism (8.1) is dual to the base change isomorphism involving $Rf_!$ while the morphism (8.2) is dual to the morphism (3.3). Let us supply the details. The proof is based on the Yoneda's Principle.

**Proposition 8.2** (Yoneda's Principle)**.** *Suppose $\mathcal{C}$ is a (small) category. The functor*

$$\mathbb{Y} : \mathcal{C} \to \mathbf{Funct}\,(\mathcal{C}^{op}, \mathbf{Set}), \quad X \longmapsto \mathrm{Hom}_{\mathcal{C}}(-, X),$$

*is fully faithful, i.e. for every objects $X_0, X_1 \in \mathcal{C}$ the induced map*

$$\mathbb{Y} : \mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \to \mathrm{Hom}_{\mathbf{Funct}}\big(\mathrm{Hom}_{\mathcal{C}}(-, X_0), \mathrm{Hom}_{\mathcal{C}}(-, X_1)\,\big)$$

*is a bijection. In particular, the functors $\mathrm{Hom}_{\mathcal{C}}(-, X_0)$ and $\mathrm{Hom}_{\mathcal{C}}(-, X_1)$ are isomorphic if and only if the objects $X_0, X_1$ are isomorphic.* □

Returning to our problem observe that the base change formula for $Rf_!$ implies have an isomorphism

$$\mathrm{Hom}_Y(g^{-1} \circ Rf_! \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_Y(R(\pi_Y)_! \circ \pi_X^{-1} \mathcal{F}, \mathcal{G})$$

natural in $\mathcal{F} \in \mathbf{Sh}_{\mathcal{R}}(X)$, $\mathcal{G} \in \mathbf{Sh}_{\mathcal{R}}(Y)$. Using the fact that $-^*$ is the right adjoint of $-^{-1}$ and $-^!$ is the right adjoint of $-_!$ we obtain the desired conclusion from the Yoneda's principle.

□

**Proposition 8.3** (Dual Projection formula)**.** *Given a continuous map $f : X \to S$ there exists a morphism*

$$f^! \mathcal{S}_0 \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} f^{-1} \mathcal{S}_1 \to f^!(\mathcal{S}_0 \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} \mathcal{S}_1)$$

*natural in $\mathcal{S}_0, \mathcal{S}_1 \in D^b(\mathcal{R}_S)$.*

**Proof** For $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S} \in D^b(\mathcal{R}_S)$ we have

$$\operatorname{Hom}_X(f^!\mathcal{S}_0 \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} f^{-1}\mathcal{S}_1, f^!\mathcal{S}) \cong \operatorname{Hom}_S\big(Rf_!(f^!\mathcal{S}_0 \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} f^{-1}\mathcal{S}_1), \mathcal{S}\big)$$

(use the projection formula for $RF_!$)

$$\cong \operatorname{Hom}_S\big((Rf_!f^!\mathcal{S}_0) \overset{\mathrm{L}}{\otimes} \mathcal{S}_1, \mathcal{S}\big)$$

The natural morphism

$$Rf_!f^! \to \mathbb{I}$$

induces a morphism

$$\operatorname{Hom}(\mathcal{S}_0 \overset{\mathrm{L}}{\otimes} \mathcal{S}_1, \mathcal{S}) \to \operatorname{Hom}_S\big((Rf_!f^!\mathcal{S}_0) \overset{\mathrm{L}}{\otimes} \mathcal{S}_1, \mathcal{S}\big).$$

If we take $\mathcal{S} = \mathcal{S}_0 \overset{\mathrm{L}}{\otimes} \mathcal{S}_1$ we obtain via the above chain of maps a natural morphism

$$f^!\mathcal{S}_0 \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} f^{-1}\mathcal{S}_1 \to f^!(\mathcal{S}_0 \overset{\mathrm{L}}{\otimes} \mathcal{S}_1).$$

□

**Proposition 8.4.** *Let $f : X \to S$ be a continuous map between admissible spaces.*
*(a) We have an isomorphism*

$$R\,\underline{Hom}(Rf_!\mathcal{G}, \mathcal{F}) \cong Rf_*R\,\underline{Hom}(\mathcal{G}, f^!\mathcal{F}). \tag{8.3}$$

*natural in $\mathcal{F} \in D^b(\mathcal{R}_S)$ and $\mathcal{G} \in D^b(\mathcal{R}_X)$.*
*(b) We have an isomorphism*

$$f^!R\,\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1) \cong R\,\underline{Hom}(f^{-1}\mathcal{S}_0, f^!\mathcal{S}_1) \tag{8.4}$$

*natural in $\mathcal{S}_0, \mathcal{S}_1 \in D^b(\mathcal{R}_S)$.*

**Proof** Observe first that for any sheaves $\mathcal{S}_0, \mathcal{S}_1$ on $X$ we have a natural morphism

$$f_*\,\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1) \to \underline{Hom}(f_!\mathcal{S}_0, f_!\mathcal{S}_1)$$

Indeed, for every open set $U \subset S$ we have

$$f_*\,\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1)(U) = \operatorname{Hom}(\mathcal{S}_0\,|_{f^{-1}(U)}, \mathcal{S}_1\,|_{f^{-1}(U)}).$$

Any morphism $\phi$ in $f_*\,\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1)(U)$ maps a section $u_1$ of $\mathcal{S}_1$ properly supported over $U$ to a section of $\mathcal{S}_1$ which is also properly supported over $U$ since

$$\operatorname{supp}\phi(u_0) \subset \operatorname{supp} u_1.$$

Since the soft dimension of $X$ and $S$ is finite so is the flabby dimension since

$$\text{flabby dimension} \leqslant \text{soft dimension} + 1.$$

Thus every bounded complex of sheaves on $X$ admits bounded flabby resolutions and thus we can pass to derived functors[10] to obtain a

$$Rf_*R\,\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1) \to R\,\underline{Hom}(Rf_!\mathcal{S}_0, Rf_!\mathcal{S}_1) \tag{8.5}$$

In particular for $\mathcal{F} \in D^b(\mathcal{R}_S)$ and $\mathcal{G} \in D^b(\mathcal{R}_X)$ we have a canonical morphism

$$Rf_*R\,\underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \to R\,\underline{Hom}(Rf_!\mathcal{G}, Rf_!f^!\mathcal{F}).$$

[10] The flabby sheaves are adapted to Hom and $f_!$. The soft ones are not adapted to Hom.

Using the canonical morphism $Rf_! f^! \mathcal{F} \to \mathcal{F}$ we obtain a morphism

$$Rf_* R\,\underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \to R\,\underline{Hom}(Rf_!\mathcal{G}, \mathcal{F}).$$

We want to show that this map is a quasi-isomorphism of complexes of sheaves. Let $V$ be an open subset of $S$. Then

$$H^j\big(\,R\Gamma(\,V, Rf_* R\,\underline{Hom}(\mathcal{G}, f^!\mathcal{F})\,)\,\big) \cong \operatorname{Hom}_{D^b(\mathcal{R}_{f^{-1}(V)})}\big(\,\mathcal{G}\,|_{f^{-1}(V)}, f^!\mathcal{F}[j]\,|_{f^{-1}(V)}\,\big)$$

(use the Poincaré-Verdier duality)

$$\cong \operatorname{Hom}_{D^b(\mathcal{R}_V)}\big(\,(Rf_!G)\,|_V, \mathcal{F}[j]\,|_V\,\big) \cong H^j\big(\,R\Gamma(V, R\,\underline{Hom}(Rf_!\mathcal{G}, \mathcal{F})\,\big).$$

(b) We use again the Yoneda Principle. Let $\mathcal{S} \in D^b(\mathcal{R}_X)$. Then

$$\operatorname{Hom}_X\big(\mathcal{S}, f^! R\,\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1)\,\big)$$

(use the Poincaré-Verdier duality)

$$\cong \operatorname{Hom}_S\big(\,Rf_!\mathcal{S}, R\,\underline{Hom}(\mathcal{S}_0, \mathcal{S}_1)\,\big)$$

(use the adjunction isomorphism (2.16))

$$\cong \operatorname{Hom}_S\big(\,Rf_!\mathcal{S} \overset{L}{\otimes} \mathcal{S}_0, \mathcal{S}_1\big)$$

(use the projection formula)

$$\cong \operatorname{Hom}_S\big(\,Rf_!(\mathcal{S} \overset{L}{\otimes} f^{-1}\mathcal{S}_0), \mathcal{S}_1\,\big)$$

(use the the Poincaré-Verdier duality)

$$\cong \operatorname{Hom}_S\big(\,\mathcal{S} \overset{L}{\otimes} f^{-1}\mathcal{S}_0, f^!\mathcal{S}_1\big)$$

(use the adjunction isomorphism (2.16))

$$\cong \operatorname{Hom}_S\big(\,\mathcal{S}, R\,\underline{Hom}(f^{-1}\mathcal{S}_0, f^!\mathcal{S}_1)\,\big).$$

□

**Definition 8.5.** (a) For every continuous map $f : X \to S$ between finite dimensional locally compact spaces and every commutative Noetherian ring $\mathcal{R}$ with $1$ of finite global dimension, we define the *relative dualizing complex* of $X$ rel $S$ with coefficients in $\mathcal{R}$ to be the object

$$\omega_{X/S} = f^!\underline{\mathcal{R}}_S \in D^b(\mathcal{R}_X).$$

Since $\mathcal{R}$ has finite global dimension, the dualizing complex is quasi-isomorphic to a *bounded* complex of injective sheaves. In particular , if $S$ is a point and $f$ is the collapse map $X \xrightarrow{c} \{pt\}$ then

$$\omega_X := \omega_{X/pt} = c^!\mathcal{R}$$

is called the *(absolute) dualizing complex* of $X$ .

(b) For $X$ and $\mathcal{R}$ as above and any $\mathcal{F} \in D^b(\mathcal{R}_X)$ we define $\mathbb{D}_X\mathcal{F} \in D^b(\mathcal{R}_X)$ as

$$\mathbb{D}_X\mathcal{F} := R\,\underline{Hom}(\mathcal{F}, \omega_X).$$

We say that $\mathbb{D}_X\mathcal{F}$ is the *Poincaré-Verdier dual* of $\mathcal{F}$.

In the sequel we assume that the coefficient ring has finite global dimension.

*Remark* 8.6. There exists a natural morphism of functors

$$\mathbb{I}_{D^b(\mathcal{R}_X)} \to \mathbb{D}_X^2$$

induced by the evaluation map

$$D^b(\mathcal{R}_X) \ni \mathcal{F}^\bullet \mapsto R\,\underline{Hom}(R\,\underline{Hom}(\mathcal{F}^\bullet, \omega_X), \omega_X).$$

In general this morphism is not an isomorphism but becomes so when restricted to full subcategory of $D^b(\mathcal{R}_X)$ consisting of *cohomologically constructible complexes*.

□

**Proposition 8.7.** *For every continuous map $f : X \to S$ between admissible spaces we have an isomorphism*

$$\mathbb{D}_X f^{-1}\mathcal{F} \cong f^!\mathbb{D}_S\mathcal{F},$$

*natural in $\mathcal{F} \in D^b(\mathcal{R}_S)$. In particular, there exists a natural morphism*

$$f^{-1}\mathcal{F} \to \mathbb{D}_X f^!\mathbb{D}_S\mathcal{F}.$$

**Proof** Observe first that $f^!\omega_S \cong \omega_X$. We have

$$R\,\underline{Hom}_X(f^{-1}\mathcal{F}, \omega_X) \cong R\,\underline{Hom}_X(f^{-1}\mathcal{F}, f^!\omega_S) \stackrel{(8.4)}{\cong} f^! R\,\underline{Hom}_S(\mathcal{F}, \omega_S) \cong f^!\mathbb{D}_S\mathcal{F}.$$

□

## 9. Alternate descriptions of $f^!$

There are some basic situations when the functor $f^!$ can be given alternate descriptions.

Suppose $i = i_U : U \to X$ denotes the inclusion of an open subset. Using (2.6) we obtain an isomorphism

$$\mathrm{Hom}_X(i_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_U(\mathcal{F}, i^{-1}\mathcal{G})$$

natural in $\mathcal{F} \in \mathbf{Sh}_{\mathcal{R}}(U)$, and $\mathcal{G} \in \mathbf{Sh}_R(X)$. We conclude that

$$i_U^! \cong i_U^{-1}.$$

**Proposition 9.1.** *Suppose $j : X \hookrightarrow S$ denotes the inclusion of a locally closed subset in a locally compact space. Assume $\mathcal{R}$ has finite global dimension. Then*

$$j^! \cong j^{-1} \circ R\Gamma_X(\mathcal{F}), \quad \forall \mathcal{F} \in D^b(\mathcal{R}_S).$$

**Proof** Let $\mathcal{F} \in D^b(\mathcal{R}_S)$, $\mathcal{G} \in D^b(\mathcal{R}_X)$. Then for every closed subset $Z \subset S$ we have

$$\Gamma_Z(-) = \underline{Hom}(\underline{\mathcal{R}}_Z, -).$$

Using the adjunction isomorphism (2.2) and the Poincaré-Verdier duality (in this case $j_!$ is exact so that $Rj_! = j_!$))

$$\mathrm{Hom}_X(\mathcal{G}, j^!\mathcal{F}) \cong \mathrm{Hom}_S(j_!\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_S(j_!\mathcal{G}, R\Gamma_X(\mathcal{F}))$$
$$\cong \mathrm{Hom}_X(j^{-1}j_*\mathcal{G}, j^{-1}R\Gamma_X(\mathcal{F})) \cong \mathrm{Hom}_X(\mathcal{G}, j^{-1}R\Gamma_X(\mathcal{F})).$$

This proves that

$$j^!\mathcal{F} \cong j^{-1}R\Gamma_X(\mathcal{F}).$$

□

Using the dual projection formula for a continuous map between admissible spaces $f : X \to S$ we obtain a morphism

$$\omega_{X/S} \overset{\mathrm{L}}{\otimes} f^{-1}(\mathcal{S}) = f^!(\underline{\mathcal{R}}_S) \overset{\mathrm{L}}{\otimes} f^{-1}(\mathcal{S}) \to f^!(\underline{\mathcal{R}}_S \otimes \mathcal{S}) = f^!(\mathcal{S}), \;\; \forall \mathcal{S} \in D^b(\mathcal{R}_S).$$

We will describe below one instance when this morphism is an isomorphism.

**Definition 9.2.** A map $f : X \to S$ between admissible spaces is called a *topological submersion of relative dimension* $\ell$ if for every $x \in X$ there exists an open neighborhood $V$ in $X$ such that $U = f(V)$ is an open neighborhood of $s = f(x)$ in $S$ and there exists a homeomorphism $h : V \to U \times \mathbb{R}^\ell$ such that the diagram below is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\;h\;} & U \times \mathbb{R}^\ell \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p_U} \\ U & \xrightarrow{\;\mathbb{I}_U\;} & U \end{array}$$

Loosely speaking a topological submersion is a fibration, where the fibers are topological manifolds of dimension $\ell$.

**Definition 9.3.** The space $S$ is called *locally contractible* if every point $s \in S$ admits a basis of contractible, open neighborhoods.

**Proposition 9.4.** *Assume $f : X \to S$ is a topological submersion of relative dimension $\ell$, and $S$ is locally contractible. Then the following hold.*
*(a) $\mathcal{H}^k(\omega_{X/S}) = 0$ for $k \neq -\ell$ and $\mathcal{H}^{-\ell}(\omega_{X/S})$ is locally isomorphic to ${}_X\underline{\mathcal{R}}$.*
*(b) The canonical morphism of functors*

$$f^!(\underline{\mathcal{R}}_S) \overset{\mathrm{L}}{\otimes} f^{-1}(-) \to f^!(-)$$

*is an isomorphism.*

**Proof** Denote by $B$ the unit open ball centered at the origin of $\mathbb{R}^\ell$. For every point $x \in X$ we can find a basis of product like open neighborhoods, i.e. open neighborhoods $W$ with the following property. There exists an open, path connected contractible neighborhood $V$ of $s = f(x) \in S$, and a homeomorphism

$$h : B \times V \to W$$

such that if

$$i_W : W \hookrightarrow X \quad \text{and} \quad i_V : V \hookrightarrow S$$

denote the natural inclusions and

$$h_W := i_W \circ h : B \times V \to W \hookrightarrow X$$

then the diagram below is commutative.

$$\begin{array}{ccc} B \times V & \overset{h_W}{\hookrightarrow} & X \\ {\scriptstyle \pi_V}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ V & \underset{i_V}{\hookrightarrow} & S \end{array}$$

Let $\mathcal{S} \in D^b(\mathcal{R}_S)$. Then

$$R\Gamma(W, f^!\mathcal{S}) = R\operatorname{Hom}_W(\mathcal{R}, i_W^{-1}f^!\mathcal{S}) \cong R\operatorname{Hom}_{B\times V}(\mathcal{R}, h_W^{-1}f^!\mathcal{S})$$

$(h_W^{-1} = h_W^!)$

$$\cong R\operatorname{Hom}_{B\times V}(\mathcal{R}, h_W^! f^!\mathcal{S}) \cong R\operatorname{Hom}_{B\times V}(\mathcal{R}, \pi_V^! i_V^!\mathcal{S}) \cong R\operatorname{Hom}_{B\times V}(\mathcal{R}, \pi_V^! i_V^{-1}\mathcal{S})$$

$$\cong R\operatorname{Hom}_V\big(R\pi_{V!}\mathcal{R}, \mathcal{S}|_V\big).$$

Using Corollary 3.8 we deduce

$$R\pi_{V!}\mathcal{R}_{B\times V} \cong R\Gamma_c(B, \underline{\mathcal{R}}_B) \otimes \underline{\mathcal{R}}_V$$

Hence we deduce

$$R\Gamma(W, f^!\mathcal{S}) \cong R\operatorname{Hom}_V\big(R\Gamma_c(B, \underline{\mathcal{R}}_B) \otimes \underline{\mathcal{R}}_V,\ \ \mathcal{S}|_V\big)$$

$$\cong R\operatorname{Hom}_V\big(R\Gamma_c(B, \underline{\mathcal{R}}_B), R\,\underline{\mathit{Hom}}(\underline{\mathcal{R}}_V, \mathcal{S})\big)$$

Above we regard $R\Gamma_c(B, \underline{\mathcal{R}}_B)$ as a complex of *constant, free, finite rank* sheaves on $V$. Using the special adjunction formula we obtain an isomorphism

$$R\operatorname{Hom}_V\big(R\Gamma_c(B, \underline{\mathcal{R}}_B), R\,\underline{\mathit{Hom}}(\underline{\mathcal{R}}_V, \mathcal{S})\big) \cong \mathrm{RHom}\big(R\Gamma_c(B, \underline{\mathcal{R}}_B), \mathcal{R}\big) \overset{\mathrm{L}}{\otimes} R\operatorname{Hom}\big(\underline{\mathcal{R}}_V, \mathcal{S}\big).$$

Using the Poincaré-Verdier duality on $B$ we deduce further

$$R\operatorname{Hom}\big(R\Gamma_c(B, \underline{\mathcal{R}}_B), \mathcal{R}\big) \cong R\Gamma(B, \omega_B).$$

Hence

$$R\Gamma(W, f^!\mathcal{S}) \cong R\Gamma(B, \omega_B) \overset{\mathrm{L}}{\otimes} R\Gamma(V, \mathcal{S}) \cong R\Gamma(B \times V, \omega_B \overset{\mathrm{L}}{\otimes} \mathcal{S}).$$

□

Assume $\mathcal{R} = \mathbb{Z}$ and $S$ is an *oriented* manifold. The orientation is given by an isomorphism

$$\mathbf{or}_S \cong \underline{\mathbb{Z}}_S.$$

In particular, we obtain an isomorphism

$$\omega_S \cong \mathbf{or}_S[\dim S] \cong \underline{\mathbb{Z}}_S[\dim S].$$

We deduce

$$\mathbb{D}_S\underline{\mathbb{Z}}_S = \underline{\mathit{Hom}}\big(\underline{\mathbb{Z}}_S, \omega_S\big) \cong \underline{\mathbb{Z}}_S[\dim S] \Longrightarrow \underline{\mathbb{Z}}_S \cong (\mathbb{D}_S\underline{\mathbb{Z}}_S)[-\dim S] = (\mathbb{D}_S\underline{\mathbb{Z}}[\dim S]).$$

Using Proposition 8.7 we deduce

$$\omega_{X/S} \cong f^!\underline{\mathbb{Z}}_S \cong f^!\mathbb{D}_S(\underline{\mathbb{Z}}_S[\dim S]) \cong \mathbb{D}_X f^{-1}\underline{\mathbb{Z}}_S[\dim S]$$

(use the fact that $f^{-1}$ is an exact functor)

$$\cong \mathbb{D}_X\underline{\mathbb{Z}}_X[\dim S] \cong \underline{\mathit{Hom}}\big(\underline{\mathbb{Z}}_X, \omega_X[-\dim S]\big) \cong \omega_X[-\dim S].$$

**Example 9.5.**

Suppose $E$

## 10. Duality and constructibility

The results established so far simplify somewhat when restricted to the special class of constructible sheaves defined in §5. For the reader's convenience we reproduce here the definition of these sheaves.

Suppose $X$ is a locally compact space of finite soft dimension and $R$ is a ring of finite global dimension. An object $\mathcal{F} \in D^b(\mathbf{Sh}_R(X))$ is called *cohomologically constructible* (c.c. for brevity) if for any point $x \in X$ the following conditions are satisfied.

(a) $\underrightarrow{\mathbf{Lim}}_{U\ni x} R\Gamma(U,\mathcal{F})$ and $\underleftarrow{\mathbf{Lim}}_{U\ni x} R\Gamma_c(U,\mathcal{F})$ exist and the canonical maps

$$\underrightarrow{\mathbf{Lim}}_{U\ni x} R\Gamma(U,\mathcal{F}) \to \mathcal{F}_x,\quad R\Gamma_{\{x\}}(X,\mathcal{F}) \to \underleftarrow{\mathbf{Lim}}_{U\ni x} R\Gamma_c(U,\mathcal{F})$$

are isomorphisms.
(b) The complexes $\mathcal{F}_x$ and $R\Gamma_x(X,\mathcal{F})$ are perfect.

**Proposition 10.1.** *Assume $\mathcal{F} \in D^b(\mathcal{R}_X)$ is cohomologically constructible. Then the following hold.*

*(i) $\mathbb{D}_X\mathcal{F}$ is cohomologically constructible.*
*(ii) The natural morphism $\mathcal{F} \longrightarrow \mathbb{D}_X\mathbb{D}_X\mathcal{F}$ is an isomorphism.*
*(iii) For any $x \in X$ we have*

$$R\Gamma_{\{x\}}(X,\mathbb{D}_X\mathcal{F}) \cong \mathrm{RHom}(\mathcal{F}_x,\mathcal{R}), \tag{10.1a}$$

$$(\mathbb{D}_X\mathcal{F})_x \cong \mathrm{RHom}(R\Gamma_{\{x\}}(X,\mathcal{F}),\mathcal{R}). \tag{10.1b}$$

**Proof** We have

$$R\Gamma(U,\mathbb{D}_X\mathcal{F}) \cong \mathrm{RHom}(\Gamma_c(U,\mathcal{F}),\mathcal{R}).$$

Applying the functor $\underrightarrow{\mathbf{Lim}}_{U\ni x}$ and using Proposition 4.15 we deduce

$$\underrightarrow{\mathbf{Lim}}_{U\ni x} R\Gamma(U,\mathbb{D}_X\mathcal{F}) \cong \mathrm{RHom}(\underleftarrow{\mathbf{Lim}}_{U\ni x}\Gamma_c(U,\mathcal{F}),\mathcal{R}) \cong \mathrm{RHom}(R\Gamma_{\{x\}}(X,\mathcal{F}),\mathcal{R}).$$

This proves (10.1b) and thus exists and it is *perfect*..

Suppose $K$ is a compact neighborhood of $x$. We have

$$R\Gamma_K(X,\mathbb{D}_X\mathcal{F}) = \mathrm{RHom}(\underline{\mathcal{R}}_K,\mathbb{D}_X\mathcal{F}) \cong \mathrm{RHom}(\mathcal{F}_K,\omega_X)$$

(Verdier duality)

$$\mathrm{RHom}(R\Gamma_c(X,\mathcal{F}_K),\mathcal{R}) \cong \mathrm{RHom}(R\Gamma(X,\mathcal{F}_K),\mathcal{R}).$$

Applying the functor $\underleftarrow{\mathbf{Lim}}$ and Proposition 4.15 we get

$$\underleftarrow{\mathbf{Lim}}_{U\ni x} R\Gamma_c(U,\mathbb{D}_X\mathcal{F}) \cong \underleftarrow{\mathbf{Lim}}_{K\ni x} R\Gamma_K(X,\mathbb{D}_X F) \cong \mathrm{RHom}(\underrightarrow{\mathbf{Lim}}_{K\ni x} R\Gamma(X,\mathcal{F}_K),\mathcal{R})$$

$$\mathrm{RHom}((\underrightarrow{\mathbf{Lim}}_{U\ni x} R\Gamma(U,\mathcal{F}),\mathcal{R}) \cong \mathrm{RHom}(\mathcal{F}_x,\mathcal{R}).$$

This proves (10.1b) and completes the proof of (i) and (iii).

We now know that $\mathbb{D}_X\mathcal{F}$ and $\mathbb{D}_X\mathbb{D}_X\mathcal{F}$ are both cohomologically constructible. Using (10.1b) we deduce

$$(\mathbb{D}_X\mathbb{D}_X\mathcal{F})_x \cong \mathrm{RHom}(R\Gamma_{\{x\}}(X,\mathbb{D}_X\mathcal{F}),\mathcal{R}) \overset{(10.1a)}{\cong} \mathrm{RHom}\big(\mathrm{RHom}(R\Gamma_{\{x\}}(X,\mathcal{F}),\mathcal{R}),\ \mathcal{R}\big) \cong \mathcal{F}_x$$

where at the last step we have used the fact that $R\Gamma_{\{x\}}(X,\mathcal{F})$ is perfect.

□

**Corollary 10.2.** *Suppose $f : X \to S$ is a continuous map between finite dimensional locally compact spaces, and $\mathcal{R}$ is a commutative Noetheriang ring with $1$ of finite global dimension. Then, for every constructible $\mathcal{F} \in D^b(\mathbf{Sh}_{\mathcal{R}}(S))$ we have*

$$f^!\mathcal{F} \cong \mathbb{D}_X f^{-1}\mathbb{D}_S\mathcal{F}.$$

**Proof** Using Proposition 8.7 we deduce

$$f^!\mathbb{D}_S^2\mathcal{F} \cong \mathbb{D}_X f^{-1}\mathbb{D}_S\mathcal{F}.$$

Since $\mathcal{F}$ is constructible we have $\mathbb{D}_S^2\mathcal{F} \cong \mathcal{F}$.

□

**Corollary 10.3.** *Suppose $f : X \to S$ is as above. Then*

$$\omega_{X/S} \cong \mathbb{D}_X(f^{-1}\omega_S).$$

*Moreover for very $\mathcal{S} \in D^b(\mathbf{Sh}_{\mathcal{R}}(S))$*

$$f^!\mathcal{S} \cong f^{-1}\mathcal{S} \overset{\mathrm{L}}{\otimes} \mathbb{D}_X(f^{-1}\omega_S).$$

## REFERENCES

[1] N. Bourbaki: *Algèbre X. Algèbre homologique*, Masson, 1980.

[2] A. Borel: *Sheaf theoretic intersection cohomology*, in the volume "*Intersection Cohomology*", A. Borel et al., Progress in Mathematics, Birkhäuser, 1984.

[3] G. Bredon: *Sheaf Theory*, 2nd Edition, Graduate Texts in Mathematics, vol. 170, Springer Verlag, 1997.

[4] S.I. Gelfand, Yu.I. Manin: *Methods of Homological Algebra*, 2nd Edition, Springer Monographs in Mathematics, Springer Verlag 2003.

[5] R. Godement: *Topologie Algébrique et Théorie des faisceaux*, Hermann 1958.

[6] P.-P. Grivel: *Categories derivées et foncteurs derivées*, in the volume "*Algebraic $D$-modules*", A.Borel et al , Perspective in Mathematics, Academic Press, 1987.

[7] B. Iversen: *Cohomology of Sheaves*, Universitext, Springer Verlag, 1986.

[8] M. Kashiwara, P. Schapira: *Sheaves on Manifolds*, Gründlehren der mathematischen Wissenschaften,vol. 292, Springer Verlag, 1990.

[9] J. Peetre: *Une caractérisation abstraite des opérateurs différentiels*, Math. Scand. **7**(1959), 211-218.

[10] J. Peetre: *Réctification l'article "Une caractérisation abstraite des opérateurs différentiels"*, Math. Scand. **8**(1960), 116–120.

[11] *Seminaire Heidelberg-Strasbourg 1966/67. Dualité de Poincaré*, Publications I.R.M.A. Strasbourg, 1969.

[12] J.-L. Verdier: *Equivalence essentielle des systémes projectifs*, C.R. Acad. Sci. Paris, **261**(1965), 4950-4954.

[13] C.A. Weibel: *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, 1994.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.
*E-mail address*: nicolaescu.1@nd.edu