ASYMPTOTICS OF OSCILLATORY INTEGRALS

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Abstract. What follows is a study of the asymptotic behavior of oscillatory integrals of the form

$$I_\xi(a) = \int_{\mathbb{R}^n} e^{i\phi(x)\xi} a(x)\, dx,$$

in the limit as $\xi \to \infty$. The asymptotic behavior depends only entirely on the stationary points of the phase function ϕ. We describe an algorithmic approach to handling such integrals which relies the concept of the Newton polygon of the phase and the associated toric resolution.

Contents

Introduction

There are a good number of situations of physical and mathematical interest which revolve around integrals of the form

$$I_\xi(a) = I_\xi(a;\phi) = \int_{\mathbb{R}^n} e^{i\phi(x)\xi} a(x)\, dx, \quad \xi \to \infty,$$

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where the amplitude a is a compactly supported smooth function. The famous Riemann-Lebesgue Lemma states that if the phase ϕ has no stationary points on the support of a then the integral $I_\xi(a)$ goes to zero as $\xi \to \infty$ faster than any power ξ^{-n}, $n > 0$. We can explain heuristically the reason behind such a dramatic vanishing.

We regard the above integral as an infinite superposition of plane waves $a(x)e^{i\phi(x)\xi}$. As the ξ becomes increasingly large, the factor $e^{i\phi(x)\xi}$ oscillates more and per unit of distance and the contributions of these plane waves tend to cancel each other out. The net result of their superposition is a very small quantity.

We can turn Riemann-Lebesgue Lemma on its head and conclude that the integral $I_\xi(a)$ has nontrivial asymptotics as $\xi \to \infty$ only if ϕ has stationary points. The present thesis is devoted to the understanding of the nature of this asymptotics when the set of stationary points of ϕ consists of isolated points. Via partitions of unity we can reduce this to the case when ϕ has a *unique* stationary point p_0 on the support of a. The asymptotics of the integral is strongly correlated with the behavior of the phase near p_0. For simplicity we concentrated on the case when the space dimension n is at most 2. The arguments work in arbitrary dimensions but the computations are more involved.

The one-dimensional case reduces to the study of asymptotics Fresnel type integrals,

$$\int_0^\infty e^{\pm ix^m\xi}a(x)dx$$

which further reduces to a study of the Fourier transform of the homogenous functions

$$M^{\pm}_{a,k}(x) = \begin{cases} |x|^a(\log|x|)^k & \pm x > 0 \\ 0 & \pm x \leq 0. \end{cases}$$

These are thoroughly investigated in the classical monograph [4].

The higher dimensional case can be reduced to the one-dimensional case via a nice trick of I.M. Gelfand and J. Leray, whereby the phase ϕ is treated as a coordinate function. We can assume that the unique stationary point of ϕ is the origin. Moreover, according to a theorem of Tougeron [1, §6.3], up to a change of coordinates near $0 \in \mathbb{R}^n$, we can assume that the phase ϕ is a polynomial. Observe that if we change ϕ by a constant c, the integral $I_\xi(a)$ changes only by the multiplicative factor $e^{ic\xi}$ which does not affect the nature of the asymptotics. Thus, in the sequel we can assume that ϕ is a polynomial in n variables which has an isolated stationary point at the origin and such that $\phi(0) = 0$.

If the support of a is sufficiently small then in the region $\operatorname{supp} a \setminus \{\phi = 0\}$ we can regard ϕ as a space variables and reduce the integral $I_\xi(a)$ to one-dimensional oscillatory integrals

$$I_\xi(a) = \int_{t<0} e^{it\xi} \underbrace{\left(\int_{\phi=t} a(x)\frac{|dx|}{|d\phi|}\right)}_{=:J_-(t)} dt + \int_{t>0} e^{it\xi} \underbrace{\left(\int_{\phi=t} a(x)\frac{|dx|}{|d\phi|}\right)}_{=:J_+(t)} dt$$

The functions $J_\pm(t)$ are called the *Gelfand-Leray functions*. It turns out that the Gelfand-Leray functions have very nice asymptotic expansion as $t \to 0^\pm$, and these asymptotics determine the asymptotics of the oscillatory integral $I_\xi(a)$. The end result can be explicitly read off a *resolution of singularity of* ϕ. To explain how this is done we restrict our attention to the case of two space dimensions, $n = 2$.

A resolution of the singularity of ϕ at 0 is a pair (X, β), where X is a smooth surface $\beta : X \to U \subset \mathbb{R}^2$, is a proper smooth map onto a small open neighborhood U of $0 \in \mathbb{R}^2$ satisfying the following conditions.

- The *exceptional locus* $E = \beta^{-1}(0)$ is a union of connected submanifolds $E_1, \dots, E_\nu$ of X of codimension 1 which intersect transversally. More explicitly, this means that for every point $p \in E$ we can find an open neighborhood $\mathcal{O}$ and local coordinates u, v on $\mathcal{O}$ such that $u(p) = v(p) = 0$ and $E \cap \mathcal{O}$ is described either by the equation $u = 0$, or by the equation $uv = 0$.
- The induced map $\beta : X \setminus E \to U \setminus 0$ is a diffeomorphism.
- If we denote by C' the punctured curve $C' = \{q \in U \setminus 0;\ \ \phi(q) = 0\}$ then the closure of $\beta^{-1}(C')$ in X is a *smooth* curve which intersects the components of E transversally. We denote this curve by E_0 and we will refer to it as the *strict transform* of the curve $\{\phi = 0\}$.

For $i = 0, 1, \dots, \nu$ we set

$$E_i' = E_0 \setminus \cup_{j \neq i} E_j.$$

In other words E_i' is obtained from E_i by removing the intersections with the other components. We denote by m_i the order of vanishing of $\phi \circ \beta$ along E_i', and by δ_i the order of vanishing of the jacobian of β along E_i'. We form the arithmetic progressions

$$\mathcal{P}_i := \{\frac{\delta_i + 1}{m_i}, \frac{\delta_i + 2}{m_i}, \cdots, \}.$$

Let us point out that

$$\mathcal{P}_0 = \{1, 2, \dots, \}$$

We can now form the *nerve* Γ_Φ of the resolution. Every vertex of this graph corresponds bijectively to a component E_i, $i = 0, 1, \dots, \nu$. We denote by v_i the vertex corresponding to the component E_i. We connect two vertices v_i, v_j by an edge $[v_i v_j]$ if the corresponding components E_i and E_j intersect.

To a vertex $\boldsymbol{v} = v_i$ we associate the arithmetic progression $\mathcal{P}_i$, and to an edge $\boldsymbol{e} = [v_i v_j]$ we associate the arithmetic progression $\mathcal{P}_i \cap \mathcal{P}_j$. We denote by $\mathcal{V}$ the set of vertices of the graph, and by $\mathcal{E}$ the set of edges. Then $I_\xi(a)$ admits an asymptotic expansion of the type

$$I_\xi(a) \sim \sum_{\boldsymbol{v} \in \mathcal{V}} \sum_{\alpha \in \mathcal{P}_{\boldsymbol{v}}} C_\alpha^{\boldsymbol{v}}(a) \xi^{-\alpha} + \sum_{\boldsymbol{e} \in \mathcal{E}} \sum_{\beta \in \mathcal{P}_{\boldsymbol{e}}} \Big(C_{\beta,0}^{\boldsymbol{e}}(a) \xi^{-\beta} + C_{\beta,1}^{\boldsymbol{e}}(a) \xi^{-\beta} \log \xi \Big), \ \ \xi \nearrow \infty.$$

The coefficients $C_\alpha^{\boldsymbol{v}}(a)$, $C_{\beta,0}^{\boldsymbol{e}}(a)$, $C_{\beta,1}^{\boldsymbol{e}}(a)$ depend linearly and continuously on the amplitude a and vanish if the stationary point $0 \in \mathbb{R}^2$ does not belong to the support of a. In more technical terms, these coefficients are distributions on $\mathbb{R}^2$ supported at the origin and thus, by a theorem of L. Schwartz, they are linear combinations of partial derivatives of the Dirac delta distribution, [5, Thm. 2.3.4].

Thus the problem of studying the asymptotic expansion of on oscillatory integral reduces to the problem of constructing a resolution of the phase. In this thesis we describe the method of Arnold-Varchenko of producing special resolutions of the phase, the so called *toric* or *monomial resolutions*. The nerve Γ_ϕ and the arithmetic progressions $\mathcal{P}_{\boldsymbol{v}}$ can be read-off the *Newton diagram* of the polynomial phase ϕ.

Here is a brief description of the organization of the paper. We begin by discussing a number of useful tools and ideas from classical analysis. In Section 1.1 we discuss the always-useful gamma function and variants such as the beta function. Discussing this function is useful primarily for its application to the Fourier transform which is introduced in Section 1.2. Our presentation of the Fourier transform follows closely the classical text [5]. Section 1.3 is devoted to the calculation the Fourier transform of the functions $M_{a,k}^{\pm}$ which play a fundamental role in the sequel. As we will come to see in the sequel, the Fourier transform of the amplitude function $a(x)$ will be used to determine the coefficients in the asymptotic expansion of I_ξ.

Having these tools at our disposal, we then begin a discussion of asymptotic approximation in general. Section 2.1 lays out the basics of the theory of *regular singular asymptotic expansions*, i.e., asymptotic expansions of the type

$$f(t) \sim \sum_{a,k} C_{a,k} t^a (\log |t|)^k, \;\; t \to 0 \;\text{ or }\; t \to \infty.$$

The series in the right hand side need not be convergent so it only has a formal meaning. Inspired by the computations in Section 1.3 we define a formal Fourier transform which associates to a formal series as above another formal series of the same type. The framework of asymptotic approximation is then applied directly to the case of oscillatory integrals in one dimension. Section 2.2 introduces many of the critical ideas of this paper, most notably the Riemann-Lebesgue Lemma (Lemma 2.4) and Proposition 2.5 which states the asymptotics at $\xi \to \infty$ of the Fourier transform of a function a is the formal Fourier transform of the asymptotic expansion of a near the origin.

Once the situation has been laid out in one dimension, we naturally turn out attention to two dimensions. Section 3.1 explains how one may exactly calculate the asymptotic expansion of oscillatory integrals with two-dimensional monomial phases using the trick of I.M. Gelfand and J. Leray in which one takes the phase function ϕ to be a variable itself. This section culminates with Corollary 3.2 outlining exactly which powers of $|\xi|$ will be present in the asymptotic approximations of these integrals.

Finally, we turn our attention to the case of non-monomial phases in two-dimensions. In sections 3.2 and 3.3 we run into some very elegant mathematics. We construct a manifold known as the *toric resolution* of our phase ϕ, using an idea due to A. Varchenko. This reduces a general oscillatory integral to a sum of oscillatory integrals with monomial phases. In order to make this process absolutely clear, we offer a concrete example and follow the process from start to finish. The end result is a very carefully constructed space in which our non-monomial phase becomes a monomial over various subspaces of the toric resolution. Section 3.4 finishes off our discussion by looking at the asymptotics of a specific instance of our concrete phase example, giving the recipe for determining exactly which powers of ξ and $\log \xi$ will appear in the asymptotic approximation. Finally, we explain an elementary way in which the leading power of the asymptotic approximation (often referred to as the oscillation index) can be easily obtained directly from the Newton polygon of our phase function ϕ.

1. Classical Analysis

1.1. **The Gamma Function.**

As is the case with many scientific calculations, the Gamma Function (also known as the factorial function) will prove to be a valuable tool in our understanding of Oscillatory Integrals. We therefore devote the first section of our Classical Analysis introduction to understanding the Gamma Function.

Definition 1.1 (The Gamma Function)**.** *The Gamma function is the function*

$$\Gamma : \{\mathbf{Re}(z) > 0\} \to \mathbb{C}, \;\; \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \qquad \square$$

It can be shown that Γ may be analytically continued as a meromorphic function on the whole complex plane. To see this, note that

$$\Gamma(z) = \int_0^1 e^{-t}t^{z-1}dt + \int_1^\infty e^{-t}t^{z-1}dt$$

The second integral is an entire function, so $\Gamma(z)$ is analytic wherever the first integral is. We can analytically continue the first integral by replacing e^{-t} with its power series expansion.

$$\begin{aligned} \int_0^1 e^{-t}t^{z-1}dt &= \int_0^1 t^{z-1}dt \sum_{k=0}^\infty \frac{(-1)^k}{k!}t^k = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_0^1 t^{k+z-1}dt \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{1}{z+k}. \end{aligned}$$

We are justified in switching the order of integration and summation above because the the integral of the sum is absolutely convergent for $\mathbf{Re}(z) > 0$. As this shows, $\Gamma(z)$ may be analytically continued to the whole complex plane as a meromorphic function with simple poles at 0 and the negative integers. As such, we could write (for $z \neq 0, -1, -2, \dots$)

$$\Gamma(z) = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{1}{z+k} + \int_1^\infty e^{-t}t^{z-1}dt$$

The Gamma function has several useful properties, with factorial relationships between Gamma functions of integers being particularly useful.

Proposition 1.2 (Properties of the Gamma Function)**.**

$$\begin{aligned} \Gamma(z+1) &= z\Gamma(z), && (1.1) \\ \Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin \pi z}, && (1.2) \\ 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) &= \sqrt{\pi}\Gamma(2z). && (1.3) \end{aligned}$$

Proof. To obtain equation 1.1 we simply integrate $\Gamma(z+1)$ by parts to see that

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z dt = -e^{-t}t^z|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z)$$

We start by temporarily assuming that $\mathbf{Re}(z) < 1$. Then

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty e^{-t}t^{z-1}dt \int_0^\infty e^{-s}s^{-z}ds = \int_0^\infty \int_0^\infty e^{-(s+t)}s^{-z}t^{z-1}ds\,dt.$$

Then, setting $u = s+t$ and $v = \frac{t}{s}$, we see that

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \int_0^\infty e^{-u}v^{z-1}\frac{du\,dv}{1+v} = \int_0^\infty \frac{v^{z-1}}{z+v}dv = \frac{\pi}{\sin \pi z}.$$

We can safely continue this function analytic into any regions where $\Gamma(z)$ and $\Gamma(1-z)$ are able to be continued, and thus equation (1.2) is valid for all z except the integers.

Equation (1.3), commonly referred to as the *doubling formula*, can again be derived directly from Definition 1.1.

$$\begin{aligned}
2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) &= 2^{2z-1}\int_0^\infty e^{-t}t^{z-1}dt\int_0^\infty e^{-s}s^{-z}ds \\
&= \int_0^\infty\int_0^\infty e^{-(s+t)}(2\sqrt{st})^{2z-1}t^{-1/2}\,ds\,dt \\
&= 4\int_0^\infty\int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}\alpha\,d\alpha\,d\beta,
\end{aligned}$$

under the substitution $\alpha=\sqrt{s}$ and $\beta=\sqrt{t}$. Since α and β are dummy variables, we may re-write this as

$$\begin{aligned}
2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) &= 2\int_0^\infty\int_0^\infty e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}(\alpha+\beta)\,d\alpha\,d\beta \\
&= 4\int_0^\infty\int_0^\alpha e^{-(\alpha^2+\beta^2)}(2\alpha\beta)^{2z-1}(\alpha+\beta)\,d\alpha\,d\beta.
\end{aligned}$$

Letting $u=\alpha^2+\beta^2$, $v=2\alpha\beta$, and $u=v+\omega^2$ we see

$$\begin{aligned}
2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) &= \int_0^\infty v^{2z-1}\,dv\int_0^\infty \frac{e^{-u}}{\sqrt{u-v}}\,du \\
&= 2\int_0^\infty e^{-v}v^{2z-1}\,dv\int_0^\infty e^{-\omega^2}\,d\omega \\
&= \sqrt{\pi}\Gamma(2z).
\end{aligned}$$

As before, this equation can be continued analytically to the region $z\neq 0,-1/2,-1,-3/2,\ldots$. □

Now that we have developed these properties, it is time to put them to use. First, simply note that equation (1.1) sets up a recursion relation for the Gamma function of integers, that is to say,

$$\Gamma(n+1)=n\Gamma(n).$$

Noting that $\Gamma(1)=1$, we clearly see that $\Gamma(n+1)=n!$ for $n=0,1,2,\ldots$.

Furthermore, equation (1.3) with $z=1/2$ gives us that

$$\Gamma(1/2)=\frac{\sqrt{\pi}\,\Gamma(1)}{\Gamma(1)}=\sqrt{\pi},$$

and equation (1.1) then implies that

$$\Gamma(n+1/2)=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi},\qquad n=1,2\ldots$$

We would now like to introduce a number of integrals which are very similar to the Gamma function in form, but differ in ways which expand our available toolbox of integration tricks.

First, we look at integrals of the form

$$\int_0^\infty e^{-pt}t^{z-1}\,dt,\qquad p\in\mathbb{R}_{>0},\quad \mathbf{Re}(z)>0.$$

Letting $s=pt$, we easily see that

$$\int_0^\infty e^{-pt}t^{z-1}\,dt=\frac{\Gamma(z)}{p^z}. \tag{1.4}$$

Of course, this formula may be analytically continued to arbitrary complex $\mathbf{Re}(p) > 0$ in a manner similar to that used before.

Another function which is very similar to the Gamma function is the aptly-named Beta function. In fact, it will turn out that Gamma and Beta are more than similar, the Beta function is simply a number of Gamma functions in disguise. Here we are getting ahead of ourselves, however, so we must start with a definition.

Definition 1.3 (The Beta Function)**.** *The Beta function is the function $B : \mathbb{C}^2 \to \mathbb{R}$ defined by*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad \mathbf{Re}(x) > 0,\ \mathbf{Re}(y) > 0.$$

If we look at the Beta function a little closer we notice that it shares some similarities with the Gamma function. In particular, for the change of variables $u = t/(1-t)$, we have that

$$B(x, y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du, \quad \mathbf{Re}(x) > 0,\ \mathbf{Re}(y) > 0.$$

Furthermore, by equation (1.4) we see that

$$\frac{\Gamma(x+y)}{(1+u)^{x+y}} = \int_0^\infty e^{-(1+u)t} t^{x+y-1} dt,$$

and thus

$$\begin{aligned} B(x, y) &= \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{x+y-1} \int_0^\infty e^{-ut} u^{x-1}\, du \\ &= \frac{\Gamma(x)}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \end{aligned}$$

1.2. **The Fourier Transform.** One of the primary reasons the Gamma function will be of use to us in our studies is because of its close connection with the Fourier transform. A large majority of our work will utilize the language of the Fourier and Laplace transforms of a function, so we will now lay out the essential facts regarding these two operations.

First, we must introduce some notation.

We define

$$\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \langle x, \xi \rangle := \sum_{i=0}^{n} x_n \xi_n.$$

Definition 1.4 (The Fourier Transform)**.** *For any absolutely integrable function $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform of f to be*

$$\widehat{f}(\xi) = \int e^{-i\langle x, \xi \rangle} f(x)\, dx. \qquad \square$$

Physically, the Fourier transform of a function f can be thought of as decomposing f into "frequency components"; that is to say, the Fourier transform calculates the contribution of a given frequency ξ to the amplitude function f. Similarly, we may derive an inverse Fourier transform which essentially reconstructs the amplitude f from these individual frequency components $\widehat{f}(\xi)$.

To develop the inversion formula we must study a specific subset of the smooth functions.

Definition 1.5. *Let $\mathcal{S}(\mathbb{R}^n)$ denote the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that*

$$\sup_x |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indices α and β.

In the following we will use the notation $D_{x_j} = -i\partial_{x_j}$, $D_{\xi_k} = -i\partial_{\xi_k}$ and for $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$D_x^\alpha = (-i)^{\alpha_1+\cdots+\alpha_n}(\partial_{x_1})^{\alpha_1}\cdots(\partial_{x_n})^{\alpha_n}.$$

Lemma 1.6. *The Fourier transform $\mathcal{S} \to \mathcal{S}$, $\phi \mapsto \widehat{\phi}$ maps $\mathcal{S}$ continuously into itself. Furthermore,*

$$\widehat{D_{x_j}\phi} = \xi_j\widehat{\phi}, \quad \widehat{x_j\phi} = -D_{\xi_j}\widehat{\phi}$$

Proof. By definition, $\widehat{\phi}(\xi) = \int e^{-i\langle x,\xi\rangle}\phi(x)\,dx$. Thus

$$\begin{aligned} D_\xi^\alpha\widehat{\phi}(\xi) &= (-i)^{\alpha_1+\cdots+\alpha_n}(\partial_{\xi_1})^{\alpha_1}\cdots(\partial_{\xi_n})^{\alpha_n}\int e^{-i\langle x,\xi\rangle}\phi(x)\,dx \\ &= (-i)^{2(\alpha_1+\cdots+\alpha_n)}\int (x_1{}^{\alpha_1}\cdots x_n{}^{\alpha_n})e^{-i\langle x,\xi\rangle}\phi(x)\,dx \\ &= \int e^{-i\langle x,\xi\rangle}(-x)^\alpha\phi(x)\,dx. \end{aligned}$$

Since this integral is uniformly convergent, we see that $\widehat{\phi} \in C^\infty$. Thus $D_\xi^\alpha\widehat{\phi} = \widehat{(-x)^\alpha}\phi$.

Furthermore,

$$\widehat{x_j\phi} = \int_{\mathbb{R}^n} x_j e^{-i\langle x,\xi\rangle}\phi(x)dx = -D_{\xi_j}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\phi(x)dx = -D_{\xi_j}\widehat{\phi}(\xi).$$

Next observe that

$$|\xi^\beta D^\alpha\widehat{\phi}(\xi)| = (-1)^{|\alpha|+|\beta|}\widehat{D_x^\beta(x^\alpha\phi)}$$

On the other hand, since $\phi \in \mathcal{S}$ there exists $C > 0$ such that

$$\sup_{x\in\mathbb{R}^n} D_x^\beta(x^\alpha\phi)| \le \frac{C}{1+|x|^{n+1}}$$

and therefore,

$$\sup_x |\xi^\beta D_\xi^\alpha\widehat{\phi}(\xi)| \le C\left(\int \frac{1}{(1+|x|)^{n+1}}\,dx\right) < \infty.$$

□

Lemma 1.7. *If $T : \mathcal{S} \to \mathcal{S}$ is a linear map such that*

$$TD_j\phi = D_jT\phi, \quad Tx_j\phi = x_jT\phi, \quad j = 1, 2, \ldots, n, \quad \phi \in \mathcal{S},$$

Then $T\phi = c\phi$ for some constant c.

Proof. Let $\phi(y) = 0$. We may write $\phi(x) = \sum(x_j - y_j)\phi_j(x)$, where the $\phi_j \in C^\infty$. Thus

$$T\phi(x) = \sum(x_j - y_j)T\phi_j(x).$$

For $x = y$, $T\phi(x) = 0$, and thus we see that for all $\phi \in \mathcal{S}$,

$$T\phi(x) = c(x)\phi(x).$$

Choosing some $\phi \in \mathcal{S}$ which is everywhere non-zero, we obtain $c \in C^\infty$. Furthermore,

$$0 = D_jT\phi - TD_j\phi = (D_jc)\phi.$$

Thus we see that $c(x)$ must be constant. □

Theorem 1.8. *The Fourier transform $\mathcal{S} \ni \phi \mapsto \widehat{\phi} \in \mathcal{S}$ is an isomorphism with inverse*

$$\phi(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x,\xi\rangle} \widehat{\phi}(\xi)\, d\xi. \tag{1.5}$$

Proof. By Lemma 1.6, we see that F^2 maps $\mathcal{S}$ into $\mathcal{S}$ and that it anticommutes with D_j and x_j. Let $R(x) = \phi(-x)$. By Lemma 1.7, $T = RF^2$ is a linear map which anticommutes with D_j and x_j, and is therefore equal to a constant, $RF^2 = c$. Since this is the case for any $\phi \in \mathcal{S}$, we choose

$$\phi = e^{-|x|^2/2}.$$

Clearly $(x_j + iD_j)\phi = 0$, and therefore $(-D_j + i\xi_j)\widehat{\phi}(\xi) = 0$ for $j = 1, 2, \ldots, n$. This implies that $\widehat{\phi} = c_1\phi$ where $c_1 = \widehat{\phi}(0) = (2\pi)^{n/2}$. Thus $F^2\phi = F(F(\phi)) = F(c_1\phi) = {c_1}^2\phi$, and therefore $c = (2\pi)^n$. We therefore have that $RF^2\phi = c\phi$, and the theorem is proved. □

While proving these results in the general case is useful, in practice we will deal with the Fourier transform on $\mathbb{R}$. From this point on, unless otherwise stated the Fourier transform will be acting on functions $f \in L^1(\mathbb{R})$, and therefore

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx.$$

Having thus defined the Fourier transform, let's take a minute to compute the following example. While it may seem random at this point, this calculation will be very useful in the future.

1.3. **A fundamental example.** First, we define the function $M_{a,k} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $M_{a,k}(x) = |x|^a(\log|x|)^k$, and we denote by $M^{\pm}_{a,k}$ its truncations ,

$$M^{\pm}_{a,k}(x) = \begin{cases} M_{a,k}(|x|) & \text{if} \quad \pm x > 0 \\ 0 & \text{if} \quad \pm x \le 0. \end{cases}$$

To get our bearings, we will start with the simplest example. Let $f_\lambda(x) = e^{-\lambda x} M^{+}_{a,0}(x)$. Then we have that

$$\widehat{f_\lambda}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f_\lambda(x)\, dx = \int_0^{\infty} e^{-ix\xi} e^{-\lambda x} x^a\, dx = \int_0^{\infty} e^{-p\,x} x^a\, dx,$$

for $p = \lambda + i\xi$. Equation (1.4) implies

$$\widehat{f_\lambda}(\xi) = \frac{\Gamma(a+1)}{p^{a+1}} = \frac{\Gamma(a+1)}{(\lambda + i\xi)^{a+1}} = \frac{\Gamma(a+1)}{(i)^{a+1}(\xi - i\lambda)^{a+1}}.$$

By convention, we define $z^\lambda := e^{\lambda \log|z|} e^{i\lambda \arg z}$ with $-\pi < \arg z < \pi$. Thus

$$(i)^{-(a+1)} = e^{-(a+1)\log 1} e^{i-(a+1)(\pi/2)} = e^{-\frac{\pi}{2} i(a+1)},$$

and

$$\widehat{f_\lambda}(\xi) = \frac{e^{-\frac{\pi}{2} i(a+1)}\Gamma(a+1)}{(\xi + i\lambda)^{a+1}}.$$

Now let us move to the more general case. Let $f_\lambda(x) = e^{-\lambda x} M^{+}_{a,k}(x)$. We first note that the general functions $M^{+}_{a,k}(x)$ satisfy the relation

$$\frac{\partial^k}{\partial a^k} M^{+}_{a,0}(x) = \frac{\partial^{k-1}}{\partial a^{k-1}} M^{+}_{a,1}(x) = \cdots = M^{+}_{a,k}(x).$$

It is therefore clear that in the general case we have

$$\begin{aligned}\widehat{f_\lambda}(\xi) &= \int_{-\infty}^{\infty} e^{-ix\xi} e^{-\lambda x} M_{a,k}^{+}(x)\,dx = \int_{0}^{\infty} e^{-ix\xi} e^{-\lambda x} M_{a,k}^{+}(x)\,dx \\ &= \int_0^\infty e^{-px} \frac{\partial^k}{\partial a^k} M_{a,0}^{+}(x)\,dx = \frac{\partial^k}{\partial a^k} \int_0^\infty e^{-px} M_{a,0}^{+}(x)\,dx \\ &= \frac{\partial^k}{\partial a^k}\left(\frac{e^{-\frac{\pi}{2}i(a+1)}\Gamma(a+1)}{(\xi+i\lambda)^{a+1}}\right)\end{aligned}$$

If we now denote $E_\lambda = e^{-\lambda x}$ and $f_{\pm\lambda}(x) = E_{\pm\lambda} M_{a,0}^{\pm}(x)$ we get the most general description, namely

$$\widehat{E_{\pm\lambda} M_{a,0}^{\pm}}(\xi) = \widehat{f_{\pm\lambda}}(\xi) = \frac{\Gamma(a+1)}{e^{\pm\frac{\pi}{2}(a+1)i}(\xi \mp i\lambda)^{a+1}}. \tag{1.6}$$

As before, this implies that

$$\widehat{E_{\pm\lambda} M_{a,k}^{\pm}}(\xi) = \frac{\partial^k}{\partial a^k}\left(\frac{\Gamma(a+1)}{e^{\pm\frac{\pi}{2}(a+1)i}(\xi \mp i\lambda)^{a+1}}\right) \tag{1.7}$$

For $a > -1$ each monomial $M_{a,k}^{\pm}(x)$ is locally integrable and thus naturally defines a generalized function (distribution) on $\mathbb{R}$. We get two real analytic maps

$$(-1,\infty) \ni a \mapsto M_{a,k}^{\pm} \in \mathcal{D}'(\mathbb{R}) = \text{the sapce of distributions on } \mathbb{R}.$$

As explained in [4], they admit meromorphic extensions to the entire plane with simple poles located at $\mathbb{Z}_{<0} \subset \mathbb{C}$. They satisfy the functional equation

$$M_{a,k}^{\pm} = \frac{\partial^k}{\partial a^k} M_{a,0}^{\pm}, \ \ \forall\, a \in \mathbb{C} \setminus \mathbb{Z}_{<0}.$$

For any $f \in L^1(\mathbb{R})$ we have defined the Fourier transform of f to be

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx.$$

For $a > -1$ and $\lambda > 0$ the distribution $E_{\pm\lambda} M_{a,0}^{\pm}$ is integrable and its Fourier transform is

$$\widehat{E_{\pm\lambda} M_{a,k}^{\pm}}(\xi) = \frac{\partial^k}{\partial a^k}\left(\frac{\Gamma(a+1)}{e^{\pm\frac{\pi}{2}(a+1)i}(\xi \mp i\lambda)^{a+1}}\right).$$

The Fourier transform of the distribution $M_{a,k}^{\pm}$ is then the distribution

$$\widehat{M_{a,k}^{\pm}}(\xi) = \frac{\partial^k}{\partial a^k} \lim_{\lambda \searrow 0} \frac{\Gamma(a+1)}{e^{\pm\frac{\pi}{2}(a+1)i}(\xi \mp i\lambda)^{a+1}} =: \frac{\partial^k}{\partial a^k}\left(\frac{\Gamma(a+1)}{e^{\pm\frac{\pi}{2}(a+1)i}(\xi \mp 0i)^{a+1}}\right). \tag{1.8}$$

The last equality should be understood as follows. For every tempered test function $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$\frac{\partial^k}{\partial a^k}\left\langle \frac{\Gamma(a+1)}{e^{\pm\frac{\pi}{2}(a+1)i}(\xi \mp i\lambda)^{a+1}} \,,\, \widehat{\phi}(\xi)\right\rangle = 2\pi \frac{\partial^k}{\partial a^k} \int_{\mathbb{R}} M_{a,0}^{\pm}(x)\phi(x)\,dx, \ \ \forall\, a > -1.$$

Because $z^\beta = \big(|z|e^{i\arg z}\big)^\beta = |z|^\beta e^{i\beta\arg z}$ we deduce

$$(\xi \pm 0i)^{-(a+1)} = \begin{cases} |\xi|^{-(a+1)} & \text{on} \quad \xi > 0 \\ e^{\pm\pi i(a+1)}|\xi|^{-(a+1)} & \text{on} \quad \xi < 0 \end{cases}$$

We conclude that the restriction of $\widehat{M^{\pm}_{a,0}}$ to $\xi > 0$ is a monomial in $\mathbb{C}[\mathcal{P}]$ and we have

$$\widehat{M^{\pm}_{a,0}}(\xi) = \left(e^{\mp\frac{\pi}{2}(a+1)i}\right)\Gamma(a+1)M_{-(a+1),0}(\xi),\ \ \forall\ a > -1,\ \ \xi > 0.$$

More generally, we get

$$\widehat{M^{\pm}_{a,k}}(\xi) = \frac{\partial^k}{\partial a^k}\left(\left(e^{\mp\frac{\pi}{2}(a+1)i}\right)\Gamma(a+1)M_{-(a+1),0}(\xi)\right),\ \ a > -1,\ \ \xi > 0. \tag{1.9}$$

Note that if a is a nonnegative half-integer, $a = n/2$ then for $\xi \gg 0$ we have

$$\widehat{M^{\pm}_{n/2,0}}(\xi) = e^{\mp\frac{\pi}{2}(n/2+1)i}\Gamma(n/2+1)\xi^{-1-n/2} = \frac{\mp ie^{\mp n\pi i/4}\Gamma(1+n/2)}{\xi^{1+n/2}} \tag{1.10}$$

When n is even we have

$$\Gamma(1+n/2) = (n/2)!.$$

When n is odd, $n = 2k-1$ we have

$$\Gamma(1+n/2) = \Gamma((2k+1)/2) = \frac{(k-1)!}{2^{k-1}}\Gamma(1/2) = \frac{(k-1)!}{2^{k-1}}\sqrt{\pi}.$$

2. Basics of asymptotic analysis

2.1. **Regular Singular Asymptotics.** We would like to study the asymptotic behavior of integrals of the type

$$\int_0^\infty e^{i\phi(x)\xi}f(x)\,dx,$$

where the amplitude f is $\equiv 0$ near ∞ and both the phase $\phi(x)$ and the amplitude $f(x)$ have certain asymptotic expansions near 0. To describe them we first discuss the building blocks of such expansions.

Set

$$\mathcal{P} = \{(a,k) \in \mathbb{R}\times\mathbb{Z} \mid k \geq 0\}.$$

The *weight* is the canonical projection

$$w : \mathcal{P} \to \mathbb{R},\ \ (a,k) \mapsto a.$$

The *multiplicity* is the natural projection

$$\nu : \mathcal{P} \to \mathbb{Z}_{\geq 0},\ \ (a,k) \mapsto k.$$

Thus, for any $\mathfrak{p} \in \mathcal{P}$ we have

$$\mathfrak{p} = (w(\mathfrak{p}), \nu(\mathfrak{p})).$$

$\mathcal{P}$ is equipped with a natural involution

$$\mathfrak{p} \mapsto \check{\mathfrak{p}} = (-w(\mathfrak{p}), \nu(\mathfrak{p})).$$

Define

$$M_{\mathfrak{p}}(x) = |x|^{w(\mathfrak{p})}(\log|x|)^{\nu(\mathfrak{p})},\ \ x \neq 0.$$

Observe that

$$M_{\mathfrak{p}}(x)\cdot M_{\mathfrak{q}}(x) = M_{\mathfrak{p}+\mathfrak{q}}(x),\ \ \ \forall\ \ \mathfrak{p},\mathfrak{q} \in \mathcal{P}. \tag{2.1}$$

and

$$\frac{d}{dx}M_{a,k} = aM_{a-1,k} + kM_{a-1,k-1},\ \ \forall\ a > 0. \tag{2.2}$$

We introduce a linear order relation on $\mathcal{P}$ as follows.

$$\mathfrak{p} \prec \mathfrak{q} \Longleftrightarrow M_{\mathfrak{q}} = o(M_{\mathfrak{p}}),\ \ \text{as}\ \ x \searrow 0 \Longleftrightarrow \lim_{x\searrow 0}\frac{M_{\mathfrak{q}}(x)}{M_{\mathfrak{p}}(x)} = 0.$$

Equivalently, if $\mathfrak{p} = (a, k)$, $\mathfrak{q} = (b, m)$ then

$$(a, k) \prec (b, m) \Longleftrightarrow a < b \text{ or } a = b, \ \ k > m.$$

Observe that

$$\mathfrak{p} \prec \mathfrak{p}', \ \ \mathfrak{q} \prec \mathfrak{q}' \Longrightarrow \mathfrak{p} + \mathfrak{q} \prec \mathfrak{p}' + \mathfrak{q}'.$$

We can now define intervals

$$\mathcal{P}^{\mathfrak{p}} := \{\mathfrak{q} \in \mathcal{P} \mid \mathfrak{q} \preceq \mathfrak{p}\},$$
$$\mathcal{P}_{\mathfrak{p}} := \{\mathfrak{q} \in \mathcal{P} \mid \mathfrak{p} \preceq \mathfrak{q}\}, \ \ \mathcal{P}^{\mathfrak{p}}_{\mathfrak{q}} = \mathcal{P}^{\mathfrak{p}} \cap \mathcal{P}_{\mathfrak{q}}.$$

A set $P \subset \mathcal{P}$ is *upper locally finite* (*u.l.f.*) if

$$\forall \, \mathfrak{p} \in \mathcal{P}, \ \ \#(P \cap \mathcal{P}^{\mathfrak{p}}) < \infty.$$

A set $P \subset \mathcal{P}$ is *lower locally finite* (*l.l.f.*) if $\check{P}$ is upper locally finite i.e.

$$\forall \, \mathfrak{p} \in \mathcal{P}, \ \ \#(P \cap \mathcal{P}_{\mathfrak{p}}) < \infty.$$

A set $P \subset \mathcal{P}$ is *locally finite* (*l.f.*) if it is the union of a l.l.f and an u.l.f. set. Observe that P is u.l.f (resp. l.l.f.) if for any $w' \in \mathbb{R}$ there are only finitely many $\mathfrak{p} \in P$ such that

$$w(\mathfrak{p}) \leq w' \ \ (\text{resp. } w(\mathfrak{p}) \geq w').$$

We define the *oscillation index* $\omega_+(P)$ of an u.l.f. set P by

$$\omega_+(P) = \min\{w(\mathfrak{p}) \mid \mathfrak{p} \in P\}.$$

The oscillation index $\omega_-(P)$ of a l.l.f. set P by

$$\omega_-(P) = \max\{w(\mathfrak{p}) \mid \mathfrak{p} \in P\}.$$

The *index* set of P is the set $w(P) \in \mathbb{R}$. Observe that the index set of a l.f. set is discrete.

Denote by $\mathbb{C}[\mathcal{P}]$ the complex vector space spanned by $\{M_{\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{P}\}$. It is naturally a $\mathbb{C}$-algebra with multiplication induced by equation (2.1). $M_{0,0}$ is the multiplication unit in this algebra and we will denote it by $\mathbf{1}$. It is equipped with a natural involution defined by

$$M_{a,k}(x) \longleftrightarrow \check{M}_{a,k}(x) = M_{a,k}(1/x) = (-1)^k M_{-a,k}(x).$$

Equivalently

$$\check{M}_{\mathfrak{p}} = (-1)^{d(\mathfrak{p})} M_{\check{\mathfrak{p}}}.$$

We now define two completions $\mathbb{C}[[\mathcal{P}]]$ of $\mathbb{C}[P]$, the upper completion $\mathbb{C}[[\mathcal{P}]]_+$ and the lower completion $\mathbb{C}[[\mathcal{P}]]_-$.

More precisely $\mathbb{C}[[\mathcal{P}]]_+$ consists of all formal sums

$$A = \sum_{\mathfrak{p} \in \mathcal{P}} a_{\mathfrak{p}} M_{\mathfrak{p}},$$

where $a : \mathcal{P} \to \mathbb{C}$, $\mathfrak{p} \mapsto a_{\mathfrak{p}}$, is a function with u.l.f. support. For every $A \in \mathbb{C}[[\mathcal{P}]]_+$ we set

$$\operatorname{supp} A := \{\mathfrak{p} \in \mathcal{P} \mid a_{\mathfrak{p}} \neq 0\}.$$

We define a multiplication on $\mathbb{C}[[\mathcal{P}]]_+$ by

$$\Big(\sum_{\mathfrak{p} \in \mathcal{P}} a_{\mathfrak{p}} M_{\mathfrak{p}}\Big) * \Big(\sum_{\mathfrak{q} \in \mathcal{P}} b_{\mathfrak{q}} M_{\mathfrak{q}}\Big) = \sum_{\mathfrak{s} \in \mathcal{P}} \Big(\sum_{\mathfrak{p}+\mathfrak{q}=\mathfrak{s}} a_{\mathfrak{p}} b_{\mathfrak{q}}\Big) M_{\mathfrak{s}}$$

Due to the equality (2.2) the derivative

$$\frac{d}{dx} : \mathbb{C}[\mathcal{P}] \to \mathbb{C}[\mathcal{P}]$$

extends to a derivation of the $\mathbb{C}$-algebra $(\mathbb{C}[[\mathcal{P}]]_+, +, *)$.

$$\frac{d}{dx} : \mathbb{C}[[\mathcal{P}]]_+ \to \mathbb{C}[[\mathcal{P}]]_+.$$

The *oscillation index* of $A \in \mathbb{C}[[\mathcal{P}]]_+$ is defined by

$$\omega_+(A) := \omega_+(\operatorname{supp} A).$$

Note that

$$\omega_+(AB) = \omega_+(A) + \omega_+(B), \;\; \omega_+\left(\frac{d}{dx}A\right) = \omega_+(A) - 1.$$

We define $\mathbb{C}[[\mathcal{P}]]_-$ in a similar fashion. Note that the involution $A \longleftrightarrow \check{A}$ on $\mathbb{C}[\mathcal{P}]$ induces an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[[\mathcal{P}]]_+ \longleftrightarrow \mathbb{C}[[\mathcal{P}]]_-, \;\; A \longleftrightarrow \check{A}.$$

Observe that

$$\mathbb{C}[\mathcal{P}] = \mathbb{C}[[\mathcal{P}]]_+ \cap \mathbb{C}[[\mathcal{P}]]_-.$$

An element $A \in \mathbb{C}[\mathcal{P}]$ is called *homogeneous* if the index set of $\operatorname{supp} A$ consists of a single point. In other words A has the form

$$A = x^w \sum_{k \in F} a_k (\log x)^k$$

where $F \subset \mathbb{Z}_{\geq 0}$ is a finite set. The real number w is called the weight of A.

Suppose $u : (0, \infty) \to \mathbb{C}$ is a smooth function. We say that u admits a *regular singular asymptotic expansion* at 0 if there exists $A = \sum_{\mathfrak{p} \in \mathcal{P}} a_{\mathfrak{p}} M_{\mathfrak{p}} \in \mathbb{C}[[\mathcal{P}]]_+$ such that

$$u \sim A \;\text{ as }\; x \searrow 0.$$

This means that for all $\ell \in \mathbb{Q}$ and for all $k \in \mathbb{Z}_{\geq 0}$ we have

$$\left| \frac{\partial}{\partial x^k} \Big(u(x) - \sum_{w(\mathfrak{p}) < \ell} a_{\mathfrak{p}} M_{\mathfrak{p}}(x) \Big) \right| = o(|x|^{\ell - k}) \;\text{ as }\; x \searrow 0.$$

The series A is uniquely determined by u and we set

$$[u]_0 := A, \;\; L^0_{\mathfrak{p}}(u) = a_{\mathfrak{p}}.$$

That is to say, we can now write

$$[u]_0 = \sum_{\mathfrak{p} \in \mathcal{P}} L^0_{\mathfrak{p}}(u) M_{\mathfrak{p}}.$$

We denote by $\mathcal{A}^+_0$ the vector space of functions $u : (0, \infty) \to \mathbb{C}$ which admit regular singular asymptotic expansion at 0, and by $\mathcal{A}^-_0$ the vector space of functions $v : (-\infty, 0) \to \mathbb{C}$ which admit regular singular asymptotics at 0.

Similarly, we say that u admits a *regular singular asymptotic expansion* at ∞ if there exists $B \in \mathbb{C}[[\mathcal{P}]]_-$ such that $u \sim B \in \mathbb{C}[[\mathcal{P}]]_-$ as $x \to \infty$. This means that for all $\ell \in \mathbb{Q}$ and every $k \in \mathbb{Z}_{\geq 0}$ we have

$$\left| \frac{\partial^k}{\partial x^k} \Big(u(x) - \sum_{w(\mathfrak{p}) < \ell} b_{\mathfrak{p}} M_{\mathfrak{p}}(x) \Big) \right| = o(|x|^{\ell - k}) \;\text{ as }\; x \to \infty.$$

Equivalently,

$$u(1/x) \sim \bar{B}(x), \;\text{ as }\; x \searrow 0.$$

We write

$$[u]_\infty := B.$$

We define $\mathcal{A}_\infty$ to be the vector space of smooth functions $u : (0,\infty) \to \mathbb{C}$ which have regular singular asymptotic expansions at ∞. Clearly, the spaces $\mathcal{A}_0^\pm$ and $\mathcal{A}_\infty$ are $\mathbb{C}$-algebras. Moreover we have morphisms of $\mathbb{C}$-algebras

$$[\bullet]_0 : \mathcal{A}_0^\pm \to \mathbb{C}[[\mathcal{P}]]_+,\;\; u \mapsto [u]_0,\;\; [\bullet]_\infty : \mathcal{A}_\infty \to \mathbb{C}[[\mathcal{P}]]_-,\;\; u \mapsto [u]_\infty.$$

Note that we have a natural isomorphism of $\mathbb{C}$-algebras

$$\check{}: \mathcal{A}_0^+ \to \mathcal{A}_\infty,\;\; u(x) \mapsto \check{u}(x) = u(1/x).$$

Using equation (1.9) as a guide we define the *formal one-sided Fourier transforms*

$$\boldsymbol{F}_\pm : \mathbb{C}[[\mathcal{P}]]_+ \to \mathbb{C}[[\mathcal{P}]]_-,\;\; \boldsymbol{F}_\pm(M_{a,k}) := \frac{\partial^k}{\partial a^k}\left(\left(e^{\mp\frac{\pi}{2}(a+1)i}\right)\Gamma(a+1)M_{-(a+1),0}\right). \tag{2.3}$$

2.2. **Large Frequency Asymptotics.** We want to prove that the Fourier transform maps a function with regular singular asymptotics at $x \to 0$ to a function with regular singular asymptotics at $\xi \to \infty$. Before we state and prove the main result we need to present some elementary facts.

For every absolutely integrable function $f : (0,\infty) \to \mathbb{C}$ which is identically zero near ∞ we denote by $I_\xi^\pm(f)$ the Fourier transform of the restriction of f to the semi-axis $\pm x > 0$, i.e.

$$I_\xi^\pm(f) := \int_{\pm x>0} e^{-ix\xi} f(x)\,dx = \int_{\mathbb{R}} e^{-ix\xi} f(x) M_{0,0}^\pm(x)\,dx,\;\; \xi > 0.$$

We would like to study the asymptotic behaviour of $I_\xi^\pm(f)$ as $\xi \to \infty$. Set

$$D := i\frac{d}{dx}.$$

We begin with a special case when the amplitude $f(x)$ is identically zero in a neighborhood of 0.

Lemma 2.1. *Suppose $\varphi : (0,\infty) \to \mathbb{C}$ is a compactly supported smooth function. Then $I_\xi(\varphi) \in \mathcal{A}_\infty$ and $[I(\varphi)]_\infty = 0$, i.e., for every integer $N > 0$ we have*

$$\lim_{\xi\to\infty} I_\xi^\pm(\varphi)\xi^{-N} = 0.$$

Proof Using the identity

$$D^n(\xi^{-n}e^{-ix\xi}) = e^{ix\xi}$$

we obtain after an integration by parts n times we deduce that

$$I_\xi^+(\varphi) = \left(\frac{1}{\xi}\right)^n I_\xi(D^n\varphi).$$

Hence for every $n > 0$ we have

$$|I_\xi(f)| \leq \xi^{-n}\sup|D^n\varphi(x)|.$$

□

We fix a smooth function $\eta : \mathbb{R} \to [0,1]$, such that $\eta(x) = 1$ if $|x| \leq 1$ and $\eta(x) = 0$ if $|x| \geq 2$.

Lemma 2.2. *Suppose $g \in \mathbb{C}[\mathcal{P}]$ is a homogeneous element of weight $w(g) < -1$. Then*

$$(1-\eta)g \in L^1(\mathbb{R}),$$
$$I_\xi^\pm((1-\eta)g) \in \mathcal{A}_\infty(\xi) \ \text{ and } \ [I_\xi^\pm((1-\eta)g]_\infty = 0.$$

Proof We prove this Lemma only for I^+. The integrability of $(1-\eta)g$ follows from the weight condition on g and the support properties of $(1-\eta)$. Observe next that for every positive integer m the derivative $D^m g$ is homogeneous of weight $w(g) - m$ and we have

$$\lim_{|x|\to\infty} D^m g(x) = 0, \ \ D^m(1-\eta)(x) \equiv 0, \ \ \forall\, |x| \gg 1.$$

Using these facts we deduce

$$I_\xi^+((1-\eta)g) = \xi^{-n}\int_0^\infty D^n(e^{i\xi})(1-\eta)g\,dx = \xi^{-n}\int_0^\infty e^{ix\xi}D^n((1-\eta)g)\,dx.$$

Using Leibniz product formula we deduce

$$I_\xi^+((1-\eta)g) = \xi^{-n}\sum_{k=0}^n \binom{n}{k} I_\xi(D^k(1-\eta)D^{n-k}g)$$

so that

$$|I_\xi^+(1-\eta g)| \le \xi^{-n}\sum_{k=0}^n \binom{n}{k}\int_0^\infty |D^k(1-\eta)D^{n-k}g|\,dx.$$

□

Lemma 2.3. *Suppose $g \in \mathbb{C}[\mathcal{P}]$ is homogeneous of weight $w(g) > -1$. Then*

$$I_\xi^\pm(\eta g) \in \mathcal{A}_\infty(\xi) \ \text{ and } \ [I^\pm(\eta g)]_\infty = \boldsymbol{F}_\pm(g),$$

where $\boldsymbol{F}_\pm$ denote the formal Fourier transforms defined in equation 2.3.

Proof We prove this Lemma only for I^+. Let $G := \boldsymbol{F}_+(g)$. We know that

$$G(\xi) = \lim_{\tau\searrow 0} I_\xi^+(E_\tau g), \ \ E_\tau(x) = e^{-\tau x}.$$

Observe that

$$I_\xi^+(\eta g) = \lim_{\tau\searrow 0} I_\xi^+(E_\tau \eta g).$$

Thus we have to show that

$$\lim_{\tau\searrow 0} I_\xi^+(E_\tau(1-\eta)g)\Big) \sim 0, \ \ \text{as } \ \xi\to\infty,$$

that is for any positive integer $N > 0$ we have

$$\lim_{\xi\to\infty} \xi^{-N}\Big(\lim_{\tau\searrow 0} I_\xi^+(E_\tau(1-\eta)g)\Big) = 0.$$

Observe that

$$De^{ix\xi-\tau x} = (\xi + i\tau)e^{ix\xi-\tau x}.$$

So that

$$e^{ix\xi-\tau x} = D^m\Big\{(\xi+i\tau)^{-m}e^{ix\xi-\tau x}\Big\}, \ \ \forall\, m > 0.$$

Set for simplicity $E_{\xi,\tau} = e^{ix\xi-\tau x}$. Suppose n is a positive integer such that

$$w(g) - n < 1 \Longleftrightarrow n > w(g) - 1.$$

We deduce that

$$I^+_\xi(E_\tau(1-\eta)g) = (\xi+i\tau)^{-n}\int_0^\infty E_{\xi,\tau}D^n((1-\eta)g).$$
$$= (\xi+i\tau)^{-n}\sum_{k=0}^n\binom{n}{k}\int_0^\infty E_{\xi,\tau}D^k(1-\eta)D^{n-k}g\,dx.$$

For $0<k\leq n$ the function $D^k(1-\eta)D_kg$ is compactly supported and from Lemma 2.1 we deduce

$$\lim_{\tau\searrow 0}\int_0^\infty E_{\xi,\tau}D^k(1-\eta)D^{n-k}g\,dx = I^+_\xi(D^k(1-\eta)D_kg)\sim 0,\ \text{ as }\ \xi\to\infty.$$

We still have to deal with the remaining term $(1-\eta)D^ng$. Observe first that $D^ng\in\mathbb{C}[\mathcal{P}]$ is homogeneous of weight $w(g)-n<-1$. We deduce that the function $(1-\eta)D^ng$ is integrable and from the dominated convergence theorem we deduce that

$$\lim_{\tau\searrow 0}I^+_\xi(E_\tau(1-\eta)g) = I^+_\xi((1-\eta)g).$$

Lemma 2.2 now implies that $I^+_\xi((1-\eta)g)\sim 0$ as $\xi\to\infty$.

□

Lemma 2.4 (Riemann-Lebesgue)**.** *Let $g\in\mathcal{A}_0(x)$ and $n\in\mathbb{Z}_{\geq 0}$ such that $\omega_+([g]_0)>n$. Then*

$$I^\pm_\xi(\eta g) = o(\xi^{-n-1}),\ \text{ as }\ \xi\to\infty$$

Proof We prove the Lemma only for I^+. The condition on the oscillation index of g implies that $D^{n+1}(\eta g)\in L^1(0,\infty)$.

$$D^kg(0)=0,\ \ \forall\, 0\leq k\leq n.$$

Let $h:=D^{n+1}(\eta g)$. The extension of $h(x)$ by zero for $x<0$ is in $L^1(\mathbb{R})$ and we have

$$I^+_\xi(\eta g) = \frac{1}{\xi^{n+1}}\int_{\mathbb{R}}D^n(e^{ix\xi})\eta g\,dx = \xi^{-n-1}\int_{\mathbb{R}}e^{ix\xi}h(x)\,dx.$$

We have to show that

$$I_\xi(h) = \int_{\mathbb{R}}e^{ix\xi}h(x)\,dx = o(1)\ \text{ as }\ \xi\to\infty.$$

This is a generalization of the classical Riemann-Lebesgue Lemma. For a very elegant proof we refer to [6, p.14].

□

Proposition 2.5. *Suppose $f\in\mathcal{A}_0(x)$, $\omega_+([f]_0)>-1$ Then*

$$I^\pm_\xi(\eta f)\in\mathcal{A}_\infty(\xi)$$

and

$$[I^\pm_\xi(\eta f)]_\infty = \boldsymbol{F}_\pm([f]_0),$$

where $\boldsymbol{F}_\pm$ denote the formal Fourier transforms $\boldsymbol{F}_\pm:\mathbb{C}[[P]]_\pm\to\mathbb{C}[[P]]_+$ defined in equation (2.3).

Proof We prove the proposition only for I_ξ^+. Set

$$A = [f]_0 = \sum_{w(\mathfrak{p})>-1} A_{\mathfrak{p}} M_{\mathfrak{p}}, \;\; B = \boldsymbol{F}_+(A) := \sum_{w(\mathfrak{q})<0} B_{\mathfrak{q}} M_{\mathfrak{q}}.$$

Recall that for every $\mathfrak{p} \in \mathcal{P}$, $w(\mathfrak{p}) > -1$ the Fourier transform of $M_{\mathfrak{p}}$ is homogeneous of weight $-w(\mathfrak{p}) - 1$. For every nonnegative integer n we set

$$A_n = \sum_{-1<w(\mathfrak{p})\leq n} A_{\mathfrak{p}} M_{\mathfrak{p}}, \;\; f_n = f - A_n, \;\; B_n = \sum_{-n-1\leq w(\mathfrak{q})<0} B_{\mathfrak{q}} M_{\mathfrak{q}}.$$

We have to prove that for every positive integer n we have

$$|I_\xi^+(\eta f) - B_n(\xi)| = o(\xi^{-n-1}) \;\text{ as }\; \xi \nearrow \infty.$$

Observe that $f_n \in \mathcal{A}_0$ and $\omega_+([f_n]_0) > n$ and $B_n = \boldsymbol{F}_+(A_n)$. By the Riemann-Lebesgue Lemma we have

$$I_\xi^+(\eta f_n) = o(\xi^{-n-1}), \;\text{ as }\; \xi \nearrow \infty.$$

On the other hand Lemma 2.3 implies

$$I_\xi^+(\eta A_n) = B_n(\xi) + o(\xi^{-n-1}) \;\text{ as }\; \xi \nearrow \infty.$$

Hence

$$I_\xi^+(\eta f) = I_\xi^+(\eta A_n) + I_\xi^+(\eta f_n) = B_n(\xi) + o(\xi^{-n-1}) \;\text{ as }\; \xi \nearrow \infty.$$

□

We would now like to investigate the asymptotics as $\hbar \searrow 0$ of the integrals

$$I_\hbar = \int_{-\infty}^{\infty} e^{\pm i x^j/\hbar} \eta(x)\, dx,$$

where η is a compactly supported smooth function, and j is a positive integer. For simplicity we will consider only the cases $j = 2, 3$ because they contain all the intricacies of the general situation. Observe that η admits an asymptotic expansion near $x = 0$,

$$\eta(x) \sim \sum_{k\geq 0} \eta_k x^k, \;\; eta_k = \frac{1}{k!}\eta^{(k)}(0).$$

We replace $1/\hbar$ with ξ and instead study $I_\xi = \int_{-\infty}^{\infty} e^{\pm i\xi x^j} \eta(x)\, dx$.

Example 2.6 ($j = 2$)**.** We shall start by stu dying the asymptotics of the integral

$$I_\xi = \int_{-\infty}^{\infty} e^{\epsilon i \xi x^2} \eta(x)\, dx, \;\text{ as }\; \xi \nearrow \infty, \;\; \epsilon = \pm 1.$$

By evenness of the integrand we have

$$I_\xi = 2\int_0^{\infty} e^{\epsilon i x^2 \xi} \eta(x)\, dx.$$

We distinguish two cases.

A. $\epsilon = 1$ We make the change in variables

$$y = -x^2, \;\; -\infty \leq y < 0 \Longleftrightarrow x = |y|^{1/2} \Longrightarrow dx = \frac{-\,dy}{2(-y)^{-1/2}}$$

Hence, as $\xi \nearrow \infty$ we have

$$I_\xi = \int_{-\infty}^{0} e^{-i\xi y}|y|^{-1/2}\eta(|y|^{1/2})\,dy \sim \eta_0 \boldsymbol{F}_-[M_{-1/2,0}] + O(|\xi|^{-3/2})$$

$$= \eta_0 e^{\frac{\pi}{4}i}\Gamma(1/2)\xi^{-1/2} + O(|\xi|^{-3/2}).$$

B. $\epsilon = -1$ We make the change in variables

$$y = x^2,\ \ 0 \le y < \infty \Longleftrightarrow x = y^{1/2} \Longrightarrow dx = \frac{dy}{2y^{-1/2}}$$

Hence

$$I_\xi = \int_0^\infty e^{-iy\xi}y^{-1/2}\eta(y^{1/2})\,dy \sim \eta_0\boldsymbol{F}_+[M_{-1/2,0}]+O(|\xi|^{-3/2}) = \eta_0 e^{-\frac{\pi}{4}i}\Gamma(1/2)\xi^{-1/2}+O(|\xi|^{-3/2}).$$

Summarizing,

$$I_\xi = \int_{-\infty}^{\infty} e^{\pm i\xi x^2}\eta(x)\,dx \sim e^{\pm i\pi/4}\sqrt{\pi}\eta_0\xi^{-1/2} + O(|\xi|^{-3/2}).$$

$\square$

Example 2.7 ($j = 3$)**.**

$$I_\xi = \int_{-\infty}^{\infty} e^{\pm i\xi x^3}\eta(x)\,dx.$$

We first note that under the change of variables $x = -t$,

$$\int_{-\infty}^{\infty} e^{i\xi x^3}\eta(x)\,dx = \int_{-\infty}^{\infty} e^{-i\xi t^3}\eta(t)\,dt,$$

so we need only consider one case. We will choose the negative case.

We must first separate the integral into two pieces, and then make our substitution.

$$\int_{-\infty}^{\infty} e^{-i\xi x^3}\eta(x)\,dx = \int_{-\infty}^{0} e^{-i\xi x^3}\eta(x)\,dx + \int_0^\infty e^{i\xi x^3}\eta(x)\,dx$$

In both cases we make the change of variable $y = x^3$, and we thus obtain

$$\int_{-\infty}^{\infty} e^{i\xi x^3}\eta(x)\,dx = \frac{1}{3}\left[\int_0^\infty e^{-iy\xi}|y|^{-2/3}\eta(y^{1/3})\,dy + \int_{-\infty}^{0} e^{-iy\xi}|y|^{-2/3}\eta(y^{1/3})\,dx\right]$$

$$\sim \frac{\eta_0}{3}\left[\boldsymbol{F}_+[M_{-2/3,0}] + \boldsymbol{F}_-[M_{-2/3,0}]\right] = \frac{\eta_0}{3}\Gamma\left(\frac{1}{3}\right)\left[e^{-\frac{\pi}{6}i} + e^{+\frac{\pi}{6}i}\right]\xi^{-1/3} + O(|\xi|^{-4/3}).$$

$\square$

As one can readily see, these calculations are not *easy*, but they are at least possible, and knowing how is a significant improvement over our previous situation.

3. Higher Dimensions

In this last section we discuss the asymptotics of oscillatory integrals of two variables. More precisely, we investigate the large ξ behavior of integrals of the type[1]

$$I_\xi(a;\phi) = \int_{\mathbb{R}^2} e^{i\xi\phi(x,y)} a(x)|dxdy|$$

where ϕ is a polynomial in two variables with an isolated stationary point at the origin, while the amplitude a is a smooth, compactly supported function.

We will follow the approach pioneered by A. Varchenko which reduces this problem to simpler oscillatory integrals with *monomial* phases via the process of *toric resolutions*.

3.1. **Basic Monomials.**

We will start our higher dimension discussion by considering oscillatory integrals of the form

$$I_pm(\xi,\rho) = \int e^{\pm i\xi x^A y^B} \rho(x,y)\,|dx\,dy|, \quad m,n \geq 0$$

where $A, B \in \mathbb{Z}$, $A \leq B$, and $\rho \in C^\infty(\mathbb{R}^2)$ is a nonnegative smooth function supported in the box

$$B = [-1,1] \times [-1,1].$$

In order to get a handle on this integral we invoke a trick of I.M. Gelfand and J. Leray, namely, replacing our y variable with the phase function itself, in a sense treating ϕ as the independent variable.

In order to avoid future complications, there are a few cosmetic changes that need to take place before we can nail down ϕ. First of all, we would like for the two powers in the expression of ϕ to be relatively prime. In order to do so, let $d = \gcd(A,B)$. Using the notation $ad = A$, $bd = B$ we can then write

$$I_\pm(\xi,\rho) = \int_B e^{\pm i\xi(x^a y^b)^d} \rho(x,y)\,|dx\,dy|.$$

We would now like to treat this not as a function of x and y but rather as a function of x and the phase variable $\phi = x^a y^b$. This is possible only away from the two coordinate axes $x = 0$, $y = 0$. Denote by B^* the complement of the two axes in the box B. This complement is a union of four boxes $B_1, \ldots, B_4$ corresponding to the four quadrants. We have

$$I(\xi) = \int_{B^*} e^{\pm i\xi(x^a y^b)^d} \rho(x,y)\,|dx\,dy| = \sum_{k=1}^4 \underbrace{\int_{B_k} e^{i\xi(x^a y^b)^d} \rho(x,y)\,|dx\,dy|}_{=:\mathcal{I}_k^\pm(\xi)}$$

We investigate only the asymptotics of $\mathcal{I}_1^+(\xi,\rho)$, the integral over the first quadrant, since the other cases are completely similar. Note that

$$y = (\phi x^{-a})^{1/b}, \quad d\phi = ax^{a-1}y^b dx + bx^a y^{b-1} dy,$$

$$dy = \frac{d\phi - ax^{a-1}y^b dx}{bx^a y^{b-1}}.$$

[1] The notation $|dxdy|$ indicates that we regard the integrand as a *density* and not as a differential form. The reason that we need to work with densities rather than differential forms is that we will be forced to work on non-orientable surfaces.

Because $dx\wedge dx=0$,

$$dx\wedge dy=\frac{1}{b}x^{-a}y^{1-b}dx\wedge d\phi=\frac{1}{b}x^{-a/b}\phi^{1/b-1}dx\wedge d\phi.$$

We have that

$$\mathcal{I}_1^{\pm}(\xi)=\frac{1}{b}\int_{B_1}e^{\pm i\xi\phi^d}\rho(x,x^{-a/b}\phi^{1/b})x^{-a/b}\phi^{1/b-1}|dxd\phi|.$$

To obtain this asymptotics we integrate along the fibers of the function

$$\phi: B_1=(0,1]\times(0,1]\to(0,\infty)$$

Note that along the level set $\phi=c$ we have $c^{1/a}\leq x\leq 1$. We set

$$J_1(\phi;\rho):=\frac{1}{b}\phi^{1/b-1}\int_{\phi^{1/a}}^{1}\rho\left(x,x^{-a/b}\phi^{1/b}\right)x^{-a/b}dx.$$

Observe that $J_1(\phi)$ is supported in the interval $[0,1]$. From Fubini's formula we deduce

$$\mathcal{I}_1^{\pm}(\xi)=\int_0^{\infty}e^{+\pm i\xi\phi^d}\phi^{1/b-1}J_1(\phi;\rho)d\phi,$$

Of course, $\mathcal{I}_1$ looks very similar to the functions we studied in Chapter 2. In fact, to use all of our previous conclusions we need only determine whether or not the function $J(\phi)$ has a regular singular asymptotic expansion near the origin.

We have by assumption that $\rho(x,y)$ has a regular singular asymptotic expansion near $(0,0)$,

$$\rho(x,y)\sim A(x,y)=\sum_{k,\ell\geq 0}a_{k\ell}\,x^ky^\ell.$$

We have

$$J_1(\phi,x^ky^\ell)=\frac{1}{b}\phi^{(\ell+1)/b-1}\int_{\phi^{1/a}}^{1}x^{k-(\ell+1)a/b}dx$$

$$=\begin{cases}\frac{1}{ab(\,(k+1)/a-(\ell+1)/b)\,)}\left(\phi^{(\ell+1)/b-1}-\phi^{(k+1)/a-1}\right) & \text{if}\quad (k+1)/a\neq(\ell+1)/b\\ -\frac{1}{ab}\phi^{(\ell+1)/b-1}\log\phi & \text{if}\quad (k+1)/a=(\ell+1)/b.\end{cases}$$

In the second case we must have

$$\frac{k+1}{\ell+1}=\frac{a}{b}$$

and since the fraction a/b is irreducible we deduce that $(\ell+1)$ must be an integer multiple of b, $\ell+1=(m+1)b$, $m\geq 0$.

We would like to introduce here a useful notation for the sets of exponents that will result from calculations of this type. From above, we will get a certain type of contribution when the value $(k+1)/a$ is not equal to the value of $(\ell+1)/b$, (and k and ℓ are both by assumption greater than or equal to zero), as well as contributions containing logarithmic terms when those two values are equal. Denote by

$$\mathcal{P}_{a,k}:=\left\{\frac{k+1}{a},\frac{k+2}{a},\dots\right\}.$$

Note in particular that $\mathcal{P}_{1,0}=\mathbb{Z}_{\geq 1}$. Using this notation, we deduce that $J(\phi,\rho)$ has an asymptotic expression of the form

$$J_1(\phi,\rho)=\sum_{\alpha\in\mathcal{P}_{a,0},\,\beta\in\mathcal{P}_{b,0}}\phi^{-1}\left(A_\alpha\phi^\alpha+B_\beta\phi^\beta\right)+\sum_{\gamma\in\mathcal{P}_{a,0}\cap\mathcal{P}_{b,0}}C_\gamma\phi^{\gamma-1}\log\phi. \tag{3.1}$$

More generally, the above argument shows that for any smooth function ρ supported in B, and any nonnegative integers μ, ν we have

$$J_1(\phi, x^\mu y^\nu \rho) = \sum_{\alpha\in\mathcal{P}_{a,\mu},\, \beta\in\mathcal{P}_{b,\nu}} \phi^{-1}\Big(A_\alpha \phi^\alpha + B_\beta \phi^\beta \Big) + \sum_{\gamma\in\mathcal{P}_{a,\mu)}\cap\mathcal{P}_{B,\nu}} C_\gamma \phi^{\gamma-1}\log\phi \tag{3.2}$$

Putting together all of the above we deduce the following result.

Proposition 3.1. *Suppose $\rho\in C^\infty(\mathbb{R}^2)$ is supported in $\mathbb{R}^2$. Then for every $A, B\in\mathbb{Z}_{>0}$ we set*

$$D = \gcd(A,B),\quad a = \frac{A}{D},\quad b = \frac{B}{D},\quad I^\pm_{A,B}(\xi;\rho) := \int_{\mathbb{R}^2} e^{\pm i\xi x^A y^B}\rho(x,y)|dxdy|.$$

Then

$$I^\pm_{A,B}(\xi;\rho) = \int_0^\infty e^{\pm i\xi\phi^D} J^+_{\mu,\nu}(\phi,\rho)d\phi + \int_{-\infty}^0 e^{\pm i\xi\phi^d} J^-_{\mu,\nu}(\phi,\rho)d\phi$$

where $J^\pm(\phi;\rho)$ are smooth functions defined on the semiaxis $\{\pm\phi>0\}$ which near $\phi=0$ admit regular singular asymptotic expansions of the form

$$J^\pm(\phi;\rho) \sim \sum_{\alpha\in\mathcal{P}_{a,0},\, \beta\in\mathcal{P}_{b,0}} |\phi|^{-1}\Big(\mathcal{A}^\pm_\alpha(\rho)|\phi|^\alpha + \mathcal{B}^\pm_\beta(\rho)|\phi|^\beta \Big) + \sum_{\gamma\in\mathcal{P}_{a,0}\cap\mathcal{P}_{b,0}} \mathcal{C}_\gamma(\rho)|\phi|^{\gamma-1}\log|\phi|.$$

Above the coefficients $\mathcal{A}^\pm_\alpha(\rho)$, $\mathcal{B}^\pm_\beta(\rho)$, $\mathcal{C}^\pm_\gamma(\rho)$ depend linearly[2] *on the amplitude ρ. Moreover, for any compactly supported smooth function $\rho:\mathbb{R}^2\to\mathbb{R}$ and any non-negative integers μ,ν we have*

$$\mathcal{A}^\pm_k(x^\mu y^\nu\rho) = \mathcal{B}^\pm_\ell(x^\mu y^\nu\rho) = \mathcal{C}^\pm_m(x^\mu y^\nu\rho) = 0$$

if $\alpha\in\mathcal{P}_{a,0}\setminus\mathcal{P}_{a,\mu}$, $\beta\in\mathcal{P}_{b,0}\setminus\mathcal{P}_{b,\nu}$, $\gamma\notin\mathcal{P}_{a,\mu}\cap\mathcal{P}_{b,\nu}$.

Using Proposition 2.5 we deduce as in Example 2.7 the following result.

Corollary 3.2. *For any positive integers A, B, and any $\alpha\in\mathcal{P}_{A,0}$, $\beta\in\mathcal{P}_{B,0}$, $\gamma\in\mathcal{P}_{A,0}\cap\mathcal{P}_{B,0}$ there exist distributions $\boldsymbol{u}_\alpha$, $\boldsymbol{v}_\beta$, $\boldsymbol{w}_\gamma$ on $\mathbb{R}^2$ such that, the following hold.*
(a) The distributions $\boldsymbol{u}_\alpha$, $\boldsymbol{v}_\beta$, $\boldsymbol{w}_\gamma$ are supported at $0\in\mathbb{R}^2$, i.e., $\boldsymbol{u}_\alpha(\rho)=\boldsymbol{v}_\beta(\rho)=\boldsymbol{w}_\gamma(\rho)=0$, for all $\rho\in C^\infty_0(\mathbb{R}^2)$, $0\notin\operatorname{supp}\rho$.
(b)For all $\rho\in C^\infty_0(\mathbb{R}^2)$ we have the regular singular asymptotic expansion as $\xi\nearrow\infty$,

$$I^\pm_{A,B}(\xi,\rho) \sim \sum_{\alpha\in\mathcal{P}_{A,0}} \boldsymbol{u}_\alpha(\rho)|\xi|^{-\alpha} + \sum_{\beta\in\mathcal{P}_{B,0}} \boldsymbol{v}_\beta(\rho)|\xi|^{-\beta} + \sum_{\gamma\in\mathcal{P}_{A,0}\cap\mathcal{P}_{B,0}} \gamma(\rho)|\xi|^{-\gamma}\log|\xi|,$$

where $D=\gcd(A,B)$.
(c)

$$\boldsymbol{u}_\alpha(x^\mu y^\nu\rho)\neq 0 \Longrightarrow \alpha\in\mathcal{P}_{A,\mu},$$

$$\boldsymbol{v}_\beta(x^\mu y^\nu\rho)\neq 0 \Longrightarrow \beta\in\mathcal{P}_{B,\nu},$$

$$\boldsymbol{w}_\gamma(x^\mu y^\nu\rho)\neq 0 \Longrightarrow \gamma\in\mathcal{P}_{A,\mu}\cap\mathcal{P}_{B,\nu}.$$

[2]More rigorously, they are distributions supported at the origin $0\in\mathbb{R}^2$. As such they are certain *universal* linear combinations of the Dirac function and its partials.

3.2. **Non-Monomials.** While the preceding discussion has allowed us to appropriately calculate the asymptotics of integrals with phase functions of monomial terms, it would certainly be nice to consider phase functions which are not monomials. It turns out that, again, we will be able to take a difficult object and convert it into one with which we are more familiar. In this instance, we will find that phase functions which are not monomials can, by a rather elegant detour, be treated as monomials in a different space. This space's fancy title (which will be made more transparent soon) is the toric resolution of the phase ϕ. But before we can start integrating, we have to figure out what each of the words in that title means.

For the purposes of this paper, we have decided that it is more enlightening to go through a specific example of this process than to demonstrate it's viability in all generality. Should the reader be interested in the full derivation of this process for all dimensions, we heartily recommend they pick up [2].

The phase with which we have chosen to work was not chosen because of some special property or coincidence which would make its computation easier; it was selected at random. We invite the reader to follow the steps with a completely different and equally random phase, and the process will be exactly the same. But enough beating around the bush! The phase we will work with for this long example is

$$\phi(x,y) = A_{(6,0)}x^6 + A_{(5,7)}x^5y^7 + A_{(2,4)}x^2y^4 + A_{(0,10)}y^{10}.$$

We should note here that there are some constraints on the coefficients for this polynomial, but that the set of numbers for which this process will not work is completely negligible. We will describe the constraints at a more appropriate time.

We have written the terms in order of decreasing values of the exponential of x for reasons that will be apparent forthwith. As we can see from Figure 1, our phase ϕ has an *isolated* stationary point at the origin which is solely responsible for the asymptotics of an integral involving ϕ.

The first step in this construction is to generate from this phase the geometric object called the *Newton Polygon* of our polynomial $\phi(x,y)$. Also note here that the subscripts on the coefficients of each individual monomial represent the powers of the x and y variables. We can think of these pairs of numbers as being points in the positive quadrant of $\mathbb{R}^2$. The Newton Polygon can be thought of as the convex hull of these points and the points $(\infty, 0)$ and $(0, \infty)$. The edges of this polygon are shown in Figure 2, and it is important to note that not every point in the polynomial will contribute to this polygon.

The next step is to calculate the slope of each of the lines of our polygon, and thereby find the slope of the normal vectors to each of them. We number the lines in the order we encounter them as we move counterclockwise from the x axis to the y axis. Letting m_i denote the slope of the line and s_i the slope of its normal, we find

$$\begin{aligned} s_1 = 0 \quad &\rightarrow \quad m_1 = \infty \\ s_2 = -1 \quad &\rightarrow \quad m_1 = 1 \\ s_3 = -3 \quad &\rightarrow \quad m_1 = \frac{1}{3} \\ s_4 = \infty \quad &\rightarrow \quad m_1 = 0 \end{aligned}$$

The purpose of finding the slopes of the normals to these lines is in order that we might construct an object known as the *fan associated to the Newton Polygon* of ϕ. Quite simply, the associated fan is the normal vectors and the regions which lie between them. If we were to write these slopes in fractional terms (where we set $0 = \frac{0}{1}$ and $\infty = \frac{1}{0}$), we obtain the x and y coordinates of our normal rays from the denominator and numerator, respectively.

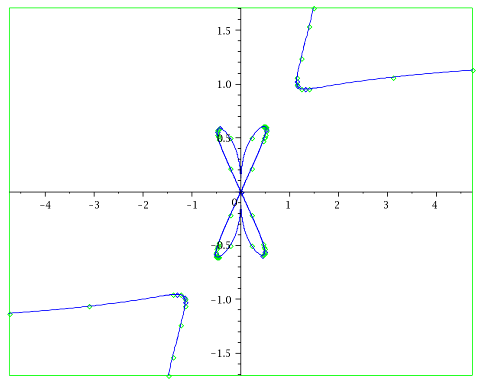


FIGURE 1. $\phi(x, y) = 0$ in $\mathbb{R}^2$.

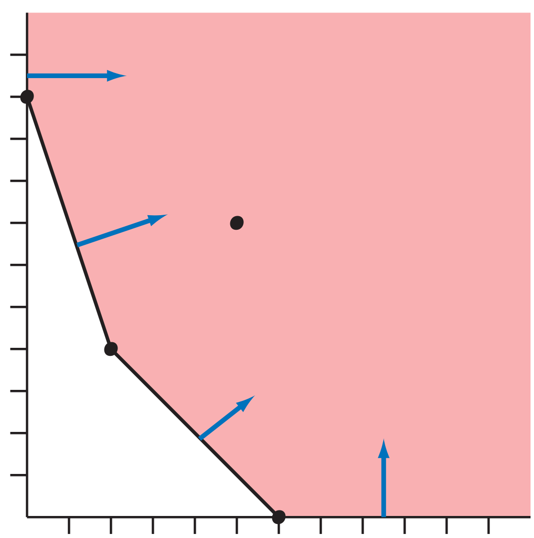


FIGURE 2. The Newton polygon associated to ϕ.

Thus

$$\begin{aligned} m_1 &= \frac{1}{0} \quad \to \quad r_1 = (0,1) \\ m_2 &= \frac{1}{1} \quad \to \quad r_2 = (1,1) \\ m_3 &= \frac{1}{3} \quad \to \quad r_3 = (3,1) \\ m_4 &= \frac{0}{1} \quad \to \quad r_4 = (1,0) \end{aligned}$$

If we plot these rays, the associated fan looks like Figure 3.

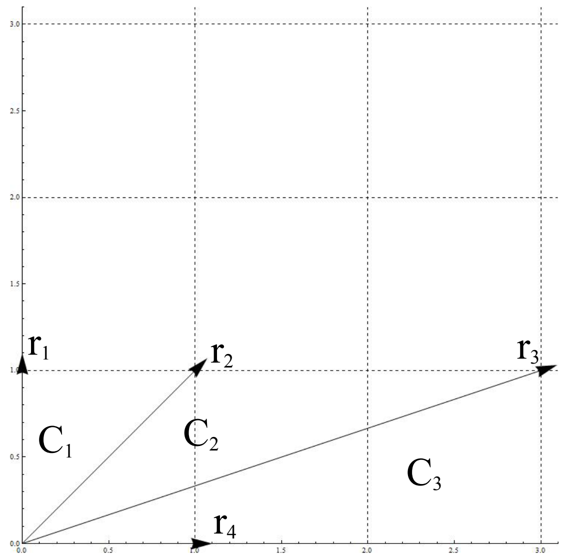


FIGURE 3. The fan associated to the Newton Polygon of ϕ.

First, we must get some more notation out of the way. Denote by C_i the cone bounded by rays r_i and r_{i+1}, and denote by W_i the matrix whose columns are the components of rays r_i and r_{i+1}. For the current fan, we have

$$W_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

We will refer to W_i as the *weight matrices.* For reasons that are not at all transparent right now, but very crucial, we need to have each of these W_i be *unimodular*, *i.e.* $\det(W_i) = 1$. While this requirement is satisfied by matrices W_1 and W_3, it is not for W_2. In order to resolve this problem, we turn to a process called *refinement* of the cone. Pictorally, this process looks to find a suitable subdivision of cone C_2. Looking at Figure 3, imagine we fix ray r_2 at it's terminal point (*i.e.* at the arrowhead) and swing the origin point clockwise. The refinement of the fan adds the first lattice point inside cone C_2 this rotating line hits. In this case, that point is the point $(2, 1)$. With this new point, we have a fan which is depicted in Figure 4. What we have done graphically can also be accomplished numerically via a formula utilizing continued fractions.

We would now like to take stock of our new W matrices and, thinking ahead a bit, list their inverses:

$$W_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \qquad W_3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad W_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$W_1^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad W_2^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, \quad W_3^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad W_4^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The reason for writing out the inverses of these matrices is to create yet another in a seemingly endless chain of matrices. We title these matrices *gluing matrices* for reasons which, again,

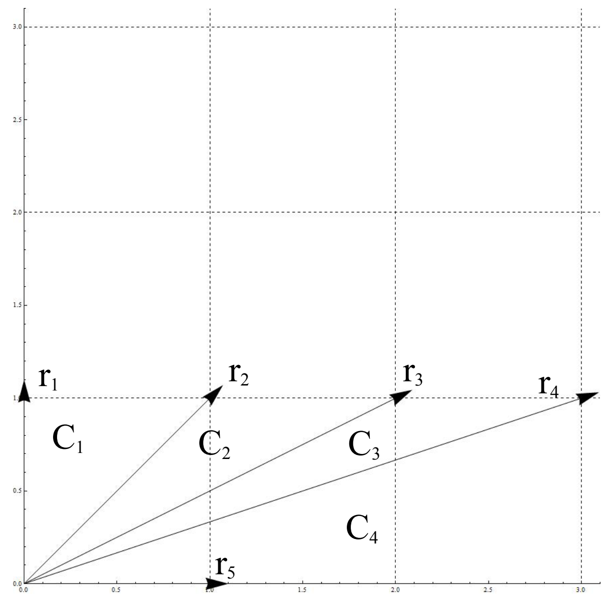


FIGURE 4. The refined fan.

will be apparent in due time. Let $G_{i+1,i} = W_{i+1}^{-1} W_i$. Then

$$G_{2,1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad G_{3,2} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad G_{4,3} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

3.3. **The Associated Toric Surface.** And so, at long last, some explanation is in order. What we have done in constructing these W and G matrices is found numbers which can be used to describe a smooth atlas for a surface which is called the *toric surface* associated to our fan. Essentially, to each cone of our refined fan we associate a copy of $\mathbb{R}^2$, and in this copy the coordinates are labeled (x_i, y_i). These copies of $\mathbb{R}^2$ are then glued together in a very particular way encoded by the matrices $G_{i+1,i}$. Because C_i and C_{i+1} share a common edge there ought to be a relationship between the ith and $(i+1)$th copies given by

$$\begin{matrix} x_2 = x_1 y_1 \\ y_2 = x_1^{-1} \end{matrix}, \quad x_1 \neq 0, \qquad \begin{matrix} x_3 = x_2^2 y_2 \\ y_3 = x_2^{-1} \end{matrix}, \quad x_2 \neq 0, \qquad \begin{matrix} x_4 = {x_3}^2 y_3 \\ y_4 = x_3^{-1} \end{matrix}, \quad x_3 \neq 0. \tag{3.1}$$

Using a rather abusive notation, the above equalities can be rewritten as

$$\log \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = G_{i+1,i} \log \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

To our fan $\mathcal{F} = \{C_1, C_2, C_3, C_4\}$, we associate the toric surface $S(\mathcal{F})$ defined by the atlas

$$U_1, \;\; U_2, \;\; U_2, \;\; U_4,$$

where each of the U_i's is a copy of $\mathbb{R}^2$ with coordinates (x_i, y_i). The copy U_i is glued to the copy U_{i+1} via the gluing maps defined in (3.1).

The weight matrices W_i define the *blowdown* map

$$\beta : S(\mathcal{F}) \longrightarrow \mathbb{R}^2(x, y) = \text{the Cartesian plane with coordinates } (x, y).$$

as follows. Letting $\beta_i = \beta|_{U_i}$, we have the following four maps in this case (note how the exponents are defined in terms of the numbers in W_i for each β_i).

$$(x_1, y_1) \xmapsto{\beta_1} (x, y) := (x_1 {y_1}^3, y_1), \quad (x_2, y_2) \xmapsto{\beta_2} (x, y) := ({x_2}^3 {y_2}^2, x_2 y_2),$$

$$(x_3, y_3) \xmapsto{\beta_3} (x, y) := ({x_3}^2 y_3, x_3 y_3), \quad (x_4, y_4) \xmapsto{\beta_4} (x, y) := (x_4, x_4 y_4).$$

Using the aforementioned "sloppy notation", this can be written in shorthand as

$$\log \begin{bmatrix} x \\ y \end{bmatrix} = W_i \log \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

It is worth our time to examine what this map tells us about our toric surface $S(\mathcal{F})$. In particular, we note what happens to the exceptional divisor of the map β (*i.e.* the zero locus $E = \beta^{-1}(0)$).

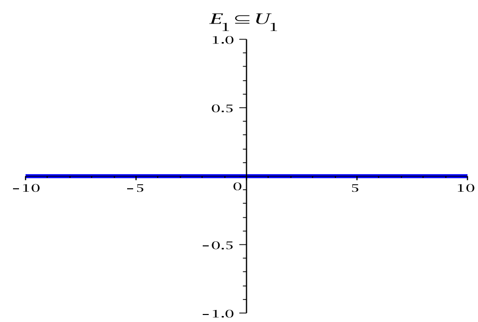


FIGURE 5. The exceptional divisor in $U_1 \subset S(\mathcal{F})$.

Let $E_i = E \cap U_i$. Then in U_1, $E_1 = \{(x_1, y_1) \mid y_1 = 0\}$. As can be seen in Figure 5, this is simply the x axis in U_1. More importantly though, we see that in U_2 we have

$$E_2 = \{(x_2, y_2) \mid x_2 = 0 \text{ or } y_2 = 0\}.$$

What is interesting here is that, according to our gluing maps defined by the matrices $G_{i+1,i}$, the portion of E_2 defined by the condition $x_2 = 0$—the y axis—is exactly the x axis of U_1, which we showed to be E_1. The correspondence is shown in Figure 6. Furthermore, the gluing maps identify the origin in U_2 with the point at infinity on the x axis in U_1, and vice-versa. In a similar fashion, it can be shown that each E_i is either one or both of the axes, and that one axis can be identified with one of the space above (when such a space exists), and the other with an axis in the space below (again, when it exists).

When we put together all of the pieces here, we can think of these overlapping axes as being rings that are attached to each other, but in a special way. Those axes which are glued together are done so with a global flip. This flip is representative of the fact that the part of the axes extending to infinity in one U_i are converging to the origin in the neighboring $U_{i\pm1}$. To get a bit of intuition about what this means, I created a three-dimensional model of this situation, and the picture has been included here as Figure 7. In particular, it is worth mentioning that the surface is *nonorientable*!

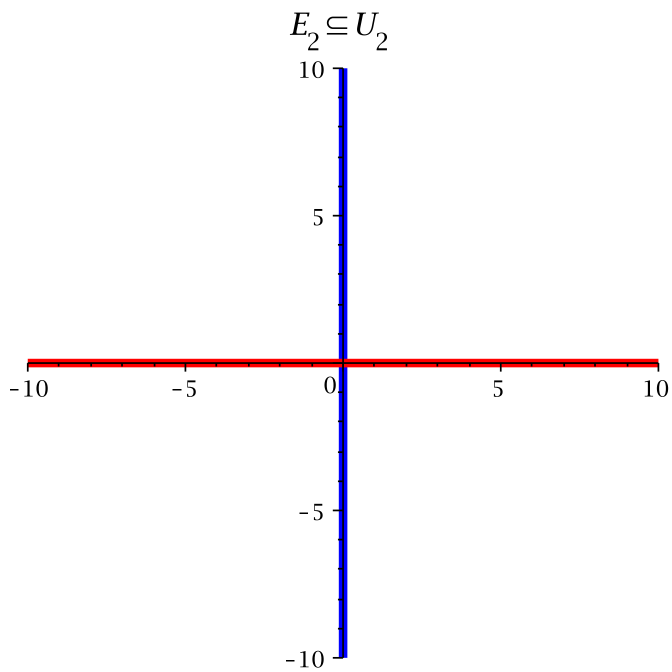


FIGURE 6. The exceptional divisor in $U_2 \subset S(\mathcal{F})$.

3.4. **Integrating on the Toric Surface.** And now finally, at long last, we can integrate. The structure we have labored so hard to create can be summarized by the following diagram

$$\begin{tikzcd} S(\mathcal{F}) \arrow[r, "\beta"] \arrow[dr, "\phi\circ\beta"'] & \mathbb{R}^2 \arrow[d, "\phi"] \\ & \mathbb{R} \end{tikzcd}$$

The blowdown map $\beta : S(\mathcal{F}) \to \mathbb{R}^2$ has the property that it induces a diffeomorphism

$$\beta : S(\mathcal{F}) \setminus E \to \mathbb{R}^2 \setminus 0, \quad E := \beta^{-1}(0).$$

Thus

$$\int_{\mathbb{R}^2} e^{i\xi\phi(x)} a(x) |dxdy| = \int_{S(\mathcal{F})\setminus E} e^{i\xi\beta^*(\phi)} \beta^*(a) \beta^* |dxdy|.$$

We will see that the integral over $S(\mathcal{F}) \setminus E$ becomes much simpler. In fact, it becomes a sum of oscillatory integrals over $\mathbb{R}^2$ with monomial phases!

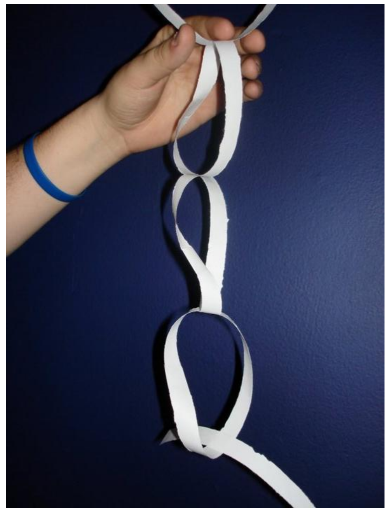


FIGURE 7. The toric resolution is a chain of three Möbius bands and two "moustaches".

First, we calculate what the function $\beta^*(\phi) = \phi \circ \beta : S(\mathcal{F}) \longrightarrow \mathbb{R}$ behaves like on each chart U_i. The zero set of the function $\phi \circ \beta$ is called the *total transform* of the curve $\{\phi = 0\}$ in the plane with coordinates x, y. The preimage $\beta^{-1}(0)$ is called the *exceptional divisor*. For simplicity we set $\phi_i := \phi \circ \beta_i$.

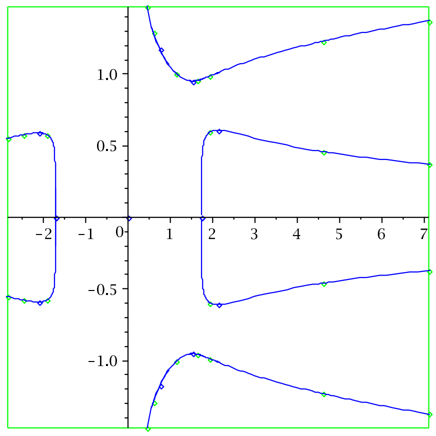
$$\begin{aligned}
\phi_1 &= A_{6,0}(x_1{y_1}^3)^6 + A_{5,7}(x_1{y_1}^3)^5(y_1)^7 + A_{2,4}(x_1{y_1}^3)^2(y_1)^4 + A_{0,10}(y_1)^{10} \\
&= {y_1}^{10}\left(A_{6,0}\,{x_1}^6{y_1}^8 + A_{5,7}\,{x_1}^5{y_1}^{12} + A_{2,4}\,{x_1}^2 + A_{0,10}\right) \\
\phi_2 &= A_{6,0}({x_2}^3{y_2}^2)^6 + A_{5,7}({x_2}^3{y_2}^2)^5(x_2y_2)^7 + A_{2,4}({x_2}^3{y_2}^2)^2(x_2y_2)^4 + A_{0,10}(x_2y_2)^{10} \\
&= {x_2}^{10}{y_2}^8\left(A_{6,0}\,{x_2}^8{y_2}^4 + A_{5,7}\,{x_2}^{12}{y_2}^9 + A_{2,4} + A_{0,10}\,{y_2}^2\right) \\
\phi_3 &= A_{6,0}({x_3}^2y_3)^6 + A_{5,7}({x_3}^2y_3)^5(x_3y_3)^7 + A_{2,4}({x_3}^2y_3)^2(x_3y_3)^4 + A_{0,10}(x_3y_3)^{10} \\
&= {x_3}^8{y_3}^6\left(A_{6,0}\,{x_3}^4 + A_{5,7}\,{x_3}^9{y_3}^6 + A_{2,4} + A_{0,10}\,{x_3}^2{y_3}^4\right) \\
\phi_4 &= A_{6,0}(x_4)^6 + A_{5,7}(x_4)^5(x_4y_4)^7 + A_{2,4}(x_4)^2(x_4y_4)^4 + A_{0,10}(x_4y_4)^{10} \\
&= {x_4}^6\left(A_{6,0} + A_{5,7}\,{x_4}^6{y_4}^7 + A_{2,4}\,{y_4}^4 + A_{0,10}\,{x_4}^4{y_4}^{10}\right)
\end{aligned}$$

The level sets $\{\phi_i = 0\}$ for the phase $\phi(x, y) = x^6 - x^5y^7 - 2x^2y^4 + 3y^{10}$ are depicted in Figure 8. They were obtained using the *Maple* package `algcurves`.

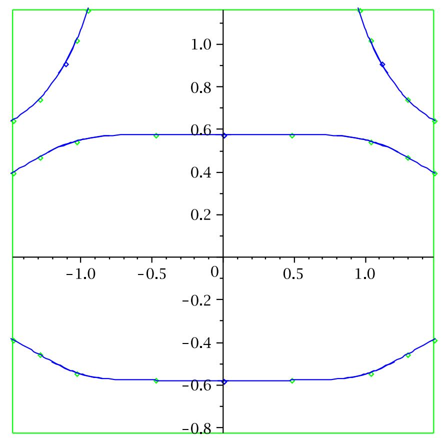
We must note now that there is one constraint on the coefficients of this polynomial, and therefore not every set of coefficients will be acceptable. For this process to work, the level set $\phi_i = 0$ must consists of connected smooth curves which intersect transversally. In particular, no three components have a point in common. This translates into conditions on the the partial derivatives of the phases ϕ_i along the axes x_i and y_i. However, random polynomials with a given Newton polygon will satisfy these conditions with probability 1, so the instances in which it is not the case are very rare. Polynomials satisfying these transversality conditions are called *Newton nondegenerate*.

Next, we compute the pullback $\beta^*|dxdy|$ of the *area density* $|dxdy|$. Observe that if we are given a monomial map

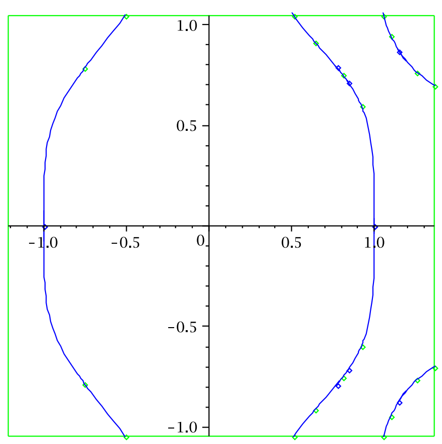
$$(v_1, v_2) \overset{\mu}{\longmapsto} (u_1, u_2) = (v_1^{m_1}v_2^{m_2}, v_1^{n_1}v_2^{n_2}), \quad \det\begin{bmatrix} m_1 & m_2 \\ n_1 & n_2 \end{bmatrix} = \pm 1,$$



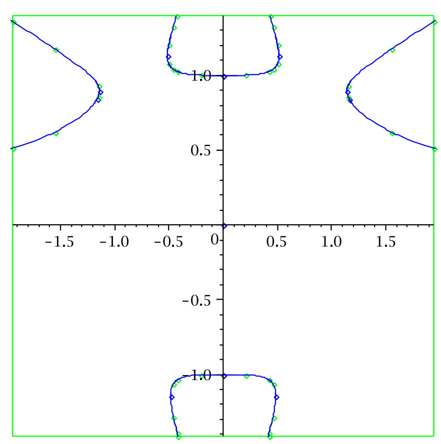
$\phi_1 = \beta_1 \circ \phi$



$\phi_2 = \beta_2 \circ \phi$



$\phi_3 = \beta_3 \circ \phi$



$\phi_4 = \beta_4 \circ \phi$

FIGURE 8. The total transform $\phi \circ \beta = 0$.

then

$$\mu^*|du_1du_2| = |m_1n_2 - n_1m_2| \cdot |v_1^{m_1+n_1-1}v_2^{m_2+n_2-1}dv_1dv_2| = \left| v_1^{m_1+n_1-1}v_2^{m_2+n_2-1}dv_1dv_2 \right|.$$

Hence we have that

$$\beta^*|dxdy||_{U_1} = |y_1^3||dx_1dy_1|, \quad \beta^*|dxdy||_{U_2} = |x_2^3y_2^2||dx_2dy_2|$$
$$\beta^*|dxdy||_{U_3} = |x_3^2y_3||dx_3dy_3|, \quad \beta^*|dxdy||_{U_4} = |x_4||dx_4dy_4|.$$

Now choose a partition of unity on the toric surface $S(\mathcal{F})$ subordinated to the open cover $U_1, \dots, U_4$. In other words we choose four smooth functions $\eta_j \in C^\infty(S)$, $j = 1, \dots, 4$ such that

$$\operatorname{supp} \eta_j \subset U_j, \ \sum_j \eta_j = 1.$$

Then

$$\int_{\mathbb{R}^2} e^{i\xi\phi}\rho|dxdy| = \int_S e^{i\xi\beta^*\phi}\beta^*\rho\beta^*|dxdy| = \sum_{j=1}^4 \int_{U_j} e^{i\xi\phi_j}\rho_j w_j|dx_jdy_j|$$

where

$$\phi_j = \eta_j\beta^*\phi, \quad \rho_j = \eta_j\rho, \quad w_j|dx_jdy_j| = \beta^*|dxdy||_{U_j}.$$

The above computations shows that the weights w_j are monomials. More precisely

$$w_1(x_1, y_1) = |y_1|^3, \quad w_2(x_2, y_2) = |x_2^3y_2^2|, \quad w_3(x_3, y_3) = |x_3^2y_3|, \quad w_4(x_4, y_4) = |x_4|.$$

Now it should be clear that the asymptotics of the integral pieces

$$I_j(\xi) := \int_{U_j} e^{i\xi\phi_j}\rho_j w_j|dx_jdy_j|$$

can be obtained using Corollary 3.2. To see how this is done, we will work with one specific portion. Let us analyze the asymptotics of $I_2(\xi)$. We start by introducing some notation.

By design, every ϕ_i is of the form ${x_i}^m{y_i}^n\widehat{\phi}_i(x_i, y_i)$. For example,

$$\phi_2 = x_2^{10}y_2^8\,\widehat{\phi}_2 \ \text{ where } \ \widehat{\phi}_2(x_2, y_2) = \left(A_{6,0}\,{x_2}^8{y_2}^4 + A_{5,7}\,{x_2}^{12}{y_2}^9 + A_{2,4} + A_{0,10}\,{y_2}^2\right).$$

Using this notation we write

$$I_2(\xi) = \int_{U_2} e^{i\xi\phi_2}\rho_2 w_2|dx_2dy_2| = \int_{U_2} e^{i\xi\,u^{10}v^8\,\widehat{\phi}_2}u^3v^2\rho_2\,du\,dv,$$

where for convenience we have renamed (x_2, y_2) as (u, v). Note that this is almost exactly the form discussed at the end of Section 3.1.

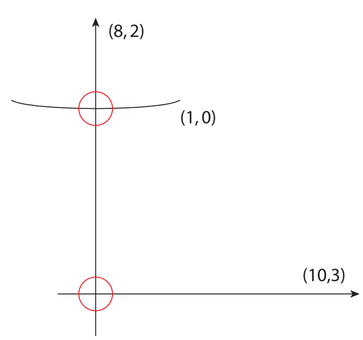


FIGURE 9. The zero set of ϕ_2 with labelled degeneracies.

A closer inspection of the zero set of ϕ_2 in U_2, also called the *total transform* of the curve $\{\phi = 0\}$, will help to illuminate our course of action, so let us examine Figure 9. As mentioned before, the important element in this picture is the exceptional divisor, in this case consisting of the axes.The level set $\{\widehat{\phi}_2 = 0\}$, which is the curvilinear arc in Figure 9, is called the *strict transform* of the curve $\{\phi = 0\}$.

The first of the two numbers on each arc in Figure 9 is the order to which the pulled back phase $\phi_2 = \phi \circ \beta_2$ vanishes along that arc. Along the strict transform curve $\widehat{\phi}_2 = 0$ we write 1 because $\widehat{\phi}_2$ vanishes to order 1: $\widehat{\phi}_2(0) = 0$ and $d\widehat{\phi}_2 \neq 0$, away from the intersection of the curve $\{\widehat{\phi}_2 = 0\}$ with the coordinate axes.

The second of the two numbers on each arc is the order of vanishing along that arc of the jacobian of the blowdown map β.

Each component of $\{\phi_2 = 0\}$ contributes terms to the asymptotic approximation which are linear combinations of powers of ξ to powers in the arithmetic progressions determined by the label of that component. Thus the horizontal axis will contribute terms of the form

$$\sum_{\alpha \in \mathcal{P}_{10,3}} C_\alpha \xi^{-\alpha},$$

where $\mathcal{P}_{10,3}$ is the arithmetic progression introduced at the end of Section 3.1). The circled intersections of the various components of $\{\phi_2 = 0\}$ contribute terms which also contain logarithms. We will look at the specifics at the origin. All other intersection points will contribute in similar fashions.

Let us first look at the origin. In this region we can reasonably absorb the variable $\widehat{\phi}$ into the variable v. While this may seem like a dubious trick, all we are saying is that

$$\left.\frac{\partial v \widehat{\phi}^{1/8}}{\partial v}\right|_{v=0} \neq 0.$$

This leaves us with an overall exponent of $i\xi u^{10} v^8$. Our integral now looks like

$$I_2(\xi) = \int e^{i\xi\, u^{10} v^8} u^3 v^2 \rho_2 \, du \, dv.$$

From Corollary 3.2 we see that there must be a contribution to the asymptotic expansion in this region of the form

$$\sum_{\alpha \in \mathcal{P}_{10,3}} u_\alpha(\rho_2)|\xi|^{-\alpha} + \sum_{\beta \in \mathcal{P}_{8,2}} v_\beta(\rho_2)|\xi|^{-\beta} + \sum_{\gamma \in \mathcal{P}_{10,3} \cap \mathcal{P}_{8,2}} w_\gamma(\rho_2)|\xi|^{-\gamma} \log|\xi|.$$

The set $\mathcal{P}_{10,3} \cap \mathcal{P}_{8,2}$ is also an arithmetic progression and its determination requires a bit of elementary number theory.

Observe first that $\operatorname{lcm}(8,10) = 40$. Then the ratio of $\mathcal{P}_{8,2}$ is $\frac{5}{40}$ and the ratio of $\mathcal{P}_{10,3}$ is $\frac{4}{40}$. The least common multiple of the numerators 4 and 5 is 20 and we deduce that the ratio of the progression $\mathcal{P}_{8,2} \cap \mathcal{P}_{10,3}$ is $\frac{20}{40} = \frac{1}{2}$. The smallest element of this progression is the smallest rational number q which admits a twofold description

$$\frac{3}{8} + \frac{k}{8} = q = \frac{4}{10} + \frac{\ell}{10}, \quad k, \ell \in \mathbb{Z}_{\geq 0}.$$

We deduce that we must have

$$15 + 5k = 16 + 4\ell \iff 5k - 4\ell = 1$$

which leads to the equality

$$k = \ell = 1.$$

Thus the first term of $\mathcal{P}_{10,3} \cap \mathcal{P}_{8,2}$ is $\frac{1}{2}$ which shows that

$$\mathcal{P}_{10,3} \cap \mathcal{P}_{8,2} = \mathcal{P}_{2,0}.$$

By similar arguments, we expect that the contribution due to the point $\{\widehat{\phi}_2 = 0\} \cap \{x - 0\}$ on the y-axis will be of the form

$$\sum_{\alpha \in \mathcal{P}_{1,0}} u_\alpha(\rho_2)|\xi|^{-\alpha} + \sum_{\beta \in \mathcal{P}_{8,2}} v_\beta(\rho_2)|\xi|^{-\beta} + \sum_{\gamma \in \mathcal{P}_{1,0}} w_\gamma(\rho_2)|\xi|^{-\gamma} \log|\xi|.$$

What we are seeing develop here is a general fact of this construction. For situations such as that presented in Figure 9, we will obtain easily recognizable sequences of numbers for our powers of ξ. For example, the asymptotic expansion of our integral in U_2 contributes terms which contain ξ raised to the following powers:

$$(10,3) \longrightarrow -\mathcal{P}_{10,3} = \left[-\frac{4}{10}, -\frac{5}{10}, -\frac{6}{10}, \ldots\right], \quad (8,2) \longrightarrow -\mathcal{P}_{8,2} = \left[-\frac{3}{8}, -\frac{4}{8}, -\frac{5}{8}, \ldots\right]$$
$$(1,0) \longrightarrow -\mathcal{P}_{1,0} = [-1, -2, -3, \ldots]$$

Of course, there will be a considerable amount of overlap between these sets, but the point remains: these are all the powers of $|\xi|$ that will appear in the U_2 contribution to the asymptotic expansion of our integral.

More generally, inspection of the other 3 phases reveals that each region contributes terms to the asymptotic approximation on the order of the positive integers as well as

$$U_1 \to -\mathcal{P}_{10,3} \cup -\mathcal{P}_{1,0}, \quad U_2 \to -\mathcal{P}_{10,3} \cup -\mathcal{P}_{8,2} \cup -\mathcal{P}_{1,0}$$
$$U_3 \to -\mathcal{P}_{8,2} \cup -\mathcal{P}_{6,1} \cup -\mathcal{P}_{1,0}, \quad U_4 \to \quad -\mathcal{P}_{6,1} \cup -\mathcal{P}_{1,0}.$$

We would like to make one final observation about this special construction. The most practical application of this theory is in examining various limits in situations of physical limit (particularly classical and high energy limits in quantum field theory), and in that case it is most useful to know the leading power of ξ.

In the case described above, the first possible (largest) exponent is $-1/3 \in -\mathcal{P}_{6,1}$, and so to first order the integral will go asymptotically as $\xi^{-1/3}$. As it turns out, there is a rather quick way of calculating this value directly from the Newton polygon. If we draw the line $y = x$ on top of the Newton polygon, it will intersect the perimeter in only one point. Denote this point t_0 (as in Figure 10). Then the leading power of ξ will be exactly $-1/t_0$. For our phase ϕ, the value of t_0 must lie on the line connecting the points (2,4) and (6,0)—*i.e.* on the line $y = 6 - x$. Clearly $t_0 = 3$, and as expected, the leading power of ξ will be $\xi^{-1/3}$. Observe that the exponent $-1/3$ appears in the expansions of the integrals over U_2, U_3 and U_4. They correspond to the cones C_2, C_3, C_4 of our fan, precisely the cones related to the edge of the Newton polygon intersected by the line $y = x$..

4. Conclusion

In conclusion, we have seen that the method of evaluating an integral of the form

$$I_\xi = \int e^{i\phi(x_1,\ldots,x_n)\xi} a(x_1,\ldots,x_n)\, dx_1 \cdots dx_n$$

(in the limit as $\xi \to \infty$, can really be looked at as an algorithmic process, each level being reduced to the previous until we have a sum of oscillatory integrals in one dimension. The various levels of approximation depend at their core on the complexity of the phase function ϕ, as this is the component which truly determines the behavior of our integral for large ξ.

Lastly, it is instructive to realize just exactly how useful integrals of this type can be. Mathematically there are an infinite number of situations that can be contrived in which integrals of this form would need to be calculated, but we hope that this topic can actually

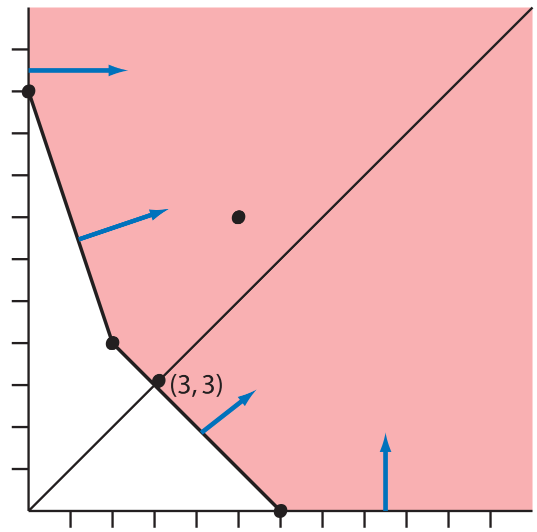


Figure 10. The Newton polygon of ϕ with the line $y = x$.

be useful in understanding the world around us. Fortunately, the topic of oscillatory integrals was not just a stab in the dark—in fact, the topic could hardly be more applicable to physics. Particularly, the path-integral formalism of modern quantum field theory relies almost exclusively on the theory of the asymptotic approximation of oscillatory integrals.

One of the major theoretical breakthroughs of the past century was the realization that the weak and strong interactions in the standard model could be described (via proper renormalization) in the path-integral formalism in the same way which electromagnetism was described in quantum chromodynamics. The asymptotic approximation provides the mathematical framework of the entire discussion of renormalization, and is therefore absolutely essential to our current understanding of the standard model and quantum field theory. However, this is not the full usefulness of this formula. Many Green's functions have essentially the same form as our general oscillatory integral, and therefore most applications of Green's functions will in some way need rely on asymptotic approximations for their validity.

In any case, one should appreciate the tremendous intricacy that is involved in discussing asymptotic approximation in even two dimensions, let alone the multiple dimensions called for by modern quantum field theory. Furthermore, it is quite frankly stunning how some notions of geometry long thought to be primarily abstract can have immediate application to new situations of physical interest. Truly, this should be seen as a subtle reminder of the vast and complex interplay between mathematics and physics which drives both subjects to a deeper and more profound understanding of the universe around us.

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