ON THE SPACE OF FREDHOLM OPERATORS

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Abstract. We compare various topologies on the space of (possibly unbounded) Fredholm selfadjoint operators and explain their K-theoretic relevance.

In memoria prietenului meu Gheorghe Ionesei, Prinţu'.

Introduction

The work of Atiyah and Singer on the index of elliptic operators on manifolds has singled out the role of the space of bounded Fredholm operators in topology. It is a classifying space for a very useful functor, the topological K-theory. This means that a continuous family $(L_x)_{x\in X}$ of elliptic pseudodifferential operators of order zero parameterized by a compact CW-complex X naturally defines an element in the group $K(X)$, the index of the family.

In most examples arising in concrete geometric situations, the elliptic operators are *partial differential operators*, and thus they are naturally unbounded. The notion of continuity has to be defined carefully.

The operator theorists have come up with a quick fix. The family $x \mapsto L_x$ of (possibly unbounded) Fredholm operators is called *Riesz continuous* if and only if the families of *bounded operators*

$$x \mapsto L_x(1 + L_x^*L_x)^{-1/2}, \;\; x \mapsto L_x^*(1 + L_xL_x^*)^{-1/2}$$

are continuous with respect to the operator norm. In concrete applications this approach can be a nuisance.

For example, consider as in [9] a *Floer family* of *elliptic* boundary value problems, parameterized by $z \in \mathbb{C}$, $|z| = 1$,

$$F_z : \mathrm{Dom}(F_s) \subset L^2([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C}),$$

$$\begin{aligned} \mathrm{Dom}(F_z) &= \Big\{ u \in L^2([0,1],\mathbb{C});\ \ \frac{du}{dt} \in L^2([0,1],\mathbb{C}),\ \ u(0)\in\mathbb{R},\ \ zu(1)\in\mathbb{R} \Big\} \\ F_z u &= \frac{du}{dt} + au. \end{aligned} \qquad (BV_z)$$

Above, Dom denotes the domain of an (unbounded) operator, and $a : [0,1] \to \mathbb{R}$ is a smooth function.

This family ought to be considered continuous, but using the above definition can be quite demanding. The first technical goal of this paper is to elucidate this continuity issue.

As observed in [1, 5], for K-theoretic purposes it suffices to investigate only (possibly $\mathbb{Z}_2$-graded) selfadjoint operators (super-)commuting with some Clifford algebra action.

For example, the space of Fredholm operators on a Hilbert space H can be identified with the space of odd, selfadjoint Fredholm operators on the $\mathbb{Z}_2$-graded space $H \oplus H$ via the correspondence

$$L \mapsto \begin{bmatrix} 0 & L^* \\ L & 0 \end{bmatrix}.$$

That is why we will focus exclusively on selfadjoint operators.

In [9] we have argued that in many instances it is much more convenient to look at the graphs of Fredholm selfadjoint operators on a Hilbert space H. If T is such an operator and $\Gamma_T \subset H \oplus H$ is its graph, then Γ_T is a Lagrangian subspace of $H \oplus H$ (with respect to a natural symplectic structure) and moreover, the pair $(H \oplus 0, \Gamma_T)$ is Fredholm. As shown in [7], the space of Fredholm pairs of Lagrangian subspaces is a classifying space for KO^1. (A similar description is valid for all the functors KO^n; see [9].)

A natural question arises. Suppose that two *families of subspaces* determined by the graphs of two families of Fredholm operators are homotopic inside the larger space of Fredholm pairs of Lagrangian subspaces. Can we conclude that the corresponding families of Fredholm *operators* are also homotopic inside the smaller space of operators?

The is the second issue we want to address in this paper. We will consider various topologies on the space of closed, *unbounded* Fredholm operators, and analyze when the above graph map $T \mapsto \Gamma_T$ from operators to subspaces is a *homotopy equivalence.* Surprisingly, to answer this question we only need to decide the continuity of Floer type families of boundary value problems. The symplectic reduction technique developed in [9] coupled with the Bott periodicity will take care of the rest.

The paper consists of three sections. In Section 1 we analyze two topologies on the space of unbounded Fredholm operators: the *gap topology*, given by the gap distance between the graphs, and the *Riesz topology*, described above. We give some simple practical criteria for convergence in these topologies. We have included a simple example of B. Fuglede showing that the gap topology is strictly weaker than the Riesz topology.

In the second section we prove a general criterion (Proposition 2.1) for recognizing when a family of first order, elliptic boundary value problems, such as (BV_z), is continuous with respect to the *Riesz topology.*

In the last section we prove (Proposition 3.1) that a certain "component" $\mathcal{F}_0$ of the space of closed, Fredholm selfadjoint operators equipped with the *Riesz topology* is a classifying space for the functor KO^1. Although we do not address it in this paper, similar descriptions exist for all the functors KO^n. Moreover, using the symplectic techniques of [9] we prove (Theorem 3.3) that the map which associates to an operator $T \in \mathcal{F}_0$ its graph induces a weak homotopy equivalence between the space $\mathcal{F}_0$ equipped with the Riesz topology, and the space of Fredholm pairs of Lagrangian subspaces in an infinite dimensional symplectic Hilbert. Here the Riesz continuity of the Floer families and the Bott periodicity play a crucial role.

This (weak) homotopy equivalence is extremely useful in applications since the space of lagrangians is much larger, and thus offers more freedom in constructing homotopies of families of operators.

In a recent article, B. Booos-Bavnbeck, M. Lesch and J. Phillips [2] investigate the space $\mathcal{F}_0$ equipped the *gap topology*, and they explicitly construct a surjection $\pi_1(\mathcal{F}_0) \to \mathbb{Z}$, the so called spectral flow. Our results show that if we equip $\mathcal{F}_0$ with the Riesz topology, then the above morphism is an isomorphism, because $\pi_1(\mathcal{F}_0) \cong \mathbb{Z}$, and any surjection $\mathbb{Z} \to \mathbb{Z}$ must be an isomorphism.

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1. Topologies on the space of selfadjoint operators

Let H be a separable complex * Hilbert space. Denote by $\mathcal{S}$ the space of densely defined, *selfadjoint* operators on H, and by $\mathcal{BS}$ the space of *bounded selfadjoint* operators $H \to H$. Set

$$[\mathcal{BS}] := \{ T \in \mathcal{BS};\ \ \|T\| < 1 \}.$$

The *Riesz map* is the injection

$$\Psi : \mathcal{S} \to \mathcal{BS},\ \ A \mapsto A(1 + A^2)^{-1/2}.$$

As explained in [2], its image consists of operators S of norm ≤ 1 such that $S \pm \mathbb{1}$ are injective. There are two natural metrics on $\mathcal{S}$.

- The *gap metric*

$$\gamma(A_0, A_1) := \|(\boldsymbol{i} + A_0)^{-1} - (\boldsymbol{i} + A_1)^{-1}\| + \|(\boldsymbol{i} - A_0)^{-1} - (\boldsymbol{i} - A_1)^{-1}\|,\ \ \boldsymbol{i} := \sqrt{-1}.$$

- The *Riesz metric*

$$\rho(A_0, A_1) := \|\Psi(A_0) - \Psi(A_1)\|.$$

Remark 1.1. According to [6, Thm. IV.2.23] we have

$$\gamma(A_n, A) \to 0 \Longleftrightarrow \delta(\Gamma_{A_n}, \Gamma_A) \to 0,$$

where Γ_T denotes the graph of the linear operator T, and δ denotes the gap between two closed subspaces, [6, IV§2]. □

Lemma 1.2. *The identity map* $(\mathcal{S}, \rho) \to (\mathcal{S}, \gamma)$ *is continuous.*

Proof. Observe that for every $A \in \mathcal{S}$ we have

$$\frac{1}{\boldsymbol{i} \pm A} = \frac{A \mp \boldsymbol{i}}{1 + A^2} = \frac{A}{1 + A^2} \mp \frac{1}{1 + A^2} = \frac{1}{(1 + A^2)^{1/2}} \Psi(A) \mp \boldsymbol{i} \frac{1}{1 + A^2}$$

and

$$\frac{1}{1 + A^2} = 1 - \Psi(A)^2$$

so that $\|\Psi(A_n) - \Psi(A)\| \to 0$ implies $\|(\boldsymbol{i} \pm A_n)^{-1} - (\boldsymbol{i} \pm A)^{-1}\| \to 0$. □

Denote by $\mathcal{A}$ the C^*-algebra of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that the limits

$$f(\pm\infty) := \lim_{\lambda \to \pm\infty} f(\lambda) \in \mathbb{C}$$

exist. Denote by $\mathcal{A}_0$ the subalgebra of $\mathcal{A}$ defined by the condition

$$f \in \mathcal{A}_0 \Longleftrightarrow f(-\infty) = f(\infty).$$

Define $P_0, P_\pm \in \mathcal{A}_0$ by

$$P_0(\lambda) \equiv 1,\ \ P_\pm(\lambda) = (\lambda \pm \boldsymbol{i})^{-1}.$$

The Stone-Weierstrass approximation theorem shows that the algebra $\mathcal{P}$ generated by $P_0, P_\pm$ is dense in $\mathcal{A}_0$.

*To deal with real operators it suffices to complexify them.

The functional calculus for selfadjoint operators show that any $A \in \mathcal{S}$ defines a continuous morphism of C^*-algebras

$$\mathcal{A} \to \mathcal{BS}, \quad f \mapsto f(A).$$

Proposition 1.3. *The following statements are equivalent.*
(i) $\gamma(A_n, A) \to 0$.
(ii) $\|f(A_n) - f(A)\| \to 0$, $\forall f \in \mathcal{A}_0$.

Proof. Clearly (ii) $\Longrightarrow$ (i) since $P_\pm \in \mathcal{A}_0$ and

$$\gamma(A_n, A) = \|P_-(A_n) - P_-(A)\| + \|P_+(A_n) - P_+(A)\|.$$

To prove (i) $\Longrightarrow$ (ii) we use an idea in [10, Chap. VIII]. Clearly if $\gamma(A_n, A) \to 0$ then

$$\|P(A_n) - P(A)\| \to 0, \quad \forall P \in \mathcal{P}.$$

Fix $f \in \mathcal{A}_0$. Since $\mathcal{P}$ is dense in $\mathcal{A}_0$, for every $\varepsilon > 0$ we can find $P \in \mathcal{P}$ such that $\|f - P\| \leq \varepsilon/3$ and then $n(\varepsilon) > 0$ such that, $\forall n \geq n(\varepsilon)$ such that

$$\|P(A_n) - P(A)\| \leq \varepsilon/3.$$

Then, $\forall n \geq n(\varepsilon)$ we have

$$\|f(A_n) - f(A)\| \leq \|f(A_n) - P(A_n)\| + \|P(A_n) - P(A)\| + \|P(A) - f(A)\| \leq \varepsilon. \qquad \square$$

Proposition 1.4. *Fix a function* $\alpha \in \mathcal{A}$ *such that* $\alpha(\lambda) \equiv 1$ *for* $\lambda \gg 1$ *and* $\alpha(\lambda) \equiv 0$ *if* $\lambda \ll -1$. *Then the following statements are equivalent.*
(i) $\rho(A_n, A) \to 0$
(ii) $\|f(A_n) - f(A)\| \to 0$, $\forall f \in \mathcal{A}$.
(iii) $\gamma(A_n, A) \to 0$ *and* $\|\alpha(A_n) - \alpha(A)\| \to 0$.

Proof. Define $r \in \mathcal{A}$ by

$$r(\lambda) := \frac{\lambda}{(1+\lambda^2)^{1/2}}.$$

The equivalence (i) $\Longleftrightarrow$ (ii) follows exactly as in the proof of Proposition 1.3 using Lemma 1.2, and the fact that the subalgebra spanned by $\mathcal{A}_0$, the constant function 1, and r is dense in A.

The equivalence (ii) $\Longleftrightarrow$ (iii) relies on Proposition 1.3, and the fact that the algebra spanned by $\mathcal{A}_0$, the constant function 1, and α is dense in $\mathcal{A}$. $\square$

From the above results we deduce the following consequence.

Corollary 1.5. *The identity maps*

$$(\mathcal{BS}, \|\bullet\|) \to (\mathcal{BS}, \rho), \quad (\mathcal{BS}, \|\bullet\|)) \to (\mathcal{BS}, \gamma),$$

are are continuous, where $(\mathcal{BF}, \|\bullet\|)$ *denotes the space* $\mathcal{BF}$ *equipped with the norm topology.* $\square$

Remark 1.6 (B. Fuglede). The topological spaces $(\mathcal{S}, \rho)$ and $(\mathcal{S}, \gamma)$ are not homeomorphic. Using Proposition 1.4 it is easy to construct an example of a sequence $A_n \xrightarrow{\gamma} A$ such that A_n does not converge to A in the Riesz metric. More precisely, consider the space

$$\ell^2 = \left\{ (x_j)_{n\geq 1};\ \ x_j \in \mathbb{R},\ \sum_j x_j^2 < \infty \right\},$$

with canonical Hilbert basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots$. For $n = 1, 2, \dots$ define

$$A_n : D(A_n) \subset \ell^2 \to \ell^2, \;\; D(A_n) = \Big\{ (x_j)_{j\geq 1} \in \ell^2;\; \sum_{j\geq 1} j^2 |x_j|^2 < \infty \Big\}$$

$$A_n \boldsymbol{e}_j = \begin{cases} j\boldsymbol{e}_j, & j \neq n \\ -n\boldsymbol{e}_j, & j = n \end{cases}$$

One can see that

$$\|(\boldsymbol{i} \pm A_n)^{-1} - (\boldsymbol{i} \pm A_0)^{-1}\| = \Big| \frac{1}{\boldsymbol{i} + n} - \frac{1}{\boldsymbol{i} - n} \Big| \to 0$$

so that $\gamma(A_n, A_0) \to 0$. On the other hand, if $\alpha \in \mathcal{A}$ is as in Proposition 1.4 then for all sufficiently large n we have

$$\|\alpha(A_n) - \alpha(A_0)\| = 1.$$

This shows that the gap topology is *strictly weaker* than the Riesz topology. □

We now want to present a simple criterion of ρ-convergence. For any closed densely defined operator we denote by $\mathcal{R}(T) \subset \mathbb{C}$ its resolvent set.

Proposition 1.7. *Suppose $A \in \mathcal{S}$ such that $\mathcal{R}(A) \cap \mathbb{R} \neq \emptyset$. Suppose S_n is a sequence of densely defined symmetric operators satisfying the following conditions.*
(a) $\mathrm{Dom}(A) \subset \mathrm{Dom}(S_n)$.
(b) There exists a sequence of positive numbers $c_n \to 0$ such that

$$\|S_n u\| \leq c_n(\|Au\| + \|u\|), \;\; \forall u \in \mathrm{Dom}(A).$$

Then $A + S_n \in \mathcal{S}$ for all $n \gg 0$ and

$$\rho(A + S_n, A) \to 0.$$

Proof. Set $A_n := A + S_n$. According to [6, Thm.IV.2.24] we have

$$\gamma(A_n, A) \to 0$$

while [6, Thm. V.4.1] implies $A + S_n \in \mathcal{S}$ for all sufficiently large n. Let $\beta \in \mathcal{R}(A) \cap \mathbb{R}$ and consider a small closed interval $I = [\beta - \varepsilon, \beta + \varepsilon]$ such that $I \subset \mathcal{R}(A)$. Then, using [6, Thm. VI.5.10] we deduce that for n sufficiently large we have

$$I \subset \mathcal{R}(A_n), \;\; \forall n \gg 0.$$

Pick now a function $\alpha \in \mathcal{A}$ such that $\alpha(\lambda) \equiv 1$ for $\lambda \geq \beta + \varepsilon$ and $\alpha(\lambda) \equiv 0$ for $\lambda \leq \beta - \varepsilon$. Using [6, Thm. VI.5.12] we deduce

$$\|\alpha(A_n) - \alpha(A)\| \to 0.$$

We can now invoke Proposition 1.4 to conclude that $\rho(A_n, A) \to 0$. □

2. Families of boundary value problems

The terminology involving Dirac operators used in this section is taken from [8, Chap. 10]. Consider now as in [9, App. A] the following data.

- A compact, oriented Riemannian manifold (M, g) with boundary $N = \partial M$ such that a tubular neighborhood of $N \hookrightarrow M$ is *isometric* to the cylinder
$$([0,1] \times N, dt^2 + g_N),$$
where g_N is a Riemann metric on N and t denotes the outgoing longitudinal coordinate.
- An Euclidean bundle of Clifford modules $E \to M$ with Clifford multiplication
$$\boldsymbol{c} : T^*M \to \operatorname{End}(E).$$
($\boldsymbol{c}(\alpha)$ is skew-symmetric for any real 1-form α.) Set $E_0 := E|_N$¿
- $D : C^\infty(E) \to C^\infty(E)$ a symmetric Dirac operator, with principal symbol $\boldsymbol{c}$, such that near N it has the form
$$D = J(\partial_t - D_0), \quad J := \boldsymbol{c}(dt),$$
where $D_0 : C^\infty(E_0) \to C^\infty(E_0)$ is symmetric and independent of t.
- A sequence of symmetric endomorphisms of E independent of t near N such that $\|T_n\|_{C^2} \to 0$, and (near N) the endomorphism JA_n is symmetric. Set $D_n := D + T_n$. Observe that near N D_n has the form
$$D_n := J(\partial_t - D_0 - JT_n).$$

Following [3], we consider the family $\mathcal{P}$ of admissible boundary conditions. It consists of zero order, formally selfadjoint pseudodifferential projectors with the same principal symbol as the Calderon projector of D_0. The symbol of any P in $\mathcal{P}$ commutes with the symbol of D_0, so that the commutator $[P, D_0]$ is a zeroth order pseudodifferential operator. We define a metric ν on $\mathcal{P}$ by setting
$$\nu(P,Q) := \Big\| P - Q \Big\| + \Big\| [P-Q, D_0] \Big\|,$$
where $\| \bullet \|$ denotes the norm on the space of bounded operators $L^2(E_0) \to L^2(E_0)$.

For every $s \in [0, \infty)$ we will denote by $H^s(E)$ (respectively $H^s(E_0)$) the Sobolev space consisting of L^2-sections of E (respectively E_0) such that all their distributional partial derivatives of order $\leq s$ are also in L^2.

If we write $Q = P + S$, where S is a pseudodifferential operator of order ≤ -1, then
$$\|[P-Q, D_0]\| \leq \Big(\|D_0\|_{E_0; H^0, H^{-1}} \cdot \|S\|_{E_0; H^{-1}, H^0} + \|D_0\|_{E_0; H^1, H^0} \cdot \|S\|_{E_0; H^0, H^1} \Big),$$
where we have denoted by $\|T\|_{E_0; H^s, H^r}$ the norm of a *bounded* operator $T : H^s(E_0) \to H^r(E_0)$. We deduce that there exists a constat $C > 0$, depending only on the geometry of M and E, such that
$$\nu(P,Q) \leq C \Big(\|P-Q\|_{E_0; H^{-1}, H^0} + \|P-Q\|_{E_0; H^0, H^1} \Big).$$

Suppose now that we are given a projector $P \in \mathcal{P}$ and a sequence $(P_n) \subset \mathcal{P}$. As in [3], we can form the Fredholm selfadjoint operators
$$A_n : \operatorname{Dom}(A_n) \subset L^2(E) \to L^2(E), \ \operatorname{Dom}(A_n) = \big\{ u \in H^1(E); \ P_n u|_N = 0 \big\}, \ A_n u = D_n u,$$

and

$$A:\operatorname{Dom}(A)\subset L^2(E)\to L^2(E),\ \ \operatorname{Dom}(A)=\big\{\,u\in H^1(E);\ \ Pu\,|_N=0,\big\},\ \ Au=Du.$$

Proposition 2.1. *If*

$$\lim_{n\to\infty}\nu(P_n,P)=0, \tag{2.1}$$

then $\lim_{n\to\infty}\rho(A_n,A)=0$.

Proof. The proof relies on the following technical result.

Lemma 2.2. *There exists a sequence of bounded, invertible operators* $U_n : L^2(E)\to L^2(E)$ *such that*
(i) $1-U_n$ *and* $1-U_n^*$ *define bounded operators* $H^s(E)\to H^s(E)$, $s=0,1$.
(ii) (U_n-1), $(U_n-1)^*\to 0$ *in the norm topology on the space of bounded operators* $H^s(E)\to H^s(E)$, $s=0,1$.
(iii) $\operatorname{Dom}(A_n)=U_n^*\operatorname{Dom}(A)$, $\forall n$.

We will prove this lemma after we have finished the proof of Proposition 2.1. Set

$$B_n:=U_nA_nU_n^*.$$

Observe that $B_n\in\mathcal{S}$, and $\operatorname{Dom}(B_n)=\operatorname{Dom}(A)$. Moreover

$$\rho(B_n,A_n)=\|\Psi(U_nA_nU_n^*)-\Psi(A_n)\|=\|U_n\Psi(A_n)U_n^*-\Psi(A_n)\|$$

$$=\Big\|\big(\,(U_n-1)+1\big)\Psi(A_n)\big(\,(U_n-1)+1\big)^*-\Psi(A_n)\Big\|\le C\|(U_n-1)\|_{L^2,L^2}\cdot\|\Psi(A_n)\|\to 0$$

Thus it suffices to show that

$$\rho(B_n,A)\to 0.$$

Observe that for all $u\in\operatorname{Dom}(A)$ we have

$$\|B_nu-Au\|=\|U_n(D+T_n)U_n^*-D\|\le\|U_nD(U_n^*u-u)\|+\|U_nT_nU_n^*u\|$$

$$\le\|U_n\|_{L^2,L^2}\|D(U_n^*u-u)\|_{L^2}+C\|T_n\|_{C^2}\|u\|_{L^2}\le C\Big(\|(U_n^*-1)u\|_{H^1}+\|T_n\|_{C^2}\|u\|_{L^2}\Big)$$

$$\le C\Big(\|(U_n^*-1)\|_{H^1,H^1}\|u\|_{H^1}+\|T_n\|_{C^2}\|u\|_{L^2}\Big)$$

(use the elliptic estimates in [3])

$$\le C\Big\{\,\|(U_n^*-1)\|_{H^1,H^1}\big(\,\|Au\|_{L^2}+\|u\|_{L^2}\,\big)+\|T_n\|_{C^2}\|u\|_{L^2}\,\Big\}\le c_n(\|Au\|+\|u\|),$$

where $c_n\to 0$. Thus, the operator $S_n=B_n-A$ satisfies all the conditions in Proposition 1.7. On the other hand, A has compact resolvent so that $\mathcal{R}(A)\cap\mathbb{R}\neq\emptyset$. We deduce

$$\rho(A,B_n)=\rho(A,A+S_n)\to 0. \qquad \square$$

Proof of Lemma 2.2 Following the constructions in [6, I.§6.4] define

$$\hat{U}_n:L^2(E_0)\to L^2(E_0)$$

by

$$\hat{U}_n=P_nP+(1-P_n)(1-P)=2P_nP-(P_n+P)+1$$
$$=2(P+R_n)P-(2P+R_n)+1=R_n(2P-1)+1.$$

$\hat{U}_n$ is a *pseudodifferential operator* of order zero with principal symbol 1. Observe that

$$\hat{U}_n^*=PP_n+(1-P)(1-P_n)$$

and, as explained in [6, I.§6.4], $\hat{U}_n^*$ is invertible and maps $\ker P$ onto $\ker P_n$. Observe moreover that

$$\|\hat{U}_n - 1\|_{L^2,L^2} \leq \|R_n\|_{L^2,L^2}\|(2P-1)\|_{L^2,L^2} \to 0. \tag{2.2}$$

Next, observe that

$$[D_0, \hat{U}_n] = [D_0, R_n](2P-1) + 2R_n[D_0, P]$$

defines a bounded operator $L^2(E_0) \to L^2(E_0)$ and, using (2.1) we deduce

$$\big\| \, [D_0, \hat{U}_n] \, \big\|_{L^2,L^2} \to 0. \tag{2.3}$$

Observe that $\hat{U}_n$ defines in an obvious fashion a bounded operator

$$\hat{U}_n : L^2(E \mid_{[0,1]\times N}) \to L^2(E \mid_{[0,1]\times N})$$

Consider now a smooth increasing function

$$\eta : [0,1] \to [0,1]$$

such that $\eta(t) \equiv 0$ for $t < 1/4$ and $\eta(t) \equiv 1$ for $t > 3/4$. We can regard η as a function on the tubular neighborhood of $N \hookrightarrow M$ and then extending it by 0 we can regard it as a smooth function on M. Notice that if u is a section of E then we can regard ηu as a section of $E \mid_{[0,1]\times N}$.

For any section of E smooth *up to the boundary* define

$$U_n u = (1-\eta)u + \hat{U}_n(\eta u).$$

It is clear that $U_n u$ is smooth up to the boundary. Notice also that there exists a constant $C > 0$ independent of n such that

$$\|U_n u\|_L^2 \leq C\|u\|_{L^2}$$

for any section u smooth up to the boundary. Thus U_n extends to a bounded operator $L^2(E) \to L^2(E)$. Using (2.2) we deduce that

$$\|(U_n - 1)\|_{L^2,L^2} \to 0.$$

We want to show that U_n induces a bounded operator $H^1(E) \to H^1(E)$ and then estimate the norm of $(U_n - 1)$ as a bounded operator $H^1 \to H^1$.

First of all observe that the elliptic estimates for D_0 imply that there exists a positive constant C such that if u is smooth up to the boundary then

$$C^{-1}\|u\|_{H^1([0,1]\times N)} \leq \|\partial_t u\|_{L^2([0,1]\times N)} + \|D_0 u\|_{L^2([0,1]\times N)} \leq C\|u\|_{H^1([0,1]\times N)}$$

Observe that for any section u smooth up to the boundary we have

$$\begin{aligned} \|U_n u - u\|_{H^1(M)} &= \|(1-\eta)u + \hat{U}_n(\eta u) - u\|_{H^1(M)} \\ &= \|\hat{U}_n(\eta u) - \eta u\|_{H^1(M)} = \|\hat{U}_n(\eta u) - (\eta u)\|_{H^1([0,1]\times N)} \\ &\leq C\Big(\|\hat{U}_n(\eta u) - (\eta u)\|_{L^2([0,1]\times N)} + \|\partial_t \hat{U}_n(\eta u) - \partial_t(\eta u)\|_{L^2([0,1]\times N)} \\ &\qquad + \|D_0 \hat{U}_n(\eta u) - D_0(\eta u)\|_{L^2([0,1]\times N)}\Big) \end{aligned} \tag{2.4}$$

Using (2.2) we deduce

$$\|\hat{U}_n(\eta u) - (\eta u)\|_{L^2([0,1]\times N)} \leq c_n\|u\|_{L^2(M)}, \;\; c_n \to 0.$$

To estimate the second term in (2.4) notice first that $[\partial_t, \hat{U}_n] = 0$ so that we have

$$\|\partial_t \hat{U}_n(\eta u) - \partial_t(\eta u)\|_{L^2([0,1]\times N)} = \|\hat{U}_n \partial_t(\eta u) - \partial_t(\eta u)\|_{L^2([0,1]\times N)}$$

$$\leq c_n \|\partial_t u\|_{L^2([0,1]\times N)} \leq c_n \|u\|_{H^1(M)}, \;\; c_n \to 0.$$

The estimate of the third term in (2.4) requires a bit more work. Observe that

$$D_0 \hat{U}_n(\eta u) - D_0(\eta u) = [D_0, \hat{U}_n](\eta u) + \hat{U}_n(D_0 \eta u) - D_0(\eta u)$$

$$= \eta \Big([D_0, \hat{U}_n]u + \hat{U}_n(D_0 u) - D_0 u \Big)$$

so that

$$\|D_0 \hat{U}_n(\eta u) - D_0(\eta u)\|_{L^2([0,1]\times N)} \leq \| \, [D_0, \hat{U}_n]u\|_{L^2([0,1]\times N)} + \|\hat{U}_n(D_0 u) - D_0 u\|_{L^2([0,1]\times N)}$$

(use (2.2))

$$\leq c_n (\|u\|_{L^2([0,1]\times N)} + \|D_0 u\|_{L^2([0,1]\times N)}) \leq c_n' \|u\|_{H^1(M)}, \;\; c_n' \to 0.$$

We have thus found a sequence of positive numbers $c_n \to 0$ such that

$$\|U_n u - u\|_{H^1(M)} \leq c_n \|u\|_{H^1(M)}$$

for every section u smooth up to the boundary. This shows that U_n induces a bounded operator $H^1(M) \to H^1(M)$ and moreover,

$$\|U_n - 1\|_{H^1, H^1} \leq c_n \to 0.$$

One can prove a similar statement concerning U_n^*. Clearly U_n is invertible being so close to 1. Since $\ker P_n = \hat{U}_n^*(\ker P)$ we deduce that $\mathrm{Dom}(A_n) = U_n^* \,\mathrm{Dom}(A)$. Lemma 2.2 is proved. □

3. Classifying spaces for K-theory

For the sake of clarity, we will consider only a special case, that of the functor KO^1. To discuss the other functors KO^n one should use the bigraded Karoubi functors $KO^{p,q}$ as we did in [9]. The proof is only notationally more complicate.

We will use the following notation.

- $\mathcal{F} \subset \mathcal{S}$ is the subspace of unbounded Fredholm selfadjoint operators.
- $\mathcal{BF} \subset \mathcal{BS}$ is the subspace of bounded, Fredholm selfadjoint operators.

The space $\mathcal{BF}$ has three connected components. Two of them $\mathcal{BF}_\pm$, are contractible while the third, $\mathcal{BF}_0$ is a classifying space for KO^1 (see [1, 3, 5]) and consists of bounded Fredholm selfadjoint operators such that the essential spectrum contains both positive and negative elements. We set

$$\mathcal{F}_0 := \Psi^{-1}(\mathcal{BF}_0)$$

Note that $\mathcal{BF}_0 \subset \mathcal{F}_0$. Set

$$[\mathcal{BF}_0] = \big\{ \, T \in \mathcal{BF}_0; \;\; \|T\| \leq 1 \, \big\}.$$

Proposition 3.1. *The inclusion map* $([\mathcal{BF}_0], \|\bullet\|) \hookrightarrow (\mathcal{F}_0, \rho)$ *is a homotopy equivalence, so that* $(\mathcal{F}_0, \rho)$ *is a classifying space for* KO^1*.*

Proof. We follow the strategy in the proof of [9, Prop. 5.1]. Corollary 1.5 shows that the identity map $(\mathcal{BF}, \|\bullet\|) \to (\mathcal{BF}, \rho)$ is a homeomorphism. In particular, the map

$$(\mathcal{BF}_0, \|\bullet\|) \hookrightarrow (\mathcal{F}_0, \rho)$$

is continuous.

For $s \in (0,1]$ define $w_s : \mathbb{R} \to \mathbb{R}$ by setting

$$w_s(\lambda) = \begin{cases} \lambda & |\lambda| \leq s^{-1} \\ s^{-1} & \lambda \geq s^{-1} \\ -s^{-1} & \lambda \leq -s^{-1} \end{cases}$$

Also, we set

$$w_0 : \mathbb{R} \to \mathbb{R}, \;\; w_0(\lambda) = \lambda, \;\; \forall \lambda.$$

Observe that the resulting map

$$W : [0,1] \times \mathbb{R} \to \mathbb{R}, \;\; (s,\lambda) \mapsto w_s(\lambda)$$

is continuous. Define

$$\Phi_s : \mathcal{F} \to \mathcal{F}, \;\; A \mapsto w_s(A).$$

Using Proposition 1.4 we deduce that the map $(\mathcal{F}, \rho) \to (\mathcal{BF}, \|\bullet\|)$, $A \mapsto \Phi_1(A)$, is continuous, and defines a retraction of $\mathcal{F}$ onto the subspace $[\mathcal{BF}]$ of bounded Fredholm, selfadjoint operators of norm ≤ 1, and a retraction of $\mathcal{F}_0$ onto $[\mathcal{BF}_0]$.

We will prove that Φ_1 is a homotopy inverse for i. We already know that $\Phi_1 \circ i = \mathbb{1}_{[\mathcal{BF}_0]}$, and we want to prove that $i \circ \Phi_1$ is homotopic to $\mathbb{1}_{\mathcal{F}_0} = \Phi_0$. This will be the case if we show that the map

$$\Phi : [0,1] \times (\mathcal{F}, \rho) \to (\mathcal{F}, \rho), \;\; (s, A) \mapsto \Phi_s(A)$$

is continuous.

The only problematic issue arises at $s = 0$. Let $A \in \mathcal{F}$. Suppose we have sequences $B_n \in \mathcal{F}$ and $s_n \in (0,1]$ such that

$$\lim_{n\to\infty} s_n = 0 = \lim_{n\to\infty} \rho(A, B_n) = 0.$$

We have

$$\rho(A, \Phi_{s_n}(B_n)\,) \leq \rho(A, B_n) + \rho(B_n, \Phi_{s_n}(B_n))$$

so it suffices to prove that

$$\lim_n \rho(B_n, \Phi_{s_n}(B_n)) = 0.$$

We have

$$\rho(B_n, \Phi_{s_n}(B_n)) = \|r(B_n) - r \circ w_{s_n}(B_n)\|, \;\; r(\lambda) = \lambda(1+\lambda^2)^{-1/2}.$$

We set

$$T_n := r(B_n), \;\; T := r(A).$$

From [2, Prop. 1.5] we deduce that $B_n = r^{-1}(T_n)$, where $r^{-1} : (-1,1) \to \mathbb{R}$ is given by $t \mapsto t(1-t^2)^{-1/2}$. Arguing as in the proof of [4, Thm. XII.2.9(c)] we deduce that for every $A \in \mathcal{S}$ we have

$$r \circ w_s(A) = r \circ w_s \circ r^{-1}(\,r(A)\,), \;\; \forall s \in (0,1].$$

Hence

$$r(B_n) - r \circ w_s(B_n) = T_n - r \circ w_s \circ r^{-1}(T_n).$$

Note that

$$u_s(t) := r \circ w_s \circ r^{-1}(t) = \begin{cases} t & |t| \leq (1+s^2)^{-1/2} \\ (1+s^2)^{-1/2} & t > (1+s^2))^{-1/2} \\ -(1+s^2)^{-1/2} & t < -(1+s^2)^{-1/2}. \end{cases}$$

As explained in [2], the image of the Riesz map $\Psi : \mathcal{F} \to \mathcal{BF}$ consists of operators $S \in \mathcal{BF}$ of norm $\|S\| \leq 1$ such that $S \pm \mathbb{1}$ is injective.

We need to prove that if $T_n \to T$ in $\Psi(\mathcal{F})$, then

$$\lim_n \|T_n - u_{s_n}(T_n)\| = 0.$$

Observe that

$$t - u_s(t) = \begin{cases} 0 & |t| \leq (1+s^2)^{-1/2} \\ t - (1+s^2)^{-1/2} & t > (1+s^2))^{-1/2} \\ t + (1+s^2)^{-1/2} & t < -(1+s^2)^{-1/2}. \end{cases}$$

so that

$$\sup_{|t|\leq 1} |t - u_s(t)| \leq 1 - (1+s^2)^{-1/2} = \frac{(1+s^2)^{1/2} - 1}{(1+s^2)^{1/2}} = \frac{s^2}{((1+s^2)^{1/2} - 1)(1+s^2)^{1/2}} \leq \frac{s^2}{2}.$$

Hence

$$\|T_n - u_{s_n}(T_n)\| \leq \frac{s_n^2}{2} \to 0 \text{ as } s_n \to 0. \qquad \square$$

Remark 3.2. In the proof of [9, Prop. 5.1] we erroneously claimed that the map

$$\mathcal{F} \ni B \mapsto w_s(B) \in (\mathcal{BF}, \|\bullet\|),$$

is continuous with respect to the gap topology on $\mathcal{F}$. Proposition 1.4 and Remark 1.6 shows that this is not the case, but we can restore the continuity by working with the Riesz topology on $\mathcal{F}$. $\square$

Observe that $H \oplus H$ is a symplectic space with complex structure

$$J = \begin{bmatrix} 0 & -\mathbb{1}_H \\ \mathbb{1}_H & 0 \end{bmatrix}$$

and $\Lambda_0 := H \oplus 0$ is a Lagrangian subspace. Define $\mathcal{FL}_0$ the set of Lagrangian subspaces $\Lambda \subset H \oplus H$ such that (Λ_0, Λ) is a Fredholm pair. We topologize $\mathcal{FL}_0$ using the gap distance δ. The space $(\mathcal{FL}_0, \delta)$ is also a classifying space for KO^1 (see [7]).

There is a natural $1-1$ map

$$\Gamma : \mathcal{F}_0 \to \mathcal{FL}_0, \;\; A \mapsto \Gamma_A.$$

According to Lemma 1.2 the map $\Gamma : (\mathcal{F}_0, \rho) \to (\mathcal{FL}_0, \delta)$ is continuous.

Theorem 3.3. *The map*

$$\Gamma : (\mathcal{F}_0, \rho) \to (\mathcal{FL}_0, \delta)$$

is a weak homotopy equivalence.

Proof. Fix $A_0 \in \mathcal{F}_0$. We have to show that for every $n > 0$ the induced map

$$\Gamma_* : \pi_n(\mathcal{F}_0, A_0) \to \pi_n(\mathcal{FL}_0, \Gamma_{A_0})$$

is an isomorphism. Observe first that, according to Bott periodicity,

$$\pi_n(\mathcal{FL}_0, \Gamma_{A_0}) \in \mathcal{G} := \Big\{0, \mathbb{Z}, \mathbb{Z}_2\Big\}.$$

The groups in the family $\mathcal{G}$ have a remarkable property. If $G \in \mathcal{G}$ and $\varphi : G \to G$ is a surjective morphism then φ is an isomorphism.

In [9, §5.3], using the symplectic reduction morphism it is shown that the morphism Γ_* is surjective provided the (general) Floer families are ρ-continuous. This continuity was established in Proposition 2.1. Theorem 3.3 is proved. $\square$

Remark 3.4. Note a curious thing. It is clear that the space $\mathcal{FL}_0$ contains subspaces of $H \oplus H$ which are note the graphs of any linear operator, e.g, $0 \oplus H$. On the other hand, $\mathcal{FL}_0$, contains subspaces which are graphs of operators of operators $T \in \mathcal{F} \backslash \mathcal{F}_0$. For example, the diagonal subspace in $H \oplus H$ is the graph of the identity map $H \to H$ which is Fredholm, but not in $\mathcal{F}_0$.

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