Some solutions to the ‘intro quiz’

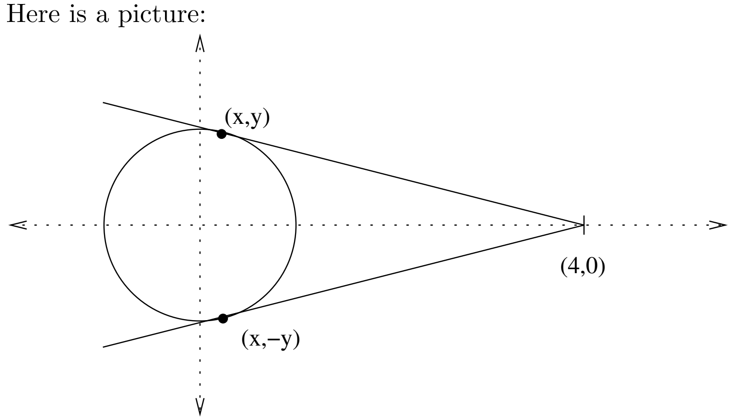
September 7, 2003

Here are some solutions to the intro quiz. The questions were:

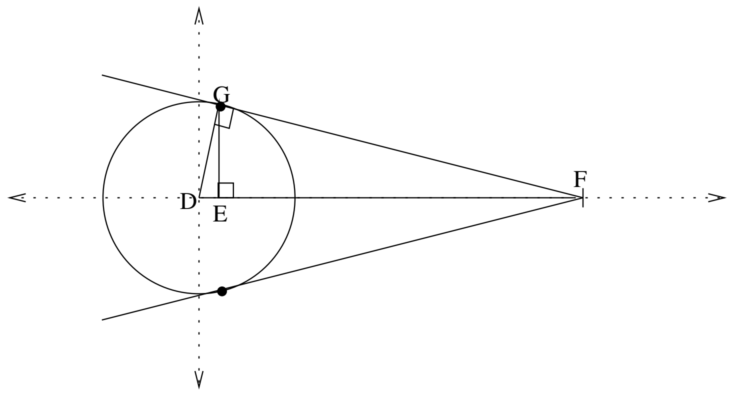
1. Draw a unit circle centered at the origin and the two lines tangent to it that pass through the point $(4, 0)$.
2. What is the relation of the point of tangency (x, y) in the first quadrant to the other point of tangency?
3. What is the point (x, y)?

Some possible solutions to these questions are given below.

1. Here is a picture:



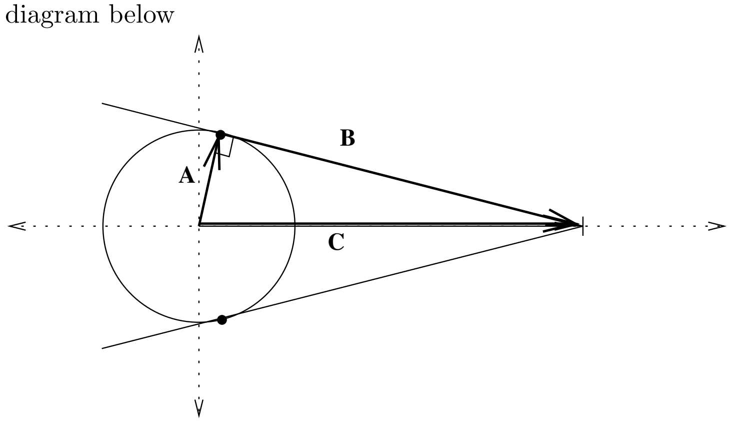
2. Here I was anticipating the observation that the other point of tangency was the point $(x, -y)$, i.e. the two points of tangency are reflections of each other across the x-axis.
3. Referring to the figure below



a good first step was to observe that the length of the line segment GF is $\sqrt{15}$, using the Pythagorean theorem on triangle DGF. Many of you then simultaneously solved for the point that was on a the line GF and circle, using algebra, and maybe some trig. The answer turned out to be $(x, y) = (1/4, \sqrt{15}/4)$. However, I'd like to highlight two other methods.

Similar Triangles Perhaps the quickest method that several of you employed was to observe that triangle DEG is similar to triangle DGF. The answer than falls out since you know that DG has length 1 and GF has length $\sqrt{15}$. Pretty slick!

Vectors I'd like to point out a method using vectors. This is not as slick as the previous method, but does avoid nasty trig. Referring to the diagram below



we identify the sides of the right triangle with vectors **A**, **B** and **C**. (I'll sometimes use boldface to represent vectors when I type.) We have the vector equation

$$\mathbf{A} + \mathbf{B} = \mathbf{C}. \tag{1}$$

We know that $\mathbf{A} = \langle x, y\rangle$ and $\mathbf{C} = \langle 4, 0\rangle$. Because of the right angle, we know that the vector **B** points in the direction of

$$\mathbf{A}^{\perp} = \langle y, -x\rangle$$

(here I rotated the vector to the **A** *right*. In the lecture, a vector was rotated to the *left* causing opposite signs.) The vector $\mathbf{A}^\perp$ is a unit vector, and we know the vector **B** has length $\sqrt{15}$. Therefore

$$\mathbf{B} = \sqrt{15}\mathbf{A}^\perp = \langle \sqrt{15}y, -\sqrt{15}x \rangle$$

Equation 1 now reads

$$\langle x, y \rangle + \sqrt{15}\langle y, -x \rangle = \langle x + \sqrt{15}y, y - \sqrt{15}x \rangle = \langle 4, 0 \rangle.$$

Equating components, we see get the system

$$\begin{aligned} x + \sqrt{15}y &= 4 \\ y - \sqrt{15}x &= 0. \end{aligned}$$

Solving, we get $(x, y) = (1/4, \sqrt{15}/4)$. I'm not saying this vector way is really faster in any way - it's probably slower, but it does illustrate some of the material on vectors that we've been covering.