Hw 2 solutions

Note Title 2/13/2009

3.1 (1c) Pick $\varepsilon > 0$. We need to show that for $n \gg 0$,

$$\frac{n}{n^2+3n+1} \approx_\varepsilon 0.$$

This is equivalent to showing that

$$\frac{n}{n^2+3n+1} = \left|\frac{n}{n^2+3n+1} - 0\right| < \varepsilon.$$

Take $n > \frac{1}{\varepsilon}$

Then

$$\frac{n}{n^2+3n+1} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon.$$

$\square$

$\boxed{3.2\,(1)}$ Let $\varepsilon > 0$.

Since
$$a_n \to L$$
$$b_n \to M$$
we deduce that
$$a_n \approx_{\varepsilon/2} L \quad \text{for} \quad n \gg 0$$
$$b_n \approx_{\varepsilon/2} M \quad \text{for} \quad n \gg 0$$
Therefor
$$a_n + b_n \approx_{\varepsilon} L + M \quad \text{for} \quad n \gg 0. \qquad \square$$

$\boxed{3.2\,(5)}$ Choose $\varepsilon = 1/2$

then $a_n \approx_{1/2} L$ for $n \gg 0$

i.e. there is an N such that for
$$n \geq N, \quad a_n \approx_{1/2} L$$
Therefor $a_N \approx_{1/2} L \approx_{1/2} a_n$ for $n \geq N$
$$\implies a_N \approx_1 a_n$$
Since a_N and a_n are both integers, this is only possible if $a_N = a_n$. Therefore $a_n = a_N$ for all $n \geq N$ $\square$

Proof of 5.1 (∞)

$\boxed{\text{proof of (a)}}$

Claim A

$$\left.\begin{array}{c} b_n \to \infty \\ \text{or} \\ b_n \to L \end{array}\right\} \Longrightarrow \quad b_n \text{ is bounded below}$$

Pf of Claim A

Indeed, if $b_n \to \infty$, then for $M = 1$

$$b_n > M \quad \text{for} \quad n \gg 0$$

i.e. there is an N such that for $n \geq N$, $b_n > M$.

Take $P = \min\{b_1, b_2, \ldots, b_{N-1}, M\}$

Then $b_n \geq P$ for all n, and b_n is bounded below.

If $b_n \to L > 0$, then for $\varepsilon = 1$ we have $b_n \approx_1 L$ $n \gg 0$.

$$\Longrightarrow \quad L-1 < b_n \quad \text{for} \quad n \gg 0$$

That is to say, there is an N such that for $n \geq N$,

$$L-1 < b_n$$

Take $K = \min\{b_1, b_2, \ldots, b_{N-1}, L-1\}$.

Then $b_n \geq K$ for all n.

$\Longrightarrow$ b_n is bounded below.

By Claim A, to prove (9), it suffices to show

$$\left\{ \begin{array}{l} a_n \to \infty \\ b_n \text{ bounded below} \end{array} \right\} \Longrightarrow a_n + b_n \to \infty$$

Let K be a lower bound for $\{b_n\}$

Let $M > 0$

Then, since $a_n \to \infty$

$$a_n > M-K \quad \text{for } n \gg 0$$

Therefore

$$a_n + b_n > M - K + K = M$$

for $n \gg 0$.

Therefore $a_n + b_n \to \infty$.

$\boxed{\text{proof of statement (10)}}$

<u>Claim B</u>

$$\left\{ \begin{array}{c} b_n \to \infty \\ \text{or} \\ b_n \to L > 0 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} b_n \geq K > 0 \\ \text{for } n \gg 0 \end{array} \right\}$$

<u>pf of Claim B</u>:

Suppose $b_n \to \infty$. Take $M = 1$.

Then $b_n > 1$ for $n \gg 0$ ✓

Suppose $b_n \to L > 0$

Take $\varepsilon = \frac{L}{2}$

Then $b_n \approx_{L/2} L$ for $n \gg 0$

$$\Longrightarrow \quad b_n > L - L/2 = L/2 \quad \text{for } n \gg 0.$$

✓

By Claim B, it suffices to show that

$$\left\{\begin{array}{l} a_n \to \infty \\ b_n \geq K > 0, \quad n \gg 0 \end{array}\right\} \Rightarrow \left\{ a_n b_n \to \infty \right\}$$

Let $M > 0$

We know, since $a_n \to \infty$, that:

$$a_n > \frac{M}{K} \quad \text{for} \quad n \gg 0$$

Therefore

$$a_n b_n > \frac{M}{K} \cdot K = M \quad \text{for} \quad n \gg 0$$

therefore $a_n b_n \to \infty$.

$\boxed{\text{proof of statement (ii)}}$

Let $\varepsilon > 0$.

Choose $M = 1/\varepsilon$

Since $a_n \to \infty$, we know

$$a_n > 1/\varepsilon \quad \text{for} \quad n \gg 0$$

$$\Longrightarrow \frac{1}{a_n} < \varepsilon \quad \text{for} \quad n \gg 0$$

therefore $\frac{1}{a_n} \to 0$.

$$\boxed{\text{proof of statement (12)}}$$

Let $M > 0$.

Since $a_n \to 0$,

$$a_n \underset{1/M}{\approx} 0 \quad \text{for} \quad n \gg 0$$

$$\Longrightarrow \quad a_n < 1/M \quad \text{for} \quad n \gg 0$$

$$\Longrightarrow \quad \frac{1}{a_n} > M \quad \text{for} \quad n \gg 0.$$

therefore $\frac{1}{a_n} \to \infty$.

5.2(1) We have

$$\frac{\sqrt{n}-1}{\sqrt{n+1}} \leq \frac{\sqrt{n}+\cos n}{\sqrt{n+1}} \leq \frac{\sqrt{n}+1}{\sqrt{n+1}}$$

$$\frac{\sqrt{n}-1}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}} - \frac{1}{\sqrt{n+1}} = \sqrt{\frac{1}{1+\frac{1}{n}}} - \sqrt{\frac{1}{n+1}} \to 1 - 0$$

(using alg limit thms and problem 5-1)

$$\frac{\sqrt{n}+1}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}} + \frac{1}{\sqrt{n+1}} = \sqrt{\frac{1}{1+\frac{1}{n}}} + \sqrt{\frac{1}{n+1}} \to 1 + 0$$

(using alg limit thms and problem 5-1)

By Squeeze thm,

$$\frac{\sqrt{n}+\cos n}{\sqrt{n+1}} \longrightarrow 1$$